# Free differential calculus I: on the $L^2$-theory for free differential operators and regularity of noncommutative distributions

Let $(\mathcal{M}, \tau)$ be a tracial $W^*$-probability space and let $X = (X_1, \dots, X_n)$ be a tuple of selfadjoint operators in $\mathcal{M}$.

1. Def: noncommutative distribution of $X$,

$$\mu_X : \mathbb{C}\langle t_1, \dots, t_n \rangle \to \mathbb{C}, \quad t_{i_1} \cdots t_{i_k} \mapsto \tau(X_{i_1} \cdots X_{i_k}).$$

2. Def: (analytic) distribution of $Y = Y^* \in \mathcal{M}$,

$$\tau(Y^k) = \int_{\mathbb{R}} t^k \, d\mu_Y(t) \qquad \forall k \in \mathbb{N}_0 .$$

Philosophy: regularity of nc distributions

$\mu_X$

| properties of $X = (X_1, \dots, X_n)$ | $\dashrightarrow$ | properties of $\mu_Y$ for $Y = f(X_1, \dots, X_n)$ |
|---|---|---|

- $\delta^*(X) = n$ $\longrightarrow$ • absence of atoms
- $\chi^*(X) > -\infty$ $\longrightarrow$ • Hölder continuity of $\mathcal{F}_Y(t) := \mu_Y((-\infty, t])$
- $\Phi^*(X) < \infty$
- dual system $\longrightarrow$ • absolute continuity

3. Def: On $\mathbb{C}\langle t\rangle := \mathbb{C}\langle t_1, \dots, t_n\rangle$, we define the nc derivatives

$$\partial_1, \dots, \partial_n : \mathbb{C}\langle t\rangle \longrightarrow \mathbb{C}\langle t\rangle \otimes \mathbb{C}\langle t\rangle$$

as the unique derivations satisfying $\partial_i t_j = \delta_{ij} 1 \otimes 1$; note that $\mathbb{C}\langle t\rangle \otimes \mathbb{C}\langle t\rangle$ forms a $\mathbb{C}\langle t\rangle$-bimodule by $P_1 \cdot (Q_1 \otimes Q_2) \cdot P_2 := (P_1 Q_1) \otimes (Q_2 P_2)$.

Ex: $\partial_1 (t_1 t_2^2 t_1) = 1 \otimes t_2^2 t_1 + t_1 t_2^2 \otimes 1$,

$\partial_2 (t_1 t_2^2 t_1) = t_1 \otimes t_2 t_1 + t_1 t_2 \otimes t_1$.

4. Def: Put $M_0 := vN(X_1, \dots, X_n) \subseteq M$. We say that

$$\xi = (\xi_1, \dots, \xi_n) \in L^2(M_0, \tau)^n$$

are the conjugate variables of $X$ if

$$\langle P(X), \xi_i \rangle_\tau = \langle (\partial_i P)(X), 1 \otimes 1 \rangle_{\tau \otimes \tau} \qquad (1)$$

for all $P \in \mathbb{C}\langle t\rangle$ and $i = 1, \dots, n$.

Remark: (1) determines $\xi$ uniquely as $\mathbb{C}\langle X\rangle$, the subalgebra of $M_0$ generated by $X$, is dense in $L^2(M_0, \tau)$.

5. Def: (non-microstates) free Fisher information

$$\Phi^*(X) := \begin{cases} \sum_{i=1}^n \|\xi_i\|_2^2, & \text{if conjugate variables } \xi = (\xi_1, \dots, \xi_n) \text{ of } X \text{ exist} \\ \infty, & \text{otherwise} \end{cases}$$

**6. Thm** (M., Speicher, Weber, '17):

If $\Phi^*(X) < \infty$, then $ev_X : \mathbb{C}\langle t\rangle \to \mathbb{C}\langle X\rangle$, $P \mapsto P(X)$ is an isomorphism.

**Thus:** $\partial_1, \dots, \partial_n$ induce densely defined unbounded linear operators, $i = 1, \dots, n$,

$$\partial_{X_i} : L^2(M_0, \tau) \supseteq \operatorname{dom} \partial_{X_i} \longrightarrow L^2(M_0,\tau) \otimes L^2(M_0,\tau)$$
$$P(X) \longmapsto (\partial_i P)(X)$$

with $\operatorname{dom} \partial_{X_i} := \mathbb{C}\langle X\rangle$.

The adjoint $\partial_{X_i}^*$ satisfies $1 \otimes 1 \in \operatorname{dom} \partial_{X_i}^*$ with $\partial_{X_i}^*(1\otimes 1) = \xi_i$ by (1); moreover (Voiculescu, '98),

$$\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle \subseteq \operatorname{dom} \partial_{X_i}^* .$$

Hence, $\partial_{X_i}^*$ is densely defined and $\partial_{X_i}$ is closable.

**7. Def.:** We say that $\xi$ are Lipschitz conjugate variables if

$$\xi_i \in \operatorname{dom} \overline{\partial_{X_i}} \quad \text{and} \quad \overline{\partial_{X_i}}\, \xi_i \in M_0 \bar{\otimes} M_0 \quad \forall\, i = 1, \dots, n.$$

**8. Thm** (Dabrowski, '05):

For all $Y \in \mathbb{C}\langle X\rangle$ and $i = 1, \dots, n$,

- $\| \partial_i (Y \otimes 1) \|_2 \leq \| \xi_i \|_2 \, \| Y \|$
- $\| (\mathrm{id} \otimes \tau)(\partial_i Y) \|_2 \leq 2 \, \| \xi_i \|_2 \, \| Y \|$

**9. Thm** (M., Speicher, Weber, '17; Charlesworth, Shlyakhtenko, '16):

If $\Phi^*(X) < \infty$ (or $\delta^*(X) = n$), then $\mu_{P(X)}$ for $P = P^* \in \mathbb{C}\langle t \rangle$ non-constant has no atoms.

**Proof:**

Suppose that $0 \neq \mu_{P(X)}(\{\alpha\}) = \tau(p)$, where $p := E_{P(X)}(\{\alpha\})$ is a spectral projection of $P(X)$; then

$$(P(X) - \alpha 1)p = 0.$$

**Idea:** prove that

$$\left.\begin{array}{l} 0 \neq P \in \mathbb{C}\langle t \rangle \\ 0 \neq p \in \mathcal{M}_0 \text{ projection} \end{array}\right\} \Longrightarrow P(X)p \neq 0. \qquad (2)$$

**Strategy:** "reduction", i.e., prove that

$$\left.\begin{array}{l} P \in \mathbb{C}\langle t \rangle \\ 0 \neq p \in \mathcal{M}_0 \text{ projection} \\ P(X)p = 0 \end{array}\right\} \Longrightarrow \begin{array}{l} \exists\, 0 \neq q \in \mathcal{M}_0 \text{ projection}: \\ (\Delta_{q,i} P)(X)p = 0 \end{array} \qquad (3)$$

$q$ projection onto $\ker P(X)^*$

(see Prop. 10)

**Def:** $\Delta_{v,i} := \tau_v \circ \partial_i$, for any $v \in \mathcal{M}_0$, where

$$\tau_v : \mathbb{C}\langle t \rangle \to \mathbb{C}, \quad P \mapsto \tau(v^* P(X)).$$

**Observation:** iterating (3) shows (2); indeed, if $\deg P = d$, we find that

$$0 = (\Delta_{q_d, i_d} \cdots \Delta_{q_1, i_1} P)(X)p = \mathrm{coeff}_{i_1, \dots, i_d}(P)\, \overbrace{\tau(q_1) \cdots \tau(q_d)}^{\neq 0}\, p \qquad \square$$

10. Prop (M, Speicher, Weber, '17):
Suppose that $\Phi^*(X) < \infty$. Then, for $P \in \mathbb{C}\langle x \rangle$, $u, v \in \mathcal{M}_0$, $Y_1, Y_2 \in \mathbb{C}\langle X \rangle$, $i = 1, \dots, n$,

$$|\langle v^* \cdot (\partial_i P)(X) \cdot u, Y_1 \otimes Y_2 \rangle| \tag{4}$$
$$\leq 4 \|\xi_i\|_2 \left( \|P(X)u\|_2 \|v\| + \|P(X)^* v\|_2 \|u\| \right) \|Y_1\| \, \|Y_2\|$$

Proof ... uses Kaplansky's density theorem and Thm 8. $\square$

$\frown$ Hölder continuity of $\mathcal{F}_{P(X)}$ ?

11. Lemma: Suppose that $\Phi^*(X) < \infty$. Take $P \in \mathbb{C}\langle x \rangle$ non-constant. Then, for every projection $p \in \mathcal{M}_0$, we find a projection $q \in \mathcal{M}_0$ such that

$$\tau(p) = \tau(q) \quad \text{and} \quad \|P(X)^* q\|_2 = \|P(X) p\|_2 .$$

$\frown$

Then: (4) yields that ($u = p$, $v = q$, $Y_1 = 1$)

$$|\langle (\Delta_{i,q} P)(X) p, Y_2 \rangle| \leq 8 \|\xi_i\|_2 \, \|P(X)p\|_2 \, \|Y_2\| .$$

By Kaplansky, this extends to $Y_2 \in \mathcal{M}_0$; take

$$Y_2 = (\Delta_{q,i} P)(X) p .$$

$$\Rightarrow \quad \underbrace{\|(\Delta_{q,i} P)(X) p\|_2^2}_{} \leq 8 \, \|\xi_i\|_2 \, \underbrace{\|P(X) p\|_2}_{} \, \|(\Delta_{q,i} P)(X) p\| \tag{5}$$

$\rightsquigarrow$ quantified version of (2) !

12 Lemma: Let $Y = Y^* \in \mathcal{M}$ be given. Suppose that

$\exists \alpha > 1, c > 0: \forall s \in \mathbb{R} \quad \forall p \in \sigma N(Y)$ spectral projection of $Y$

$$c \, \|(Y-s)p\|_2 \geq \|p\|_2^\alpha , \tag{6}$$

then $\mathcal{F}_Y(t) := \mu_Y((-\infty, t])$ is Hoelder continuous with exponent $\beta = \frac{2}{\alpha - 1}$.

Proof: Use $p = E_Y((s,t])$; then

$$\|(Y-s)p\|_2 \leq |t-s| \, \mu_Y((s,t])^{1/2} \text{ and}$$

$$\|p\|_2 = \mu_Y((s,t])^{1/2}$$

$$\overset{(6)}{\Longrightarrow} \quad |\mathcal{F}_Y(t) - \mathcal{F}_Y(s)| = \mu_Y((s,t]) \leq c^\beta |t-s|^\beta .$$

□

13. Thm (Banna, M., '18)

If $\Phi^*(X) < \infty$, then $\mathcal{F}_{P(X)}$, for $P = P^* \in \mathbb{C}\langle x \rangle$ with $d := \deg P \geq 1$, is Hölder continuous with exponent $\beta = \frac{2}{3(2^d - 1)}$.

If $X$ admits Lipschitz conjugate variables, we even get $\beta = \frac{1}{2^d - 1}$.

Proof: Iterate (5) to get a bound like (6); use that

$$\|(\Delta_{q_k, i_k} \cdots \Delta_{q_k, i_k} P)(X)\| \leq \frac{d!}{(d-k)!} \frac{\tau(q_k) \cdots \tau(q_1)}{R^k} \|P\|_R ,$$

where $R > 0$ is chosen such that

$$R \geq \max_{i=1,\dots,n} \|X_i\|$$

and where

$$\|P\|_R := \sum_{k=0}^{d} \sum_{1 \leq i_1,\dots,i_k \leq n} |a_{i_1,\dots,i_k}| R^k$$

for every $P = \sum_{k=0}^{d} \sum_{1 \leq i_1,\dots,i_k \leq n} a_{i_1,\dots,i_k} t_{i_1} \cdots t_{i_k}$.

$\square$

**14. Corollary**

If $\Phi^*(X) < \infty$, then $\chi^*(P(X)) < \infty$ for every non-constant $P = P^* \in \mathbb{C}\langle t \rangle$.

Question: Does this remain true if $\chi^*(X) < \infty$ ?