Free differential calculus II: characterization of free and cyclic gradients

1. Reminder (nc derivatives)

On $\mathbb{C}\langle t\rangle := \mathbb{C}\langle t_1,\dots,t_n\rangle$, there are the unique derivations

$$\partial_1,\dots,\partial_n : \mathbb{C}\langle t\rangle \to \mathbb{C}\langle t\rangle \otimes \mathbb{C}\langle t\rangle$$

satisfying $\partial_i t_j = \delta_{ij}\, 1\otimes 1$.

2. Def (cyclic derivatives)

$$\mathcal{D}_1,\dots,\mathcal{D}_n : \mathbb{C}\langle t\rangle \to \mathbb{C}\langle t\rangle, \qquad \mathcal{D}_i := \mu\circ\sigma\circ\partial_i$$

where
- $\sigma: \mathbb{C}\langle t\rangle\otimes\mathbb{C}\langle t\rangle \to \mathbb{C}\langle t\rangle\otimes\mathbb{C}\langle t\rangle$, $P_1\otimes P_2 \mapsto P_2\otimes P_1$ (flip)
- $\mu: \mathbb{C}\langle t\rangle\otimes\mathbb{C}\langle t\rangle \to \mathbb{C}\langle t\rangle$, $P_1\otimes P_2 \mapsto P_1P_2$ (multiplication)

3. Ex: $P := t_1 t_2^2 t_1 \in \mathbb{C}\langle t_1,t_2\rangle$

(i)
$$\partial_1 P = 1\otimes t_2^2 t_1 + t_1 t_2^2 \otimes 1$$
$$\partial_2 P = t_1\otimes t_2 t_1 + t_1 t_2\otimes t_1$$

(ii)
$$\mathcal{D}_1 P = t_2^2 t_1 + t_1 t_2^2$$
$$\mathcal{D}_2 P = t_2 t_1^2 + t_1^2 t_2$$

4. Def:
- $\partial P := (\partial_1 P,\dots,\partial_n P)$ free gradient
- $\mathcal{D} P := (\mathcal{D}_1 P,\dots,\mathcal{D}_n P)$ cyclic gradient

5. Remark

- $(\mathcal{M},\tau)$ tracial W^*-prob. space, $X=(X_1,\dots,X_n)\in\mathcal{M}_{sa}^n$,
  $\mathcal{M}_0 := vN(X)$; then

$$\Phi(X)<\infty \iff \exists\, \xi=(\xi_1,\dots,\xi_n)\in L^2(\mathcal{M}_0,\tau)^n :$$
$$\langle \partial_i P(X), 1\otimes 1\rangle_{\tau\otimes\tau} = \langle P(X),\xi_i\rangle_\tau$$

- <u>Schwinger</u> <u>-</u> <u>Dyson</u> <u>equation</u> with potential $V\in\mathbb{C}\langle \underline{t}\rangle$

selfadjoint, (c,∞)-convex

then: $\xi_i = (\mathcal{D}_i V)(X)$, $i=1,\dots,n$

Important question: $\underline{\xi}\in\overline{\{(\mathcal{D}P)(X)\mid P\in\mathbb{C}\langle t\rangle\}}$?

6. <u>Theorem</u> (Voiculescu, 2000; M., Speicher, 2019)

Let $P=(P_1,\dots,P_n)\in\mathbb{C}\langle t\rangle^n$. TFAE

(i) $\exists\, Q\in\mathbb{C}\langle t\rangle$: $P=\mathcal{D}Q$

(ii) $\sum_{j=1}^n [t_j,P_j]=0$

(iii) $\partial_i P_j = \sigma(\partial_j P_i)\quad \forall i,j=1,\dots,n$

(iv) $\mathcal{D}_i\Big(\sum_{j=1}^n t_jP_j\Big) = (N+id)P_i\quad \forall i=1,\dots,n$

(<u>number operator</u>: $N\, t_{i_1}\cdots t_{i_k} := k\, t_{i_1}\cdots t_{i_k}$, $N1=0$)

Then: find Q by solving $NQ=\sum_{j=1}^n t_jP_j$

7. Ex: $\underline{P} = (t_2^2 t_1 + t_1 t_2^2,\ t_2 t_1^2 + t_1^2 t_2)$

- $NQ = t_1 P_1 + t_2 P_2 = t_1 t_2^2 t_1 + t_1^2 t_2^2 + t_2^2 t_1^2 + t_2 t_1^2 t_2$

  $\rightsquigarrow Q = \frac{1}{4}(t_1 t_2^2 t_1 + t_1^2 t_2^2 + t_2^2 t_1^2 + t_2 t_1^2 t_2)$

- $CQ = t_1 P_1 + t_2 P_2$

  $\rightsquigarrow Q = t_1 t_2^2 t_1$

  (cyclic symmetrization operator
  $C\, t_{i_1} \cdots t_{i_k} = \sum_{p=1}^{k} t_{i_{p+1}} \cdots t_{i_k} t_{i_1} \cdots t_{i_p}$
  $C\,1 = 0$)

7. Remark: We always have (Voiculescu, 1999)

$$\sum_{j=1}^{n} [X_j, \xi_j] = 0$$

and, if $\xi_1, \dots, \xi_n \in \operatorname{dom} \overline{\partial_{X_j}}$ for $j = 1, \dots, n$, (Dabrowski, 2014)

$$\overline{\partial_{X_i}} \xi_j = \sigma(\overline{\partial_{X_j}} \xi_i) \quad \forall i,j = 1, \dots, n$$

8. Theorem (M., Speicher, 2019)

Let $U = (U_1, \dots, U_n) \in (\mathbb{C}\langle t \rangle \otimes \mathbb{C}\langle t \rangle)^n$. TFAE

(i) $\exists\, Q \in \mathbb{C}\langle t \rangle : \quad U = \partial Q$

$\boxed{(P_1 \otimes P_2) \# Q := P_1 Q P_2}$

(ii) $(\mathrm{id} \otimes \partial_i)(U_j) = (\partial_j \otimes \mathrm{id})(U_i) \quad \forall i,j = 1, \dots, n$

(iii) $\partial_i \left(\sum_{j=1}^{n} U_j \# t_j \right) = (N \otimes \mathrm{id} + \mathrm{id} \otimes N + \mathrm{id} \otimes \mathrm{id})(U_i) \ \forall i = 1, \dots, n$

Then: find Q by solving $NQ = \sum_{j=1}^{n} U_j \# t_j$.

9. Ex: $\underline{U} = (1 \otimes t_2^2 t_1 + t_1 t_2^2 \otimes 1,\ t_1 \otimes t_2 t_1 + t_1 t_2 \otimes t_1)$

$$NQ = U_1 \# t_1 + U_2 \# t_2 = 4\, t_1 t_2^2 t_1$$

$$\rightsquigarrow Q = t_1 t_2^2 t_1$$

10. Def (Voiculescu 2000/2004)

A (multivariable) generalized difference quotient ring (A, μ, ∂) (GDQ ring) consists of

- a complex unital algebra,
- the induced multiplication map
$$\mu : A \otimes A \to A,\quad a_1 \otimes a_2 \mapsto a_1 a_2,$$
- and $\partial = (\partial_1, \dots, \partial_n)$, $\partial_1, \dots, \partial_n : A \to A \otimes A$ linear,

such that $\partial_1, \dots, \partial_n$

(i) satisfy the (joint) coassociativity relation
$$(\partial_i \otimes id) \circ \partial_j = (id \otimes \partial_j) \circ \partial_i \qquad \forall i, j = 1, \dots, n,$$

(ii) are derivations on (A, μ), i.e.,
$$\partial_j \circ \mu = (\mu \otimes id) \circ (id \otimes \partial_j) + (id \otimes \mu) \circ (\partial_j \otimes id) \quad \forall j = 1, \dots, n$$

11. Def:

- $\sigma : A \otimes A \to A \otimes A,\ a_1 \otimes a_2 \mapsto a_2 \otimes a_1$
- $\mathcal{D}_i := \mu \circ \sigma \circ \partial_i : A \to A;\ \mathcal{D} := (\mathcal{D}_1, \dots, \mathcal{D}_n)$
- $\partial : A \to (A \otimes A)^n,\ \mathcal{D} : A \to A^n$

Fact: $\sigma \circ \partial_i \circ \mathcal{D}_j = \partial_j \circ \mathcal{D}_i \quad \forall i, j = 1, \dots, n$

12. Def:

(A, μ, ∂) a GDG ring. We call $\partial^* = (\partial_1^*, \dots, \partial_n^*)$ with

$$\partial_1^*, \dots, \partial_n^* : A \otimes A \to A \text{ linear}$$

a divergence for (A, μ, ∂) if, for $i, j = 1, \dots, n$,

$$\partial_j \circ \partial_i^* = (\partial_i^* \otimes id) \circ (id \otimes \partial_j) + (id \otimes \partial_i^*)(\partial_j \otimes id) + \delta_{ij}\, id \otimes id.$$

Then: • $N := \partial^* \circ \partial = \sum_{i=1}^{n} \partial_i^* \circ \partial_i$ number operator

• $L := N + id_A$ grading operator

13. Lemma: L is a coderivation w.r.t. each ∂_j, i.e.

$$\partial_j \circ L = (L \otimes id + id \otimes L) \circ \partial_j \qquad \forall j = 1, \dots, n.$$

Proof: $\partial_j \circ N = \sum_{i=1}^{n} (\partial_j \circ \partial_i^*) \circ \partial_i$

$$= \partial_j + \sum_{i=1}^{n} \Big((\partial_i^* \otimes id) \circ \underbrace{(id \otimes \partial_j) \circ \partial_i}_{= (\partial_i \otimes id) \circ \partial_j} + (id \otimes \partial_i^*) \circ \underbrace{(\partial_j \otimes id) \circ \partial_i}_{= (id \otimes \partial_i) \circ \partial_j} \Big)$$

$$= \underbrace{(id \otimes id + N \otimes id + id \otimes N)}_{=: N_2} \circ \partial_j$$

$\square$

14. Theorem (M., Speicher, 2019)

Suppose that N_2 is injective and $\operatorname{ran} \partial^* \subseteq \operatorname{ran} N$.

Let $u = (u_1, \dots, u_n) \in (A \otimes A)^n$ be given. TFAE:

(i) $\exists a \in A: \quad \partial a = u$

(ii) $(id \otimes \partial_i)(u_j) = (\partial_j \otimes id)(u_i) \quad \forall i,j = 1,\dots,n$

(iii) $\partial_i(\partial^* u) = N_2 u_i \quad \forall i = 1,\dots,n$

Proof: (i) $\Rightarrow$ (ii) coassociativity (Def 10 (i))

(ii) $\Rightarrow$ (iii) see Lemma 13

(iii) $\Rightarrow$ (i): Since $\partial^* u \in \operatorname{ran} N$, we find $a \in A$ s.t. $Na = \partial^* u$. By Lemma 13,

$$N_2(\partial_i a) = \partial_i(Na) = \partial_i(\partial^* u) \overset{(iii)}{=} N_2 u_i.$$

Since N_2 injective, we get $\partial_i a = u_i$. $\square$

15. Example:

(i) $(\mathbb{C}\langle t\rangle, \mu, \partial)$, then $\partial^* = (\partial_1^*, \dots, \partial_n^*)$ with

$$\partial_j^* : \mathbb{C}\langle t\rangle \otimes \mathbb{C}\langle t\rangle \to \mathbb{C}\langle t\rangle, \quad U \mapsto U \# t_j$$

is a divergence; in fact $N = \partial^* \circ \partial$ is the number operator that we already know. Note

- $\mathbb{C}\langle t\rangle = \bigoplus_{k \geq 0} \mathbb{C}^{(k)}\langle t\rangle$, $\operatorname{ran} N = \bigoplus_{k \geq 1} \mathbb{C}^{(k)}\langle t\rangle \supseteq \operatorname{ran} \partial^*$

- $\mathbb{C}\langle t\rangle \otimes \mathbb{C}\langle t\rangle = \bigoplus_{k,l \geq 0} \underbrace{\mathbb{C}^{(k)}\langle t\rangle \otimes \mathbb{C}^{(l)}\langle t\rangle}_{\ni U}$

and $N_2 U = (k+l+1) U$ if U; hence, N_2 injective

(ii) $\emptyset \neq K \subseteq \mathbb{C}$, compact

$$\partial : \mathcal{O}(K) \to \mathcal{O}(K \times K), \quad (\partial f)(z,w) := \frac{f(z) - f(w)}{z - w},$$

then $\partial^* : \mathcal{O}(K \times K) \to \mathcal{O}(K)$, $(\partial^* F)(z) = z\, F(z,z)$

yields a "topological" divergence;

$$(N f)(z) = z f'(z), \quad (L f)(z) = z f'(z) + f(z)$$

16. Remark

Given ∂^*, we call $\mathcal{D}^* = (\mathcal{D}_1^*, \dots, \mathcal{D}_n^*)$, $\mathcal{D}_i^* \; A \to A$,
a cyclic divergence if

$$\mathcal{D}_j \circ \mathcal{D}_i^* = \partial_i^* \circ \sigma \circ \partial_j + \delta_{ij}\, \mathrm{id}$$

We have $\mathcal{D}^* : A^n \to A$ and put

$$C := \mathcal{D}^* \circ \mathcal{D} = \sum_{i=1}^{n} \mathcal{D}_i^* \circ \mathcal{D}_i$$

Facts:
- $\mathcal{D}_j \circ C = L \circ \mathcal{D}_j \qquad \forall j = 1, \dots, n$
  (analogous to Lemma 13)
- $\mathcal{D}_j \circ N = L \circ \mathcal{D}_j \qquad \forall j = 1, \dots, n$
  if N is a derivation

Ex: For $\mathbb{C}\langle t \rangle$, we find a cyclic divergence by

$$\mathcal{D}_i^* P := P t_i$$