Spectral distributions of noncommutative functions: from free probability to random matrix theory

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Random matrices

Random matrices

Definition (Random matrices)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Elements in the complex $*$-algebra

$$\mathcal{A}_N := M_N(L^{\infty-}(\Omega, \mathbb{P})), \quad \text{where} \quad L^{\infty-}(\Omega, \mathbb{P}) := \bigcap_{1 \leq p < \infty} L^p(\Omega, \mathbb{P}),$$

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Definition (Gaussian random matrix)

A standard Gaussian random matrix (of size $N \times N$) is a hermitian random matrix $X = (X_{k,l})_{k,l=1}^N \in \mathcal{A}_N$ for which

$$\{\mathrm{Re}(X_{k,l}) \mid 1 \leq k \leq l \leq N\} \cup \{\mathrm{Im}(X_{k,l}) \mid 1 \leq k < l \leq N\}$$

are independent Gaussian random variables such that

$$\mathbb{E}[X_{k,l}] = 0 \quad \text{and} \quad \mathbb{E}[|X_{k,l}|^2] = N^{-1} \quad \text{for } 1 \leq k \leq l \leq N.$$

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$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$

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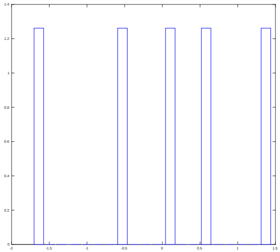

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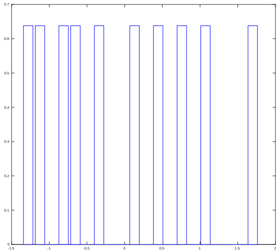

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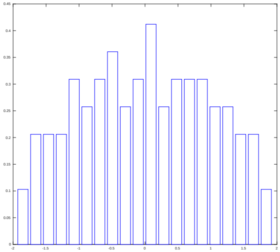

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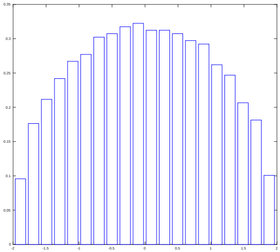

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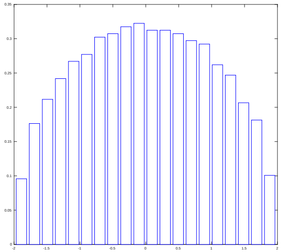

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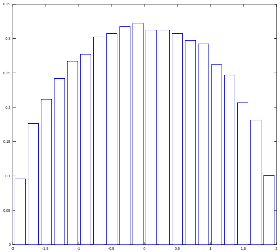

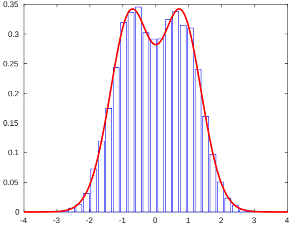

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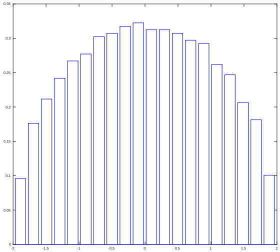

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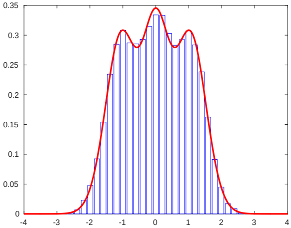

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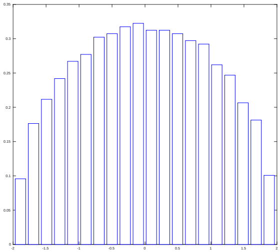

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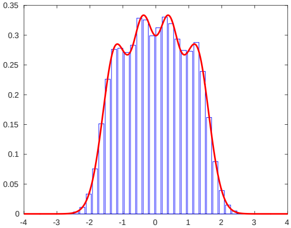

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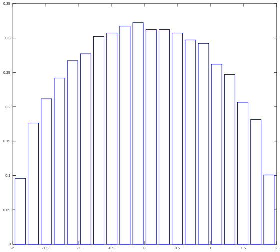

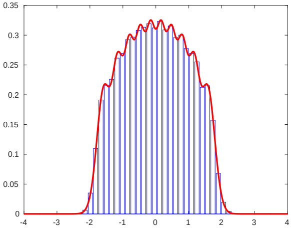

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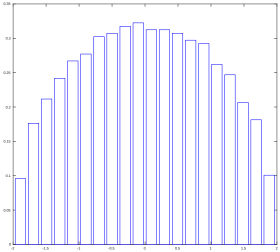

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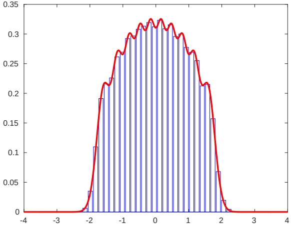

Wigner's semicircle theorem

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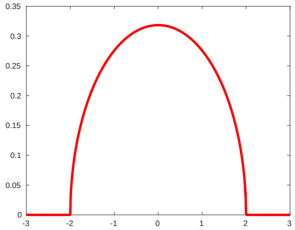
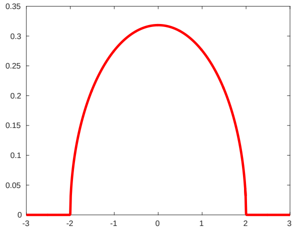


Wigner's semicircle theorem

Semicircular distribution

$$d\mu_S(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,1_{[-2,2]}(t)\,dt$$



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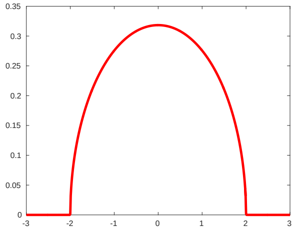

Theorem (Wigner (1955/1958))

Consider a sequence $(X^{(N)})_{N\in\mathbb{N}}$ of standard Gaussian random matrices $X^{(N)} \in \mathcal{A}_N$. Then, for all integers $k \geq 0$, it holds true that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_{\mathbb{R}} t^k\, d\mu_{X_n}(t)\Big] = \int_{\mathbb{R}} t^k\, d\mu_S(t)$$

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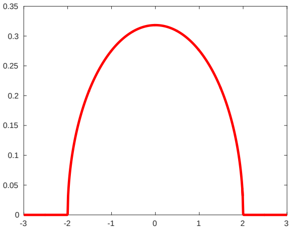

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and in fact even

$$\lim_{n\to\infty} \int_{\mathbb{R}} t^k\, d\mu_{X_n}(t) = \int_{\mathbb{R}} t^k\, d\mu_S(t) \qquad \text{almost surely.}$$

“Functions” in independent random matrices

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First Question

For each $N \in \mathbb{N}$, let independent standard Gaussian random matrices

$$X_1^{(N)}, \dots, X_n^{(N)} \in \mathcal{A}_N$$

be given and suppose that f is "some kind of noncommutative function". What can we say about the asymptotic behavior of the empirical eigenvalue distribution of

$$Y^{(N)} := f(X_1^{(N)}, \dots, X_n^{(N)}) \ ?$$

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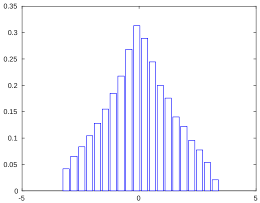

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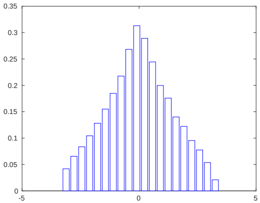

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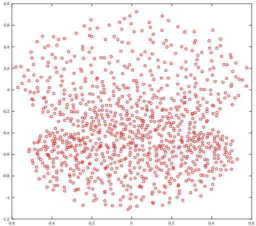

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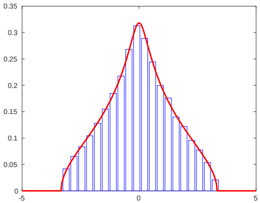

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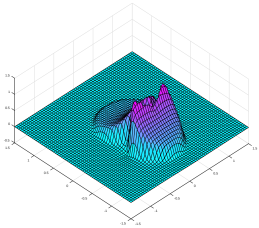

$\rightsquigarrow$ **Free Probability!**

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Noncommutative probability spaces

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A noncommutative probability space $(\mathcal{A}, \phi)$ consists of

- a complex algebra $\mathcal{A}$ with unit $1_{\mathcal{A}}$ and
- a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ satisfying $\phi(1_{\mathcal{A}}) = 1$ (expectation).

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- $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a classical probability space and $\mathbb{E}$ the usual expectation that is given by $\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$.

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- $(M_N(\mathbb{C}), \operatorname{tr}_N)$, where tr_N is the normalized trace on $M_N(\mathbb{C})$.
- $(\mathcal{A}_N, \phi_N)$, with $\mathcal{A}_N = M_N(L^{\infty-}(\Omega, \mathbb{P}))$ and expectation given by

$$\phi_N(X) := \mathbb{E}[\operatorname{tr}_N(X)] = \int_\Omega \operatorname{tr}_N(X(\omega))\, d\mathbb{P}(\omega).$$

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Definition ("analytic distribution")

Let $(\mathcal{A}, \phi)$ be a C^*-probability space and consider $X = X^* \in \mathcal{A}$. The (analytic) distribution of X is the unique Borel probability measure μ_X on $\mathbb{R}$ such that

$$\phi(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all integers } k \geq 0.$$

Free independence

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Let $(\mathcal{A}, \phi)$ be a noncommutative probability space.

(i) Unital subalgebras $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{A}$ are called freely independent (or just free), if

$$\phi(a_1 \cdots a_k) = 0$$

holds, whenever

- $a_j \in \mathcal{A}_{i(j)}$ with $i(j) \in I$ for all $j = 1, \ldots, k$,
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Free probability theory is a highly noncommutative analogue of classical probability theory.

Asymptotic freeness of random matrices

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We have the following multivariate version of Wigner's semicircle law.

Theorem (Voiculescu (1991))

For all $N \in \mathbb{N}$, realize independent standard Gaussian random matrices $X_1^{(N)}, \dots, X_n^{(N)} \in \mathcal{A}_N$. Then, for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$,

$$\lim_{N \to \infty} \mathbb{E}[\mathrm{tr}_N(P(X_1^{(N)}, \dots, X_n^{(N)}))] = \phi(P(S_1, \dots, S_n))$$

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for freely independent semicircular elements $S_1, \dots, S_n$ in some noncommutative probability space $(\mathcal{A}, \phi)$.

This means: Asymptotic freeness relates (in this case)

- the limiting eigenvalue distribution of $Y^{(N)} = P(X_1^{(N)}, \dots, X_n^{(N)})$ and
- the distribution of $Y = P(S_1, \dots, S_n)$ for freely independent semicircular elements $S_1, \dots, S_n$.

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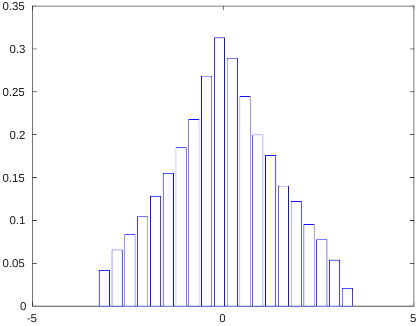

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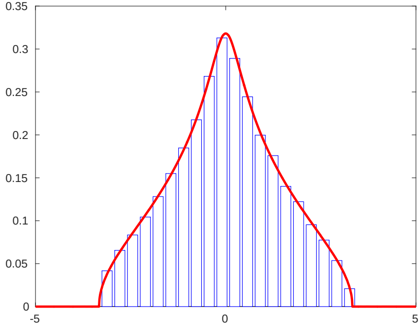

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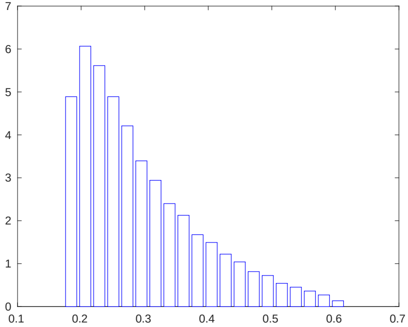

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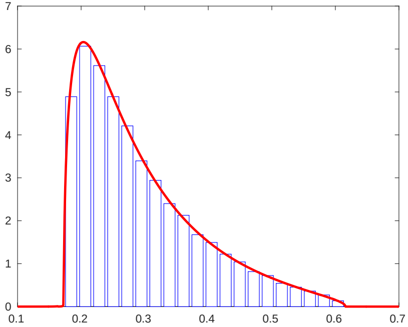

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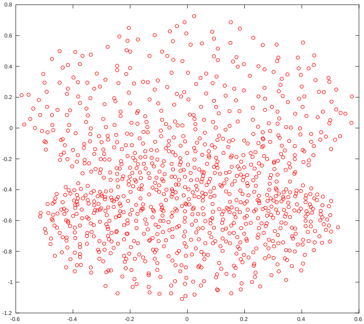

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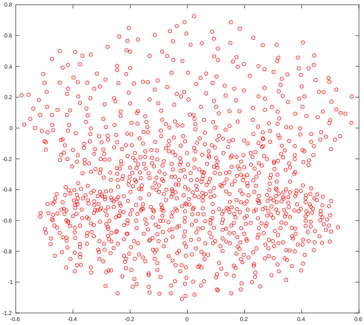

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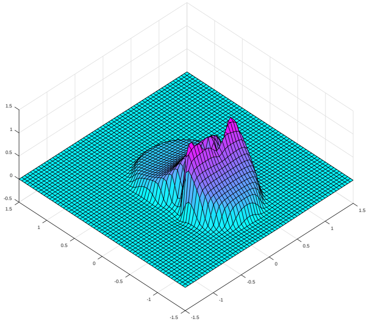

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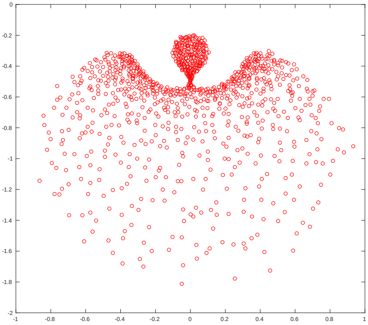

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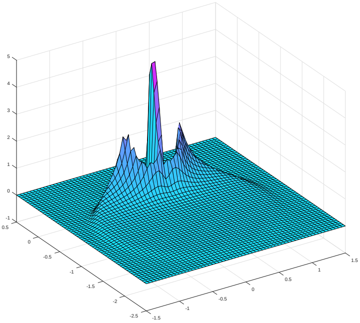

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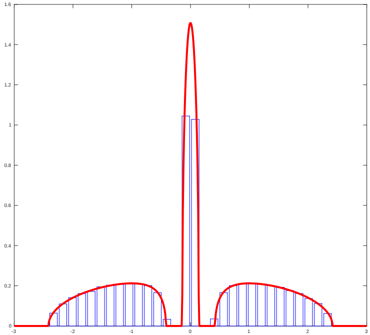

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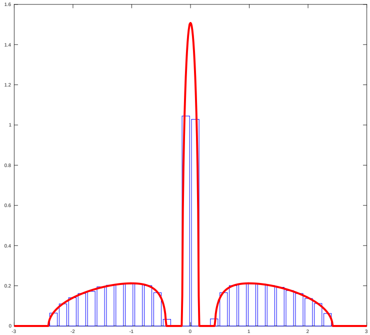

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Example

A standard Gaussian random matrix follows the law of the GUE, which is the probability measure Λ_N on $M_N(\mathbb{C})_{\mathrm{sa}} \cong \mathbb{R}^{N^2}$ that is determined by

$$d\Lambda_N(X) := \frac{1}{Z_N} e^{-\frac{N}{2}\operatorname{Tr}(X^2)}\, dX \quad \text{with} \quad Z_N := 2^{N/2}\Big(\frac{\pi}{N}\Big)^{N^2/2}$$

and $\quad dX := \prod_{k=1}^{N} dX_{k,k} \prod_{1\leq k<l\leq N} d\,\mathrm{Re}(X_{k,l})\, d\,\mathrm{Im}(X_{k,l})$.

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- A single standard Gaussian random matrix $X^{(N)}$ of size $N \times N$ follows

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Idea: replace $\frac{1}{2}\sum_{j=1}^n X_j^2$ by another selfadjoint potential $V(X_1, \dots, X_n)$

Definition (Gibbs laws)

$$d\Lambda_N^V(X_1, \dots, X_n) = \frac{1}{Z_N^V} e^{-N\operatorname{Tr}(V(X_1,\dots,X_n))}\, dX_1 \, \dots \, dX_n$$

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Theorem (Guionnet, Shlyakhtenko (2009))

Let $V \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ with law Λ_N^V. Then, for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$,

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for selfadjoint operators $X_1, \dots, X_n$ in some W^*-probability space $(\mathcal{M}, \tau)$ that satisfy the Schwinger-Dyson equation, i.e.

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Noncommutative and cyclic derivatives

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Definition

(i) The noncommutative derivatives are the linear mappings

$$\partial_1, \ldots, \partial_n : \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}\langle x_1, \ldots, x_n \rangle \otimes \mathbb{C}\langle x_1, \ldots, x_n \rangle$$

which are uniquely determined by the two conditions

- $\partial_j(P_1 P_2) = (\partial_j P_1) \cdot P_2 + P_1 \cdot (\partial_j P_2)$ for all $P_1, P_2 \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$,
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Noncommutative and cyclic derivatives

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(ii) The cyclic derivatives are the linear mappings

$$D_1, \dots, D_n : \ \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle$$

that are defined by $D_j := \tilde{m} \circ \partial_j$, where

$$\tilde{m} : \ \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle$$

denotes the flipped multiplication defined as $\tilde{m}(P_1 \otimes P_2) := P_2 P_1$.

Conjugate variables and free Fisher information

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If $\xi_1, \dots, \xi_n \in L^2(X_1, \dots, X_n; \tau)$ are such that for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$

$$(\tau \otimes \tau)((\partial_j P)(X_1, \dots, X_n)) = \tau(\xi_j P(X_1, \dots, X_n)), \quad j = 1, \dots, n,$$

then $(\xi_1, \dots, \xi_n)$ is called the conjugate system for $(X_1, \dots, X_n)$.

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The (non-microstates) free Fisher information is defined by

$$\Phi^*(X_1, \dots, X_n) := \begin{cases} \sum_{j=1}^{n} \|\xi_j\|_2^2, & \text{if a conjugate system } (\xi_1, \dots, \xi_n) \text{ for } (X_1, \dots, X_n) \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

A useful variant of free entropy dimension

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Suppose that $S_1, \dots, S_n$ are freely independent semicircular elements that are also free from $\{X_1, \dots, X_n\}$, then $(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n)$ admits a conjugate system for each $t > 0$. More precisely, we have

$$\frac{n^2}{C^2 + nt} \leq \Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n) \leq \frac{n}{t} \quad \text{for all } t > 0,$$

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Philosophy

If $\delta^\star(X_1, \dots, X_n) = n$, then $(X_1, \dots, X_n)$ has no “atomic part”.

Results about atoms I

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Theorem (Shlyakhtenko, Skoufranis (2015))

Suppose that

- the operators $X_1, \dots, X_n$ are freely independent and
- the individual analytic distributions $\mu_{X_1}, \dots, \mu_{X_n}$ are all non-atomic,

then the tuple $X = (X_1, \dots, X_n)$ has the strong Atiyah property.

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If $(X_1, \ldots, X_n)$ has the strong Atiyah property, then the following holds:

- For every selfadjoint $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$, the measure of each atom in the analytic distribution $\mu_{\mathbf{Y}}$ of the selfadjoint operator $\mathbf{Y} = \mathbf{P}(X_1, \ldots, X_n)$ is an integer multiple of $\frac{1}{N}$.

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- In particular, if $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ is a non-constant selfadjoint polynomial, then the analytic distribution μ_Y of the selfadjoint operator $Y = P(X_1, \dots, X_n)$ cannot have atoms.

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Let $P \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ be a selfadjoint non-constant noncommutative polynomial and consider the selfadjoint operator

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- Both location and measure of all atoms are fully determined by $\mathbf{P}$.

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Let $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be selfadjoint with degree $d \geq 1$ and consider

$$Y := P(X_1, \dots, X_n).$$

Then there exists some constant $C > 0$ such that

$$|\mathcal{F}_Y(t) - \mathcal{F}_Y(s)| \leq C|t-s|^{\frac{2}{3(2^d-1)}} \qquad \text{for all } s, t \in \mathbb{R}.$$

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In fact, for every $R > \max_{i=1,\dots,n} \|X_i\|$, we can take

$$C = \left(8\Phi^*(X)^{1/2}R\right)^{\frac{2}{3}} \rho_R(P)^{-\frac{2^d}{3(2^d-1)}} \|P\|_R^{-\frac{2}{3(2^d-1)}} \prod_{k=1}^{d-1} \left(\frac{d!}{(d-k)!}\right)^{\frac{2^k}{3(2^d-1)}},$$

where $\|P\|_R$ and $\rho_R(P)$ are quantities that depend only on P and R.

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(i) The empirical eigenvalue distribution $\mu_{Y^{(N)}}$ of

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(ii) There is a constant $C > 0$ such that

$$d_{\mathrm{Kol}}(\overline{\mu}_{Y^{(N)}}, \mu) \leq C N^{-\frac{1}{13 \cdot 2^{d+2} - 60}} \qquad \text{for all } N \in \mathbb{N}.$$

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Thank you!