What actually is free probability theory?

Tobias Mai
Saarland University

January 21, 2021

" free probability = nc probability + freeness "

1) The language of noncommutative probability theory

classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$

set σ-field prob. measure

expectation: $\mathbb{E}[X] := \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$

e.g. on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$

$(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E}) \rightsquigarrow (\mathcal{A}, \varphi)$

Definition: A noncommutative probability space $(\mathcal{A}, \varphi)$ consists of

elements are nc rand. var.

* a unital complex algebra $\mathcal{A}$, $1_\mathcal{A} \in \mathcal{A}$,
* a linear map $\varphi: \mathcal{A} \to \mathbb{C}$ s.t. $\varphi(1_\mathcal{A}) = 1$.

"expectation" on $\mathcal{A}$

Examples: 1) $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$

2) $(M_N(\mathbb{C}), tr_N)$, $tr_N := \frac{1}{N} \mathrm{Tr}$, i.e.

$$tr_N(a) := \frac{1}{N} \sum_{i=1} a_{ii}$$

Definition: The **noncommutative distribution** of $X = (X_1, \dots, X_n) \in \mathcal{A}^n$ is

$$\mu_X : \underbrace{\mathbb{C}\langle t_1, \dots, t_n \rangle}_{=: \mathbb{C}\langle t \rangle,\ t = (t_1, \dots, t_n)} \longrightarrow \mathbb{C}, \quad t_{i_1} \cdots t_{i_k} \longmapsto \varphi(X_{i_1} \cdots X_{i_k}).$$

Example: $X = (X_1, \dots, X_n) \in L^\infty_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})^n$, then

$$\mathbb{E}[p(X)] = \int_{\mathbb{R}^n} p(t)\, d\mu_X(t) \quad \forall\, p \in \mathbb{C}[t],$$

where $\mu_X(B) := \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in B\})$, $B \in \mathcal{B}(\mathbb{R}^n)$.

joint distribution of X

Noncommutative probability spaces with more structure

Definition: We call $(\mathcal{A}, \varphi)$ a

... C^*-probability space if

* $\mathcal{A}$ is a unital C^*-algebra and
* $\varphi : \mathcal{A} \to \mathbb{C}$ is a state (i.e., positive and unital).

... W^*-probability space if

* $\mathcal{A}$ is a von Neumann algebra and
* $\varphi : \mathcal{A} \to \mathbb{C}$ is a faithful normal tracial state.

Definition: Let $(\mathcal{A}, \varphi)$ be a C^*-probability space.

For $X = X^* \in \mathcal{A}$, there exists a unique Borel probability measure μ_X on $\mathbb{R}$ such that

$$\varphi(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \forall\, k = 0, 1, 2, \ldots$$

We call μ_X the **analytic distribution of X**.

Note: If $(\mathcal{A}, \varphi)$ is a W^*-probability space, then $\mu_X = \varphi \circ E_X$, where E_X is the resolution of identity for X.

An important analytic tool: the Cauchy transform

μ Borel prob measure on $\mathbb{R}$, $\mathbb{C}^{\pm} := \{ z \in \mathbb{C} \mid \pm \operatorname{Im}(z) > 0 \}$

$$G_\mu : \mathbb{C}^+ \to \mathbb{C}^-, \quad z \mapsto \int_{\mathbb{R}} \frac{1}{z-t} \, d\mu(t) \quad \text{analytic}$$

Note: $G_{\mu_X}(z) = \varphi\left((z-X)^{-1} \right) \quad \forall z \in \mathbb{C}^+$, $G_X := G_{\mu_X}$

Theorem: (Stieltjes inversion formula)

Put $d\mu_\varepsilon(t) := -\frac{1}{\pi} \operatorname{Im}\left(G_\mu(t + i\varepsilon) \right) dt$; then

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} f(t) \, d\mu_\varepsilon(t) = \int_{\mathbb{R}} f(t) \, d\mu(t) \qquad \forall f \in C_b(\mathbb{R}).$$

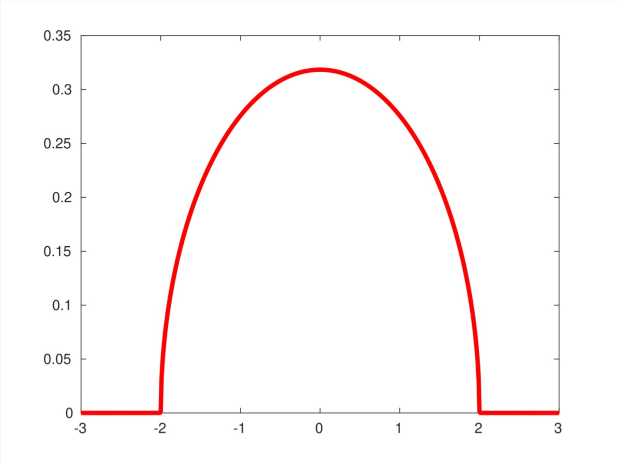
The semicircular distribution and semicircular elements

Definition:

1) The probability measure

$$d\sigma(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,\mathbb{1}_{[-2,2]}(t)\,dt$$

is the **semicircular distribution.**



2) An operator $S = S^*$ in a C^*-probability space is called **semicircular element**, if $\mu_S = \sigma$.

$$\Longleftrightarrow \quad \forall z \in \mathbb{C}^+ : \quad z\,G_S(z) = 1 + G_S(z)^2$$

2) The notion of free independence

$X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ classically independent

$\iff \quad \mathbb{E}[X^n Y^m] = \mathbb{E}[X^n]\,\mathbb{E}[Y^m] \quad \forall m, n \geq 0.$

Definition: $(\mathcal{A}, \varphi)$ nc prob. space

1) A family $(\mathcal{A}_i)_{i \in I}$ of unital subalgebras of $\mathcal{A}$ is called **free** if $\forall n \in \mathbb{N}\ \forall i_1, \dots, i_n \in I$, $i_1 \neq i_2 \neq \dots \neq i_n$

$$\left.\begin{array}{l} a_k \in \mathcal{A}_{i_k} \quad \forall k = 1, \dots, n \\ \varphi(a_k) = 0 \end{array}\right\} \implies \varphi(a_1 \cdots a_k) = 0.$$

2) A family $(X_i)_{i\in I}$ of nc rand. var. in $\mathcal{A}$ is called **free** if $(\mathrm{alg}(1_{\mathcal{A}}, X_i))_{i\in I}$ are free in the sense of 1).

Notice: free independence provides a "rule" to compute mixed moments

e.g. 1) X, Y free $\Rightarrow$ $\underbrace{\varphi\big((X-\varphi(X)1_{\mathcal{A}})(Y-\varphi(Y)1_{\mathcal{A}})\big)}=0$

$$= \varphi(XY) - \varphi(X)\varphi(Y)$$

$$\Rightarrow \varphi(XY) = \varphi(X)\varphi(Y)$$

2) $\varphi(XYXY) = \varphi(X^2)\varphi(Y)^2 + \varphi(X)^2\varphi(Y^2) - \varphi(X)^2\varphi(Y)^2$

Group algebras and algebraically free groups

G (discrete) group, $e \in G$ neutral element

$$\mathbb{C}[G] := \left\{ \alpha : G \to \mathbb{C} \;\middle|\; \#\{ g \in G \mid \alpha(g) \neq 0 \} < \infty \right\}$$

$$(\alpha * \beta)(g) := \sum_{h \in G} \alpha(g h^{-1}) \beta(h)$$

$$\varphi_G : \mathbb{C}[G] \to \mathbb{C}, \quad \alpha \mapsto \alpha(e) \qquad \rightsquigarrow \quad (\mathbb{C}[G], \varphi_G)$$

Notice: $\mathbb{C}[G] = \operatorname{span}\{ \delta_g \mid g \in G \}$, $\delta_g(h) = \begin{cases} 1, & h = g \\ 0, & h \neq g \end{cases}$

$$\delta_g * \delta_h = \delta_{gh}$$

Theorem: G_1, G_2 subgroups of G, which are algebraically free, then $\mathbb{C}[G_1], \mathbb{C}[G_2] \subseteq \mathbb{C}[G]$ are freely independent in $(\mathbb{C}[G], \varphi_G)$.

Proof:

$$\left.\begin{array}{l} i_1 \neq i_2 \neq \dots \neq i_n, \\ \alpha_k \in \mathbb{C}[G_{i_k}], \\ \alpha_k(e) = \varphi_G(\alpha_k) = 0 \end{array}\right\}$$

$$\varphi(\alpha_1 * \cdots * \alpha_n) = (\alpha_1 * \cdots * \alpha_n)(e)$$

$$= \sum_{\substack{g_1, \dots, g_n \in G \\ g_1 \cdots g_n = e}} \alpha_1(\underbrace{g_1}_{g_1 \in G_{i_1}}) \cdots \alpha_n(\underbrace{g_n}_{g_n \in G_{i_n}}) = 0$$

$$\Rightarrow \exists k : \; g_k = e$$

$\square$

Group von Neumann algebras

$\ell^2(G)$ = completion of $\mathbb{C}[G]$ w.r.t.

$$\langle \alpha, \beta \rangle := \sum_{g \in G} \alpha(g)\overline{\beta(g)} \qquad \forall\, \alpha, \beta \in \mathbb{C}[G]$$

i.e., $\ell^2(G)$ Hilbert space with ONB $(\delta_g)_{g \in G}$

left regular representation

$\lambda : G \to B(\ell^2(G))$, $g \mapsto \lambda_g$, $\lambda_g \delta_h = \delta_{gh}$

$\rightsquigarrow \quad L(G) := \lambda(G)'' \subset B(\ell^2(G))$,

$\tau : L(G) \to \mathbb{C}, \quad X \mapsto \langle Xe, e \rangle$

faithful normal tracial state

Notice: freeness goes over to the closure, i.e.

$L(G_1), L(G_2)$ are free in $(L(G), \tau)$

Example: $\mathbb{F}_n$ = free group with n generators $g_1, \dots, g_n$

$\left(\text{e.g., } \mathbb{F}_1 = \langle g \rangle = \{ g^n \mid n \in \mathbb{Z} \} \cong \mathbb{Z} \right)$

$L(\mathbb{F}_n)$ are the free group factors

The isomorphism problem for the free group factors

??? $\boxed{m,n \geq 2 : \mathcal{L}(\mathbb{F}_m) \cong \mathcal{L}(\mathbb{F}_n) \Rightarrow m = n}$???

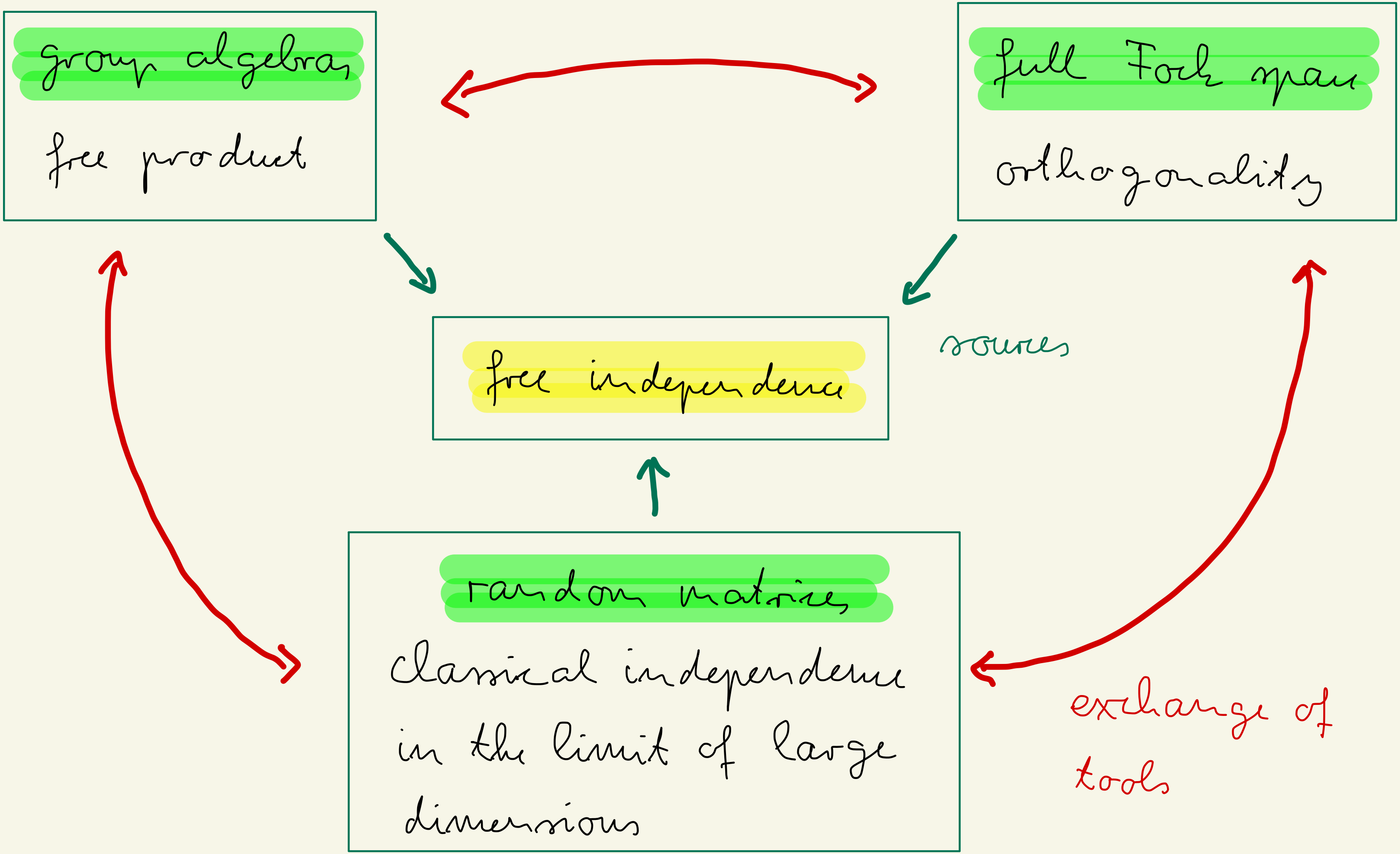
Still open, but we know ...

Theorem (Dykema 1994, Radulescu 1994)

Either $\mathcal{L}(\mathbb{F}_m) \cong \mathcal{L}(\mathbb{F}_n) \quad \forall m,n \geq 2$

or $\qquad \mathcal{L}(\mathbb{F}_m) \not\cong \mathcal{L}(\mathbb{F}_n) \quad \forall m,n \geq 2,\ m \neq n.$

!


group algebras
free product
full Fock space
orthogonality
free independence
sources
random matrices
classical independence in the limit of large dimensions
exchange of tools

Random matrices

... are elements in $\Big(\underbrace{M_N(\mathbb{C}) \otimes L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})}_{=: \mathcal{A}_N}, \underbrace{\mathrm{tr}_N \otimes \mathbb{E}}_{=: \varphi_N}\Big)$

where $$L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) := \bigcap_{p \geq 1} L^p(\Omega, \mathcal{F}, \mathbb{P}).$$

Example: GUE

$X = (X_{k,l})_{k,l=1}^N$ random matrix, $X^* = X$, s.t.

$\{X_{k,l} \mid k,l = 1, \ldots, N\}$ is a complex Gaussian family with

$$\mathbb{E}[X_{ij} X_{kl}] = \frac{1}{N} \delta_{il} \delta_{jk}.$$

Definition: $X = X^* \in \mathcal{A}_N$, $\lambda_1, \dots, \lambda_N$ eigenvalues

$$\mu_X := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}$$

random probability measure

is the empirical eigenvalue distribution of X

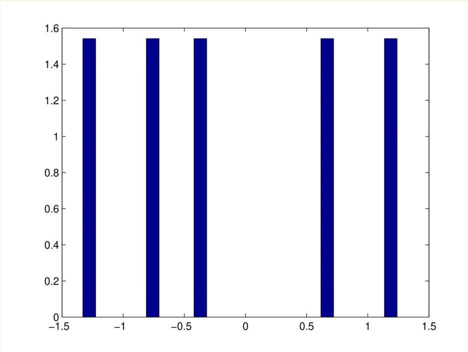
Note: The averaged eigenvalue distribution

$$\bar{\mu}_X := \mathbb{E}[\mu_X]$$

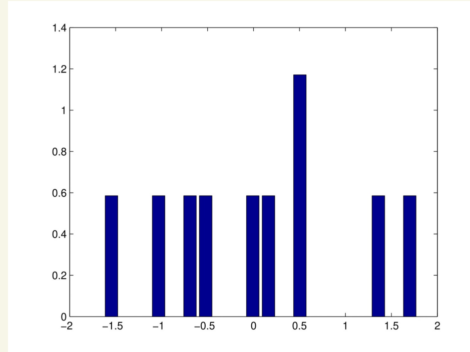
deterministic probability measure

satisfies

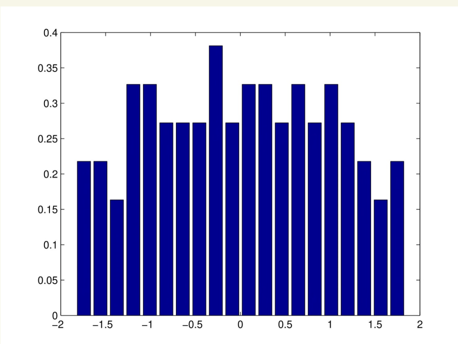
$$\varphi_N(X^k) = \mathbb{E}[tr_N(X^k)] = \mathbb{E}\left[\frac{1}{N}\sum_{j=1}^{N} \lambda_j^k\right] = \mathbb{E}\left[\int_{\mathbb{R}} t^k \, d\mu_X(t)\right] = \int_{\mathbb{R}} t^k \, d\bar{\mu}_X(t)$$



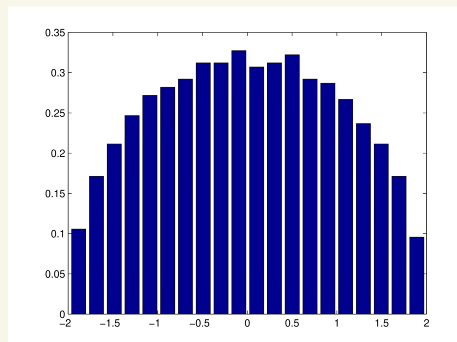
$N = 5$



$N = 10$

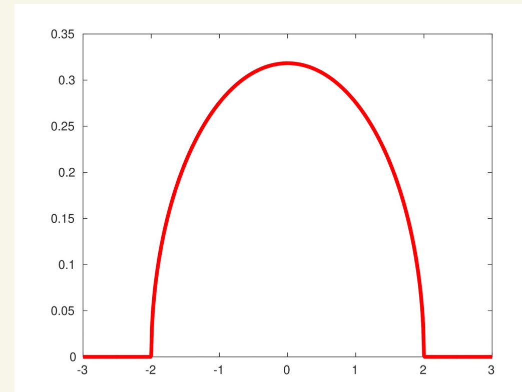


$N = 100$



$N = 1000$

Wigner's theorem (1955)



$$\mu_{X^{(N)}} \xrightarrow[\text{dist}]{N\to\infty} \sigma$$

$$X^{N} \xrightarrow[\text{dist}]{N\to\infty} S$$

Theorem (Voiculescu, 1991)

Consider $X_1^{(N)}, \dots, X_n^{(N)} \in \mathcal{A}_N$ independent GUEs. Then

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{tr}_N\left(p\left(X_1^{(N)}, \dots, X_n^{(N)}\right)\right)\right] = \varphi\left(p\left(S_1, \dots, S_n\right)\right) \quad \forall p \in \mathbb{C}\langle t \rangle$$

i.e. $$\left(X_1^{(N)}, \dots, X_n^{(N)}\right) \xrightarrow[\text{distr}]{N\to\infty} \left(S_1, \dots, S_n\right),$$

where $S_1, \dots, S_n$ are freely independent semicircular elements.

Consequence

f "nc function", $Y^{(N)} := f(X_1^{(N)}, \dots, X_n^{(N)}) \xrightarrow{N \to \infty} f(S_1, \dots, S_n)$

↳ e.g. * nc polynomial

* nc rational function

* linear pencil, i.e.

$$Y^{(N)} = a_1 \otimes X_1^{(N)} + \dots + a_n \otimes X_n^{(N)} \in M_k(\mathbb{C}) \otimes \mathcal{A}_N$$

$a_1, \dots, a_n \in M_k(\mathbb{C})$ s.a.

Fact: $\mathbb{S} := a_1 \otimes S_1 + \ldots + a_n \otimes S_n \in M_k(\mathbb{C}) \otimes \mathcal{A}$

is an operator-valued semicircular element

Define $G_{\mathbb{S}} : \mathbb{H}^+(M_k(\mathbb{C})) \to \mathbb{H}^-(M_k(\mathbb{C})),$

$$b \longmapsto (\mathrm{id} \otimes \varphi)\left((b \otimes 1 - \mathbb{S})^{-1} \right)$$

for $\mathbb{H}^{\pm}(M_k(\mathbb{C})) := \{ b \in M_k(\mathbb{C}) \mid \pm \operatorname{Im}(b) \geq 0, \operatorname{Im}(b) \text{ inv.} \}$.

Then: $b\, G_{\mathbb{S}}(b) = 1 + \eta(G_{\mathbb{S}}(b))\, G_{\mathbb{S}}(b) \quad \forall\, b \in \mathbb{H}^+(M_k(\mathbb{C}))$

when $\eta : M_k(\mathbb{C}) \to M_k(\mathbb{C}), \; b \mapsto \sum_{j=1}^{n} a_j\, b\, a_j$

Theorem: (Helton, Rashidi Far, Speicher, 2007)

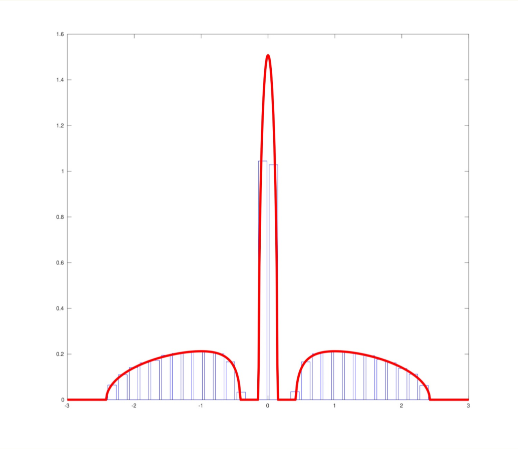
$$h_b : \mathbb{H}^-(M_k(\mathbb{C})) \to \mathbb{H}^-(M_k(\mathbb{C})), \quad w \mapsto (b - \eta(w))^{-1}$$

Then, for every $w_0 \in \mathbb{H}^-(M_k(\mathbb{C}))$, $h_b^{\circ n}(w_0) \xrightarrow{n \to \infty} G_S(b)$

Notice: $G_S(z) = \mathrm{tr}_k(G_S(z 1_k)) \quad \forall z \in \mathbb{C}^+$

Example:

$$S = \begin{pmatrix} 0 & S_1 & S_2 \\ S_1 & \frac{1}{10} S_3 & 0 \\ S_2 & 0 & \frac{1}{10} S_3 \end{pmatrix}$$



μ_S

$N = 1000$