Noncommutative rational functions evaluated in random matrices

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joint work with

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A standard selfadjoint Gaussian random matrix is a selfadjoint random matrix $X = (X_{k,l})_{k,l=1}^N$, for which

$$\{\mathrm{Re}(X_{k,l}) \mid 1 \leq k \leq l \leq N\} \cup \{\mathrm{Im}(X_{k,l}) \mid 1 \leq k < l \leq N\}$$

are independent centered Gaussian random variables, such that

$$\mathbb{E}[\mathrm{Re}(X_{k,k})^2] = \tfrac{1}{N} \quad \text{for } 1 \leq k \leq N \qquad \text{and}$$
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The law of a standard selfadjoint Gaussian random matrix X^N is the probability measure μ^N on $M_N(\mathbb{C})_{\mathrm{sa}} \cong \mathbb{R}^{N^2}$ that is determined by

$$d\mu^N(X) := \frac{1}{Z_N} e^{-\frac{N}{2}\mathrm{Tr}_N(X^2)}\, dX \quad \text{with} \quad Z_N := 2^{\frac{N}{2}} \left(\frac{\pi}{N}\right)^{\frac{N^2}{2}}$$

and $\quad dX := \prod_{k=1}^N dX_{k,k} \prod_{1 \le k < l \le N} d\,\mathrm{Re}(X_{k,l})\, d\,\mathrm{Im}(X_{k,l}).$

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$$\mu_{X^N} = \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j(X^N)}$$

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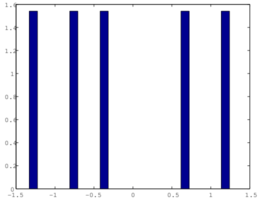

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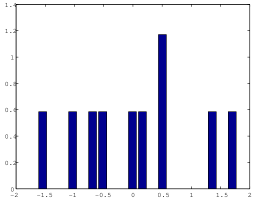

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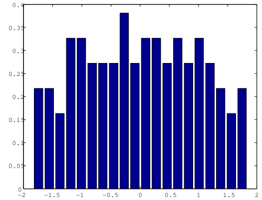

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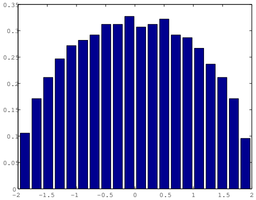

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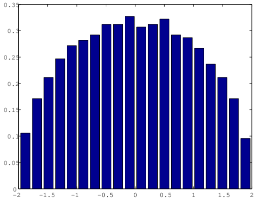

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Theorem (Wigner (1955/1958))

Consider $(X^N)_{N\in\mathbb{N}}$. For all $k \in \mathbb{N}_0$, it holds true that

$$\lim_{N\to\infty} \mathbb{E}\Big[\int_{\mathbb{R}} t^k \,\mathrm{d}\mu_{X^N}(t)\Big] = \int_{\mathbb{R}} t^k \,\mathrm{d}\sigma(t)$$

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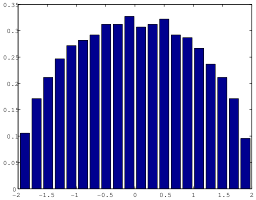

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A tracial W^*-probability space is a tuple $(\mathcal{M}, \tau)$ consisting of

- a von Neumann algebra $\mathcal{M}$, (nc random variables)
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- $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a classical probability space and $\mathbb{E}$ the usual expectation that is given by $\mathbb{E}[X] = \int_\Omega X(\omega)\, \mathrm{d}\mathbb{P}(\omega)$.

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For $X = X^* \in \mathcal{M}$, let E_X be the associated resolution of the identity. The Borel probability measure $\mu_X := \tau \circ E_X$ is called the (analytic) distribution of X; it is uniquely determined by

$$\tau(X^k) = \int_{\mathbb{R}} t^k \, \mathrm{d}\mu_X(t) \qquad \text{for all } k \in \mathbb{N}_0.$$

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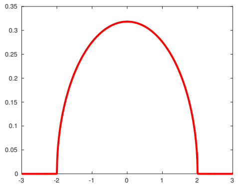

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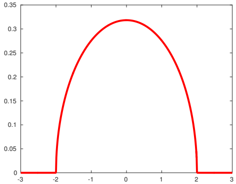

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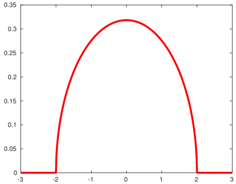

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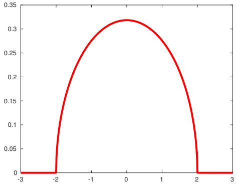

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Let $\xi_1, \dots, \xi_d$ be a orthonormal system in H. We put $S_i := l_{\xi_i} + l_{\xi_i}^*$ for $i = 1, \dots, d$ and

$$\mathcal{M} := W^*(S_1, \dots, S_d) \subset B(\mathcal{F}(H)) \quad \text{and} \quad \tau : \mathcal{M} \to \mathbb{C}, X \mapsto \langle X\Omega, \Omega \rangle.$$

Then $(\mathcal{M}, \tau)$ is a tracial W^*-probability space and $S_1, \dots, S_d$ are freely independent semicircular elements.

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By $\mathbb{C}\langle x_1, \dots, x_d\rangle$, we denote the algebra of noncommutative polynomials

$$p = a_0 + \sum_{k=1}^{m} \sum_{i_1,\dots,i_k=1}^{d} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}.$$

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- ☞ Previous results concern the particular case of bounded evaluations such as $(1 - iS_1)^{-1} S_2 (1 + iS_1)^{-1}$;
  see [Yin (2018)] and [Erdős, Krüger, Nemish (2020)].

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☞ They can be realized as *equivalence classes* of noncommutative rational expressions which are non-degenerate.

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Definition

A (noncommutative) rational expression r in d formal variables $x_1, \ldots, x_d$ is a syntactically valid combination of

- scalars $\lambda \in \mathbb{C}$ and the variables $x_1, \ldots, x_d$,
- the arithmetic operations $+, \cdot, ^{-1}$, and
- parentheses $($, $)$.

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A (noncommutative) rational expression r in d formal variables $x_1, \ldots, x_d$ is a syntactically valid combination of

- scalars $\lambda \in \mathbb{C}$ and the variables $x_1, \ldots, x_d$,
- the arithmetic operations $+, \cdot, ^{-1}$, and
- parentheses $(,)$.

Example

- $r(x_1, x_2) = (x_1 \cdot x_2 - 4)^{-1} \cdot x_1 \cdot (x_2 \cdot x_1 - 4)^{-1}$
- $r(x_1, x_2) = (i - x_1)^{-1} \cdot x_2 + x_1 \cdot (i - x_2)^{-1}$
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- We say that r is non-degenerate if $\operatorname{dom}_{M(\mathbb{C})}(r) \neq \emptyset$, where

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According to [Kaliuzhnyi-Verbovetskyi, Vinnikov (2012)], we have that

$$\mathbb{C}\langle\!\langle x_1, \ldots, x_d \rangle\!\rangle = \big\{ [r] \,\big|\, r \text{ non-degenerate nc rational expression} \big\},$$

where $[r]$ are equivalence classes with respect to the equivalence relation

$$r_1 \sim r_2 \quad :\Longleftrightarrow \quad \forall\, X \in \mathrm{dom}_{M(\mathbb{C})}(r_1) \cap \mathrm{dom}_{M(\mathbb{C})}(r_2):\; r_1(X) = r_2(X).$$

Back to our question . . .

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Theorem (Voiculescu (1991), Hiai & Petz (2000))

For all noncommutative polynomials $p \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$, it holds true that

$$\lim_{N\to\infty} \mathrm{tr}_N(p(X_1^N, \ldots, X_d^N)) = \tau(p(S_1, \ldots, S_d)) \quad \text{almost surely}$$

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Main result I: evaluations in random matrices

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Suppose that $X^N = (X_1^N, \ldots, X_d^N)$ for $N \in \mathbb{N}$ are d-tuples of selfadjoint random matrices with laws μ_d^N on $M_N(\mathbb{C})_{\text{sa}}^d$ which are absolutely continuous with respect to the Lebesgue measure on $M_N(\mathbb{C})_{\text{sa}}^d$.

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Let r be a noncommutative rational expression in d formal variables which is non-degenerate. Then there exists some $N_0 \in \mathbb{N}$ such that almost surely

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Recall that independent standard selfadjoint Gaussian random matrices of size $N \times N$ follow the law

$$d\mu_d^N(X) = \frac{1}{Z_N^d} e^{-\frac{N}{2}\mathrm{Tr}_N(X_1^2 + \cdots + X_d^2)}\, dX_1 \dots dX_d.$$

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Definition (Connes, Shlyakhtenko (2005))

$$\Delta(X) := d - \dim_{\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}} \overline{\Big\{T \in \mathrm{FR}(L^2(\mathcal{M}))^d \;\Big|\; \sum_{j=1}^{d} [T_j, JX_jJ] = 0\Big\}}^{\mathrm{HS}}$$

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- For a single operator $X = X^*$, we have $\Delta(X) = 1 - \sum_{t \in \mathbb{R}} \mu_X(\{t\})^2$.

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- If $\{X_1, \dots, X_k\}$ and $\{X_{k+1}, \dots, X_d\}$ are freely independent, then

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In particular, for a d-tuple $S = (S_1, \dots, S_d)$ of freely independent semicircular elements, we have that $\Delta(S) = \Delta(S_1) + \dots + \Delta(S_d) = d$.

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Theorem (M., Speicher, Yin (2019))

The canonical evaluation homomorphism

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \ldots, x_d \rangle \to \mathcal{M}, \qquad x_{i_1} x_{i_2} \cdots x_{i_k} \mapsto X_{i_1} X_{i_2} \cdots X_{i_k}$$

extends to an injective homomorphism $\mathrm{Ev}_X : \mathbb{C}\langle\!\!\!\langle x_1, \ldots, x_d \rangle\!\!\!\rangle \to \widetilde{\mathcal{M}}$.

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Theorem (Collins, M., Miyagawa, Parraud, Yin (2021))

Let r be a noncommutative rational expression in d formal variables which is non-degenerate. Then

$$X \in \mathrm{dom}_{\widetilde{\mathcal{M}}}(r) \qquad \text{and} \qquad r(X) = \mathrm{Ev}_X([r]).$$

Main result III: convergence in law

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Definition

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Theorem (Collins, M., Miyagawa, Parraud, Yin (2021))

Let $X^N = (X_1^N, \ldots, X_d^N)$ be a d-tuple of selfadjoint random matrices. Further, let r be a non-degenerate noncommutative rational expression in d variables which is selfadjoint. Suppose the following:

1. $X^N \xrightarrow{\mathrm{dist}} X$ almost surely as $N \to \infty$ for a d-tuple X in some tracial W^*-probability space $(\mathcal{M}, \tau)$ satisfying $\Delta(X) = d$.
2. For N large enough, $r(X^N)$ is well-defined almost surely.

Then $r(X)$ is well-defined, and the empirical measure of $r(X^N)$ converges almost surely in law towards the analytic distribution of $r(X)$.

Glimpse behind the scenes I: fullness

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Definition

Let $A \in M_k(\mathbb{C}\langle x_1, \dots, x_d \rangle)$ be given.

- The (inner) rank of A, denoted by $\rho(A)$, is the least integer $r \geq 1$ for which A can be written as $A = R_1 R_2$ with some rectangular matrices

$$R_1 \in M_{k\times r}(\mathbb{C}\langle x_1, \dots, x_d \rangle) \quad \text{and} \quad R_2 \in M_{r\times k}(\mathbb{C}\langle x_1, \dots, x_d \rangle).$$

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- If there is $X = (X_1, \ldots, X_d) \in M_N(\mathbb{C})^d$ for some $N \in \mathbb{N}$ such that $A(X)$ is invertible in $M_{kN}(\mathbb{C})$, then A is full.

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- Conversely, if A is full, then there exists $N_0 \in \mathbb{N}$ such that for each $N \geq N_0$ there is $X \in M_N(\mathbb{C})^d$ for which $A(X)$ is invertible.

Glimpse behind the scenes II: linearization

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Definition (Helton, M., Speicher (2018))

Let r be a noncommutative rational expression in the formal variables $x_1, \dots, x_d$. A formal linear representation $\rho = (u, A, v)$ of r (of dimension k) consists of

- an affine linear pencil $A = A_0 \otimes 1 + A_1 \otimes x_1 + \cdots + A_d \otimes x_d$ with matrix coefficients $A_0, A_1, \dots, A_d \in M_k(\mathbb{C})$,
- and a row vector u and a column vector v of dimension k over $\mathbb{C}$,

and satisfies the following property:

For any unital complex algebra $\mathcal{A}$ and each $X \in \operatorname{dom}_{\mathcal{A}}(r)$, we have that $A(X)$ is invertible in $M_k(\mathcal{A})$ and $r(X) = uA(X)^{-1}v$.

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- an affine linear pencil $A = A_0 \otimes 1 + A_1 \otimes x_1 + \cdots + A_d \otimes x_d$ with matrix coefficients $A_0, A_1, \dots, A_d \in M_k(\mathbb{C})$,
- and a row vector u and a column vector v of dimension k over $\mathbb{C}$,

and satisfies the following property:

For any unital complex algebra $\mathcal{A}$ and each $X \in \operatorname{dom}_{\mathcal{A}}(r)$, we have that $A(X)$ is invertible in $M_k(\mathcal{A})$ and $r(X) = uA(X)^{-1}v$.

Theorem (Helton, M., Speicher (2018))

Each noncommutative rational expression r admits a formal linear representation $\rho = (u, A, v)$. If r is non-degenerate, then A is full.

Glimpse behind the scenes III: invertibility

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in $M_k(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \ldots, x_d \rangle \cong M_k(\mathbb{C}\langle x_1, \ldots, x_d \rangle)$ which is full.

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Theorem (M., Speicher, Yin (2019))

Suppose that $X = (X_1, \ldots, X_d)$ is a d-tuple of selfadjoint operators in a tracial W^*-probability space $(\mathcal{M}, \tau)$ satisfying $\Delta(X) = d$. Then

$A(X) \in M_N(\mathcal{M})$ is invertible in $M_N(\widetilde{\mathcal{M}})$.

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Theorem (Collins, M., Miyagawa, Parraud, Yin (2021))

Suppose that $X^N = (X_1^N, \ldots, X_d^N)$ for $N \in \mathbb{N}$ are d-tuples of selfadjoint random matrices with laws μ_d^N on $M_N(\mathbb{C})_{\mathrm{sa}}^d$ which are absolutely continuous with respect to the Lebesgue measure on $M_N(\mathbb{C})_{\mathrm{sa}}^d$. Then there exists some $N_0 \in \mathbb{N}$ such that almost surely

$A(X^N)$ is invertible in $M_{kN}(\mathbb{C})$ for all $N \geq N_0$.

Glimpse behind the scenes IV: recursive structure

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Fact

Let $\mathfrak{R}_0$ be the set of all non-degenerate noncommutative rational expressions in d formal variables. Suppose that $\mathfrak{R} \subseteq \mathfrak{R}_0$ satisfies:

1. $\mathbb{C} \cup \{x_1, \dots, x_d\} \subseteq \mathfrak{R}$.
2. For $r_1, r_2 \in \mathfrak{R}$, we have $r_1 + r_2 \in \mathfrak{R}$ and $r_1 \cdot r_2 \in \mathfrak{R}$.
3. If $r \in \mathfrak{R}$ is such that r^{-1} is non-degenerate, then $r^{-1} \in \mathfrak{R}$.

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The evaluation part is proven by applying the previous argument to

$$\mathfrak{R}_0 := \{r \mid X \in \mathrm{dom}_{\widetilde{\mathcal{M}}}(r) \text{ and } r(X) = \mathrm{Ev}_X([r])\} \quad \text{respectively}$$
$$\mathfrak{R}_0 := \{r \mid \exists N_0 \in \mathbb{N} : \text{ almost surely } \forall N \geq N_0 : \ X^N \in \mathrm{dom}_{M_N(\mathbb{C})}(r)\}.$$

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For 3, use linearization $\rho = (u, A, v)$ of r and the Schur complement formula for the full affine linear pencil $\begin{pmatrix} 0 & u \\ v & A \end{pmatrix} \in M_{k+1}(\mathbb{C}\langle x_1, \dots, x_d\rangle)$.

Glimpse behind the scenes V: selfadjoint linearization

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Definition (Helton, M., Speicher (2018))

Let r be a selfadjoint noncommutative rational expression. A selfadjoint formal linear representation $\rho = (Q, w)$ of r (of dimension k) consists of

- an affine linear pencil $Q = Q_0 \otimes 1 + Q_1 \otimes x_1 + \cdots + Q_d \otimes x_d$ with selfadjoint matrix coefficients $Q_0, Q_1, \ldots, Q_d \in M_k(\mathbb{C})$,
- and a column vector w of dimension k over $\mathbb{C}$,

and satisfies the following property:

For any unital complex $*$-algebra $\mathcal{A}$, if $X \in \mathcal{A}_{\mathrm{sa}}^d \cap \mathrm{dom}_{\mathcal{A}}(r)$, then $Q(X)$ is invertible in $M_k(\mathcal{A})$ and $r(X) = w^* Q(X)^{-1} w$.

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Theorem (Helton, M., Speicher (2018))

Each selfadjoint noncommutative rational expression r admits a selfadjoint formal linear representation $\rho = (Q, w)$. If r is non-degenerate, then Q is full.

Glimpse behind the scenes VI: convergence in law

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For $X \in \widetilde{\mathcal{M}}_{\mathrm{sa}}$, define the cumulative distribution function

$$\mathcal{F}_X : \mathbb{R} \to [0,1], \qquad \mathcal{F}_X(t) := \mu_X\big((-\infty, t]\big).$$

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By Portmanteau's theorem, we need to show that (almost surely)

$$\limsup_{N\to\infty} \left|\mathcal{F}_{w^* Q(X^N)^{-1} w}(t) - \mathcal{F}_{w^* Q(X)^{-1} w}(t)\right| = 0$$

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Step 1: For $\varepsilon > 0$, let $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ be continuous such that $f_\varepsilon(t) = t^{-1}$ for all $t \in \mathbb{R} \setminus [-\varepsilon, \varepsilon]$; then, with $Q_N := Q(X^N)$ and $Q_\infty := Q(X)$,

$$\begin{aligned}\left|\mathcal{F}_{w^*Q_N^{-1}w}(t) - \mathcal{F}_{w^*Q_\infty^{-1}w}(t)\right| &\leq \left|\mathcal{F}_{w^*f_\varepsilon(Q_N)w}(t) - \mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(t)\right| \\ &\quad + (\mathrm{Tr}_k \otimes \mathrm{tr}_N)(\mathbf{1}_{[-\varepsilon,\varepsilon]}(Q_N)) + (\mathrm{Tr}_k \otimes \mathrm{tr}_N)(\mathbf{1}_{[-\varepsilon,\varepsilon]}(Q_\infty)).\end{aligned}$$

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Step 2: Prove that $w^* f_\varepsilon(Q_N) w \xrightarrow{\mathrm{dist}} w^* f_\varepsilon(Q_\infty) w$ as $N \to \infty$, by approximating f_ε by polynomials, and use Portmanteau's theorem again.

Extension of the main results

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So far: $X^N \xrightarrow{\text{dist}} X$ almost surely as $N \to \infty$

- $X^N = (X_1^N, \dots, X_d^N)$ are tuples of selfadjoint random matrices with absolutely continuous laws on $M_N(\mathbb{C})_{\text{sa}}^d$,
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In such situations, we studied the (convergence in law of) evaluations of (selfadjoint) non-degenerate noncommutative rational expressions.

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It is possible to generalize this to ...

1. tuples (X^N, U^N) of selfadjoint and unitary random matrices with absolutely continuous laws on $M_N(\mathbb{C})_{\text{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}$ which are convergent in $*$-distribution to (X, U) with $\Delta(X, U) = d_1 + d_2$;

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2. (convergence in law of) evaluations of (selfadjoint) matrix-valued non-degenerate noncommutative rational expressions.

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Thank you!