Regularity properties of spectral distributions: from free probability to random matrix theory

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Noncommutative probability spaces

Noncommutative probability spaces

Definition

A noncommutative probability space $(\mathcal{A}, \phi)$ consists of

- a complex algebra $\mathcal{A}$ with unit $1_\mathcal{A}$ and
- a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ with $\phi(1_\mathcal{A}) = 1$ (expectation).

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Definition

A noncommutative probability space $(\mathcal{A}, \phi)$ is called

- C^*-probability space if
  - $\mathcal{A}$ is a unital C^*-algebra and
  - ϕ is a state on $\mathcal{A}$.
- tracial W^*-probability space, if
  - $\mathcal{A}$ is a von Neumann algebra and
  - ϕ is a faithful normal tracial state on $\mathcal{A}$.

Noncommutative distributions

Noncommutative distributions

Definition ("combinatorial distribution")

Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. For any given family $X = (X_i)_{i \in I}$ of noncommutative random variables, we call

$$\mu_X : \mathbb{C}\langle x_i \mid i \in I \rangle \to \mathbb{C}, \quad x_{i_1} \cdots x_{i_k} \mapsto \phi(X_{i_1} \cdots X_{i_k})$$

the (joint) noncommutative distribution of X.

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Definition ("analytic distribution")

Let $(\mathcal{A}, \phi)$ be a C^*-probability space. For any given $X = X^* \in \mathcal{A}$, the noncommutative distribution of X can be identified with the unique Borel probability measure μ_X on the real line $\mathbb{R}$ that satisfies

$$\phi(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all integers } k \geq 0.$$

Noncommutative distribution $\mu_{X_1,\dots,X_n}$

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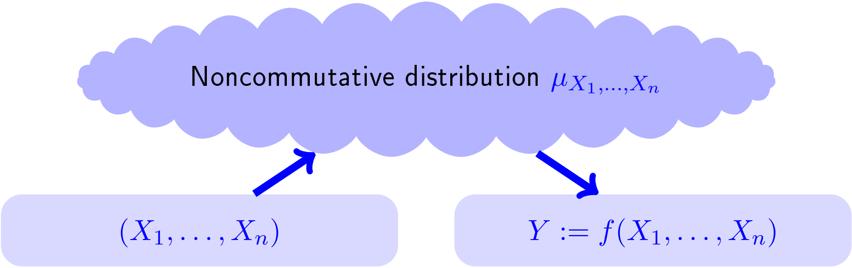
$(X_1, \dots, X_n)$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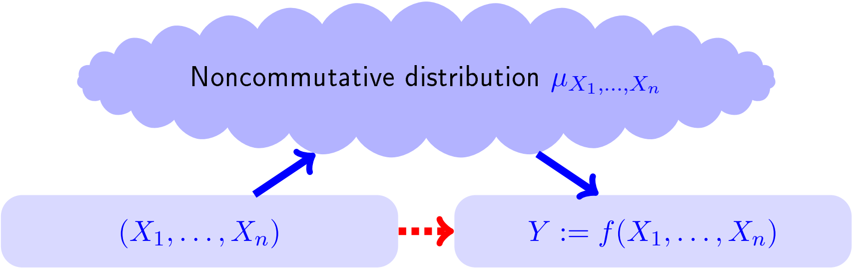
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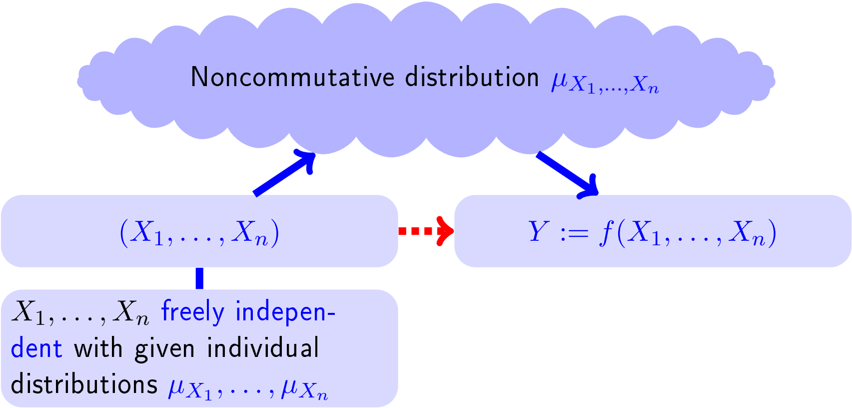
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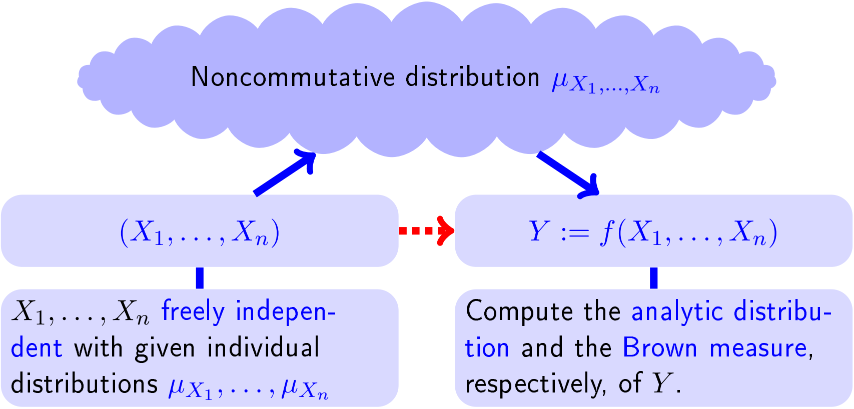
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[Belinschi-M.-Speicher, '17], [Belinschi-Sniady-Speicher, '18], [Helton-M.-Speicher, '18]

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[Shlyakhtenko-Skoufranis, '15], [Ajanki-Erdös-Krüger, '16], [Alt-Erdös-Krüger, '18]

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[Charlesworth-Shlyakhtenko, '16], [M.-Speicher-Weber, '17], [M.-Speicher-Yin, '18], [Banna-M., '18]

Important classes of “noncommutative test functions”

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1. Noncommutative polynomials, i.e., expressions of the form

$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$

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3. Affine linear pencils, i.e., matrices of noncommutative polynomials that are of the particular form

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4. Noncommutative rational functions. i.e., elements of the universal field of fractions $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ for $\mathbb{C}\langle x_1, \dots, x_n \rangle$.

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$$\partial_1, \ldots, \partial_n : \ \mathbb{C}\langle x_1, \ldots, x_n\rangle \to \mathbb{C}\langle x_1, \ldots, x_n\rangle \otimes \mathbb{C}\langle x_1, \ldots, x_n\rangle$$

which are uniquely determined by the two conditions

- $\partial_j(P_1 P_2) = (\partial_j P_1) \cdot P_2 + P_1 \cdot (\partial_j P_2)$ for all $P_1, P_2 \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$,
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$\mathbb{C}\langle x_1, \ldots, x_n\rangle \otimes \mathbb{C}\langle x_1, \ldots, x_n\rangle$ becomes a $\mathbb{C}\langle x_1, \ldots, x_n\rangle$-bimodule via

$$P_1 \cdot (Q_1 \otimes Q_2) \cdot P_2 := (P_1 Q_1) \otimes (Q_2 P_2).$$

Conjugate variables and free Fisher information

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Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and consider any selfadjoint operators $X_1, \dots, X_n \in \mathcal{M}$; we put $\mathcal{M}_0 := \mathrm{vN}(X_1, \dots, X_n)$.

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Definition (Voiculescu (1998))

If $\xi_1,\dots,\xi_n \in L^2(\mathcal{M}_0,\tau)$ are such that for all $P \in \mathbb{C}\langle x_1,\dots,x_n\rangle$

$$(\tau\otimes\tau)((\partial_j P)(X_1,\dots,X_n)) = \tau(\xi_j P(X_1,\dots,X_n)), \quad j=1,\dots,n,$$

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The (non-microstates) free Fisher information is defined by

$$\Phi^*(X_1, \dots, X_n) := \begin{cases} \sum_{j=1}^{n} \|\xi_j\|_2^2, & \text{if a conjugate system } (\xi_1, \dots, \xi_n) \text{ for } (X_1, \dots, X_n) \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

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Definition (Dabrowski (2014); Dabrowski & Ioana (2016))

We say that $(\xi_1, \dots, \xi_n)$ are Lipschitz conjugate variables for X if

$$\xi_j \in \mathrm{dom}(\overline{\partial}_j) \quad \text{and} \quad \overline{\partial}_j \xi_j \in \mathcal{M}_0 \mathbin{\overline{\otimes}} \mathcal{M}_0 \qquad \text{for } j = 1, \dots, n.$$

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Consider $Y = Y^*$ in $(\mathcal{M}, \tau)$. Let μ_Y be the analytic distribution of Y and let $\mathcal{F}_Y$ be its cumulative distribution function, i.e., $\mathcal{F}_Y(t) := \mu_Y((-\infty, t])$.

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Lemma (M., Speicher, Yin (2018))

If there exist $c > 0$ and $\alpha > 1$ such that

$$c\|(Y - s)p\|_2 \geq \|p\|_2^\alpha$$

for all $s \in \mathbb{R}$ and each spectral projection p of Y, then $\mathcal{F}_Y$ is Hölder continuous with exponent $\beta := \frac{2}{\alpha-1}$; more precisely, we have that

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Proof – following ideas of [Charlesworth, Shlyakhtenko (2016)].

Take $p = E_Y((s,t])$ for the spectral measure E_Y of Y and observe that

$$\|p\|_2 = \mu_Y((s,t])^{1/2} \qquad \text{and} \qquad \|(Y-s)p\|_2 \leq |t-s|\mu_Y((s,t])^{1/2}.$$

□

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Theorem (Banna, M. (2018))

Let $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be selfadjoint with degree $d \geq 1$ and consider

$$Y := P(X_1, \dots, X_n).$$

1. Suppose that $\Phi^*(X_1, \dots, X_n) < \infty$. Then there exists some constant $C > 0$ such that

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2. If $(X_1, \ldots, X_n)$ admits even Lipschitz conjugate variables, then there exists some constant $C' > 0$ such that

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In fact, we can give explicit values for the constants C and C'.

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Definition (Random matrices)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Elements in the complex $*$-algebra

$$\mathcal{A}_N := M_N(L^{\infty-}(\Omega, \mathbb{P})), \quad \text{where} \quad L^{\infty-}(\Omega, \mathbb{P}) := \bigcap_{1 \leq p < \infty} L^p(\Omega, \mathbb{P})$$

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Definition (Empirical eigenvalue distribution)

Given $X \in \mathcal{A}_N$, the empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

$$\omega \mapsto \mu_{X(\omega)} = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)},$$

where $\lambda_1(\omega), \ldots, \lambda_N(\omega)$ are the eigenvalues of $X(\omega)$ with multiplicities.

Standard Gaussian random matrices

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A standard Gaussian random matrix (of size $N \times N$) is a hermitian random matrix $X = (X_{k,l})_{k,l=1}^{N} \in \mathcal{A}_N$ for which

$$\{\mathrm{Re}(X_{k,l}) \mid 1 \leq k \leq l \leq N\} \cup \{\mathrm{Im}(X_{k,l}) \mid 1 \leq k < l \leq N\}$$

are independent Gaussian random variables such that

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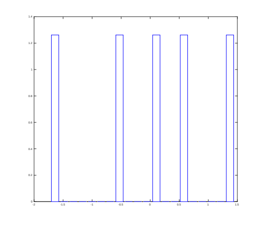
Eigenvalue histograms

$n = 5$



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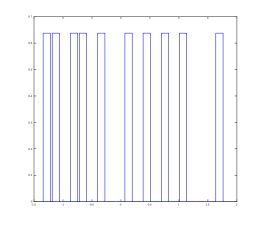
Eigenvalue histograms

$n = 10$



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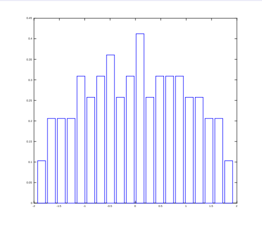
Eigenvalue histograms

$n = 100$



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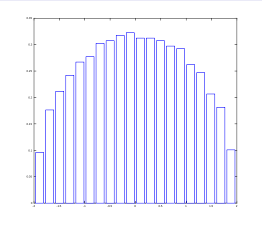
Eigenvalue histograms

$n = 1000$



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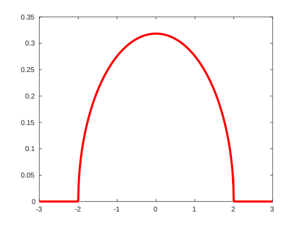
Eigenvalue histograms

$n \to \infty$



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Theorem (Wigner (1955/1958))

Let $(X^{(N)})_{N \in \mathbb{N}}$ be a sequence of self-adjoint Gaussian random matrices $X^{(N)} \in \mathcal{A}_N$. Then, for all $k \in \mathbb{N}_0$, it holds true that

$$\lim_{n \to \infty} \mathbb{E}\Big[\int_{\mathbb{R}} t^k \, d\mu_{X_n}(t)\Big] = \int_{\mathbb{R}} t^k \, d\mu_S(t)$$

for the semicircular distribution $\quad d\mu_S(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\, 1_{[-2,2]}(t)\, dt.$

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Asymptotic freeness of random matrices

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We have the following multivariate version of Wigner's semicircle law.

Theorem (Voiculescu (1991))

For all $N \in \mathbb{N}$, realize independent standard Gaussian random matrices $X_1^{(N)}, \ldots, X_n^{(N)} \in \mathcal{A}_N$. Then, for all $P \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$,

$$\lim_{N \to \infty} \mathbb{E}[\mathrm{tr}_N(P(X_1^{(N)}, \ldots, X_n^{(N)}))] = \phi(P(S_1, \ldots, S_n))$$

for freely independent semicircular elements $S_1, \ldots, S_n$ in some noncommutative probability space $(\mathcal{A}, \phi)$.

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This means: Asymptotic freeness relates (in this case)

- the limiting eigenvalue distribution of $Y^{(N)} = P(X_1^{(N)}, \dots, X_n^{(N)})$ and
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☞ Note that $(S_1, \dots, S_n)$ has Lipschitz conjugate variables.

Gibbs laws and the Schwinger-Dyson equation

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Theorem (Guionnet, Shlyakhtenko (2009))

Let $V \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ following the Gibbs law

$$d\Lambda_N^V(X_1^{(N)}, \dots, X_n^{(N)}) = \frac{1}{Z_N^V} e^{-N \operatorname{Tr}(V(X_1^{(N)}, \dots, X_n^{(N)}))} \, dX_1^{(N)} \dots dX_n^{(N)}.$$

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☞ Due to the Schwinger-Dyson equation, $(X_1, \dots, X_n)$ has Lipschitz conjugate variables; they are given by $\xi_j = (\mathcal{D}_j V)(X_1, \dots, X_n)$.

Polynomial evaluations for Gibbs laws

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Corollary (Banna, M. (2018))

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(i) The empirical eigenvalue distribution $\mu_{Y^{(N)}}$ of

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(ii) We have almost surely that

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$$d_{\mathrm{Kol}}(\overline{\mu}_{Y^{(N)}}, \mu) \leq C N^{-\frac{1}{13 \cdot 2^{d+2} - 60}} \qquad \text{for all } N \in \mathbb{N}.$$

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