The free field meets free probability theory

I. The free group factors

G discrete group, $\ell^2(G) := \left\{ \xi : G \to \mathbb{C} \;\middle|\; \sum_{x\in G} |\xi(x)|^2 < \infty \right\}$

inner product: $\langle \xi, \eta \rangle := \sum_{x\in G} \xi(x)\overline{\eta(x)}$

orthonormal basis: $(\delta_g)_{g\in G}$, $\delta_g(x) := \begin{cases} 1, & x = g \\ 0, & \text{otherwise} \end{cases}$

(left regular) representation: $\lambda : G \to B(\ell^2(G))$, $g \mapsto \lambda_g$,

where $(\lambda_g \xi)(x) := \xi(g^{-1}x)$, i.e. $\lambda_g \delta_h = \delta_{gh}$.

Def: • $L(G) := \overline{\operatorname{span} \lambda(G)}^{WOT} \subseteq B(\ell^2(G))$

• $\tau : L(G) \to \mathbb{C}$, $X \mapsto \langle X\delta_e, \delta_e \rangle$,

faithful normal tracial state

Thm (Murray, von Neumann, 1943)

$L(G)$ is a factor, i.e., $L(G) \cap L(G)' = \mathbb{C}1$ $\iff$ G is icc, i.e. $\forall g \neq e : |\{hgh^{-1} \mid h \in G\}| = \infty$

Ex: $G = \mathbb{F}_n = \langle g_1, \dots, g_n \rangle$ free group, $n \geq 2$;

$\rightsquigarrow$ $L(\mathbb{F}_n)$ free group factor; $L(\mathbb{F}_n) = vN(u_1, \dots, u_n)$, $u_j := \lambda_{g_j}$

Open problem: $m, n \geq 2$, $L(\mathbb{F}_m) \cong L(\mathbb{F}_n) \implies m = n$???

II. Free probability theory

Voiculescu, 1985

$$\begin{array}{ccc} \mathbb{F}_{m+n} \cong & \mathbb{F}_m * & \mathbb{F}_n \\ & \cup & \cup \\ & \mathbb{F}_m & \mathbb{F}_n \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} L(\mathbb{F}_m) *_{\mathbb{C}} & L(\mathbb{F}_n) & \cong L(\mathbb{F}_{m+n}) \\ \cup & \cup & \\ L(\mathbb{F}_m) & L(\mathbb{F}_n) & \text{freely indep.} \end{array}$$

<u>Def</u>: (M, τ) <u>W^*-probability space</u>

- M von Neumann algebra
- $\tau : M \to \mathbb{C}$ faithful normal tracial state

<u>Ex</u>: $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E}) \ni X \rightsquigarrow \mu_X(A) := \mathbb{P}(X \in A)$

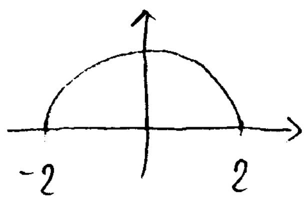
<u>Def</u>: $X \in M$ normal, E_X spectral measure on $\mathcal{B}(\sigma(X))$,

then $\mu_X := \tau \circ E_X$, the <u>distribution of X</u>

<u>Ex</u>: $L(\mathbb{F}_n) \ni u_j \rightsquigarrow \mu_{u_j} =$ Haar measure on $\{z \in \mathbb{C} \mid |z| = 1\}$

<u>Def</u>: $S = S^* \in (M, \tau)$ <u>semicircular</u> (<u>element</u>) if

$$d\mu_S(t) = \frac{1}{2\pi} \sqrt{4 - t^2} \, \mathbb{1}_{[-2,2]}(t) \, dt$$



<u>Fact</u>: $L(\mathbb{F}_n) = vN(u_1, \dots, u_n)$ $\quad u_1, \dots, u_n$ free

$\cong vN(S_1, \dots, S_n)$ $\quad S_1, \dots, S_n$ free

III. <u>Zero divisors and division rings</u>

(M, τ) W^*-prob. space, $M = vN(X_1, \dots, X_n)$, $\underline{X} := (X_1, \dots, X_n)$

$$(*) \quad \left. \begin{array}{l} Y \in \mathbb{C}\langle \underline{X} \rangle, \; Y \neq 0 \\ \xi \in L^2(M, \tau), \; \xi \neq 0 \end{array} \right\} \implies Y\xi \neq 0 \; ?$$

(motivation: atoms of μ_Y for $Y = Y^*$)

<u>Linnell (1991)</u>: $L(\mathbb{F}_n) = vN(u_1, \dots, u_n)$

Shlyakhtenko - Skoufranis (2015): $L(\mathbb{F}_n) \cong vN(S_1, \dots, S_n)$

N. - Speicher - Weber (2017), Charlesworth - Shlyakhtenko (2016)

$\mathcal{M} = vN(X_1, \dots, X_n)$ with $\delta^*(X_1, \dots, X_n) = n$ resp.
$\Delta(X_1, \dots, X_n) = n$

Proof strategies:

① Let $\mathcal{M} \subset \mathcal{A}$ be the $*$-algebra of all densely defined and closed linear operators affiliated with $\mathcal{M}$.

Show: $\exists$ division ring (skew field) D: $\mathbb{C}\langle \underline{X} \rangle \subseteq D \subseteq \mathcal{A}$

Then: $p :=$ projection onto $\ker(Y)$; $p \neq 0$ if $\exists\, 0 \neq \xi \in \ker(Y)$

If $Y \neq 0$, then $0 = Y^{-1}(Yp) = p$. ↯ $\Rightarrow (*)$

② Show: $\underline{X}$ has the Strong Atiyah Property (SAP), i.e., $\forall\, Y \in M_N(\mathbb{C}\langle \underline{X} \rangle)$: $\operatorname{rank}(Y) \in \mathbb{N}_0$,

where $\operatorname{rank}(Y) := N - (\mathrm{Tr}_N \circ \tau^{(N)})(p)$.

Then: $Y \in \mathbb{C}\langle \underline{X} \rangle \implies \operatorname{rank}(Y) \in [0,1]$
$\overset{SAP}{\implies} \operatorname{rank}(Y) \in \{0,1\}$

Hence, if $Y \neq 0$, then $\operatorname{rank}(Y) = 1$, $\ker(Y) = \{0\}$. $\Rightarrow (*)$

IV. The free field

Let $\mathbb{C}\langle \underline{t} \rangle$, $\underline{t} = (t_1, \dots, t_n)$, be the algebra of all noncommutative polynomials; e.g., $P = t_1 t_2 t_1^2 + t_2^2 t_1$.

$\mathbb{C}\langle\!\langle \underline{t} \rangle\!\rangle$:= "universal field of fractions for $\mathbb{C}\langle \underline{t} \rangle$"

$\rightsquigarrow$ n.c. rational function, e.g., $r = (t_1 t_2^{-1} - t_2^{-1} t_1)^{-1} t_2$.

Def: Let $Q \in M_N(\mathbb{C}\langle \underline{t} \rangle)$

(i) $\rho(Q) := \min\{k \geq 1 \mid \exists R_1 \in M_{N \times k}(\mathbb{C}\langle \underline{t} \rangle), R_2 \in M_{k \times N}(\mathbb{C}\langle \underline{t} \rangle) : Q = R_1 R_2\}$

(inner) rank of Q

(ii) Q full if $\rho(Q) = N$

Fact: (i) Q full $\iff$ Q invertible in $M_N(\mathbb{C}\langle\!\langle \underline{t} \rangle\!\rangle)$

(ii) $r \in \mathbb{C}\langle\!\langle \underline{t} \rangle\!\rangle$ admits a linear representation

$$\rho = (u, Q, v),$$

i.e., • $Q = Q_0 + Q_1 t_1 + \ldots + Q_n t_n \in M_N(\mathbb{C}\langle \underline{t} \rangle)$ full

• u, v scalar vectors

• $r = u Q^{-1} v$

Def: $\mathrm{dom}_{\mathcal{A}}(\rho) := \{\underline{X} \in \mathcal{A}^n \mid Q(\underline{X}) \text{ invertible in } M_N(\mathcal{A})\}$

Thm: $\mathcal{A}$ is "stably finite"

(i) ρ_1, ρ_2 lin. rep. of r, then

$\underline{X} \in \mathrm{dom}_{\mathcal{A}}(\rho_1) \cap \mathrm{dom}_{\mathcal{A}}(\rho_2) \implies u_1 Q_1(\underline{X})^{-1} v_1 = u_2 Q_2(\underline{X})^{-1} v_2$

(ii) On $\mathrm{dom}_{\mathcal{A}}(r) := \bigcup \{\mathrm{dom}_{\mathcal{A}}(\rho) \mid \rho \text{ lin. rep. of } r\}$, we have that $\mathrm{dom}_{\mathcal{A}}(r) \to \mathcal{A}$, $\underline{X} \mapsto \mathrm{Ev}_{\underline{X}}(r) := u Q(\underline{X})^{-1} v$

is well-defined

V. The main result

<u>Theorem</u> (M.-Speicher-Yin, 2019)

$(\mathcal{M},\tau)$, $\underline{X}=(X_1,\dots,X_n)$, and $\mathcal{A}$ as before. TFAE

(i) $\forall r \in \mathbb{C}\langle\!\langle \underline{t} \rangle\!\rangle$: $\underline{X} \in \mathrm{dom}_{\mathcal{A}}(r)$

and

$\mathrm{Ev}_{\underline{X}}: \mathbb{C}\langle\!\langle \underline{t} \rangle\!\rangle \to \mathcal{A}, \quad r \mapsto \mathrm{Ev}_{\underline{X}}(r)$

is an <u>injective</u> homomorphism extending

$\mathrm{ev}_{\underline{X}}: \mathbb{C}\langle \underline{t} \rangle \to \mathcal{M}, \quad t_{i_1}\cdots t_{i_k} \mapsto X_{i_1}\cdots X_{i_k}.$

[In this case,

$$D := \mathrm{Ev}_{\underline{X}}(\mathbb{C}\langle\!\langle \underline{t} \rangle\!\rangle) \cong \mathbb{C}\langle\!\langle \underline{t} \rangle\!\rangle$$

is a <u>division ring</u> $\mathbb{C}\langle \underline{X} \rangle \subseteq D \subseteq \mathcal{A}$; in fact, it is the <u>division closure</u> of $\mathbb{C}\langle \underline{X} \rangle$ in $\mathcal{A}$.]

(ii) $\forall N \in \mathbb{N}\ \forall P \in M_N(\mathbb{C}\langle \underline{t} \rangle)$ (linear)

$$P \text{ full} \implies P(\underline{X}) \in M_N(\mathcal{A}) \text{ is invertible}$$

(iii) $\forall N \in \mathbb{N}\ \forall P \in M_N(\mathbb{C}\langle \underline{t} \rangle)$: $\mathrm{rank}(P(\underline{X})) = \rho(P)$

[In particular, $\underline{X}$ has the SAP]

(iv) $\Delta(\underline{X}) = n$

<u>Def</u> (Connes - Shlyakhtenko, 2005)

$$\Delta(\underline{X}) := n - \dim_{M \bar{\otimes} M^{op}} \overline{\left\{ (T_1, \ldots, T_n) \in FR(L^2(M,\tau)) \;\middle|\; \sum_{j=1}^{n} [T_j, JX_j^* J] = 0 \right\}}^{HS}$$

<u>Facts</u>:

(i) $\underline{X}$ selfadjoint; then [CS05] $\delta^*(\underline{X}) \leq \Delta(\underline{X}) \leq n$

(ii) If we find $D_1, \ldots, D_n \in B(L^2(M,\tau))$ satisfying

$$[X_i, D_j] = \delta_{ij} P_\Omega \quad \forall\, i,j = 1, \ldots, n,$$

where $P_\Omega :=$ projection onto trace vector $\Omega \in L^2(M,\tau)$,

then $\Delta(\underline{X}) = n$

[We call $\underline{D} = (D_1, \ldots, D_n)$ a <u>dual system</u> of $\underline{X}$]

<u>Note</u>: $u_1, \ldots, u_n$, $u_j = \lambda_{g_j}$, admit a dual system

<u>Corollary</u>:

Suppose that $\Delta(\underline{X}) = n$. Let $P \in M_N(\mathbb{C}\langle \underline{t} \rangle)$ be given for which $Y := P(\underline{X}) \in M_N(M)$ is normal. Then

$$\mu_Y(\{\lambda\}) = \frac{1}{N}\left(N - \rho(P - \lambda 1_N)\right) \quad \forall \lambda \in \mathbb{C}.$$