The free field meets free probability theory

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Workshop on Free Probability: the theory, its extensions
Centre de recherches mathematiques Montreal

March 5, 2019

Supported by the ERC Advanced Grant "Non-commutative distributions in free probability"

European Research Council
Established by the European Commission

Noncommutative probability spaces

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Definition

A noncommutative probability space $(\mathcal{A}, \phi)$ consists of

- a complex algebra $\mathcal{A}$ with unit $1_\mathcal{A}$ and
- a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ with $\phi(1_\mathcal{A}) = 1$ (expectation).

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Definition

A noncommutative probability space $(\mathcal{A}, \phi)$ is called

- C^*-probability space if
  - $\mathcal{A}$ is a unital C^*-algebra and
  - ϕ is a state on $\mathcal{A}$.
- tracial W^*-probability space, if
  - $\mathcal{A}$ is a von Neumann algebra and
  - ϕ is a faithful normal tracial state on $\mathcal{A}$.

Noncommutative distributions

Noncommutative distributions

Definition ("combinatorial distribution")

Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. For any given family $X = (X_i)_{i \in I}$ of noncommutative random variables, we call

$$\mu_X : \mathbb{C}\langle x_i \mid i \in I \rangle \to \mathbb{C}, \quad x_{i_1} \cdots x_{i_k} \mapsto \phi(X_{i_1} \cdots X_{i_k})$$

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Definition ("analytic distribution")

Let $(\mathcal{A}, \phi)$ be a C^*-probability space. For any given $X = X^* \in \mathcal{A}$, the noncommutative distribution of X can be identified with the unique Borel probability measure μ_X on the real line $\mathbb{R}$ that satisfies

$$\phi(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all integers } k \geq 0.$$

"Atoms" of noncommutative distributions

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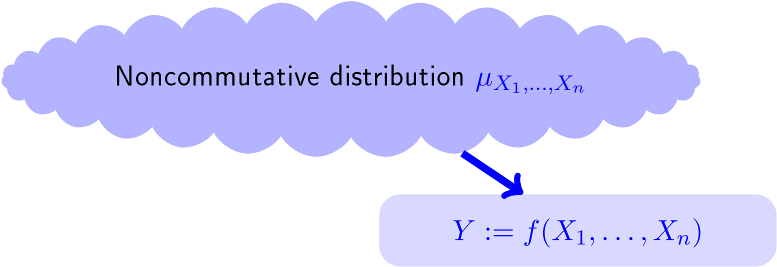

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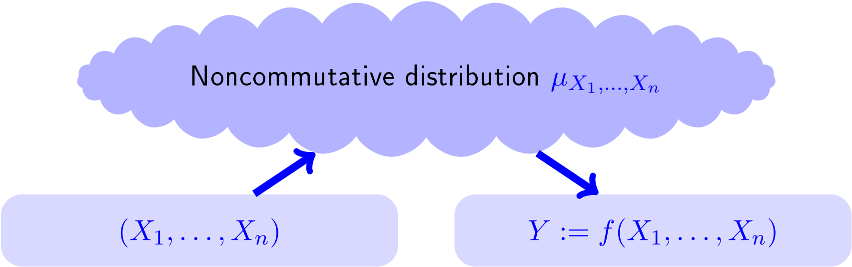

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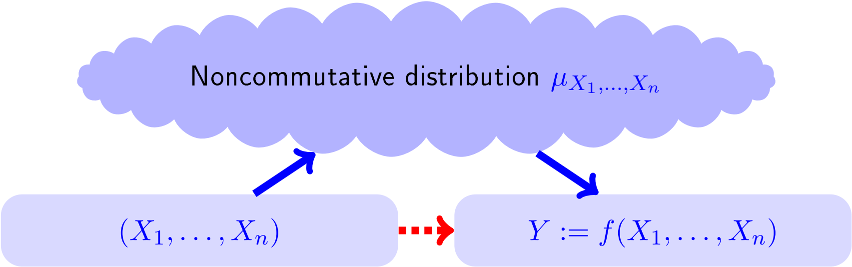

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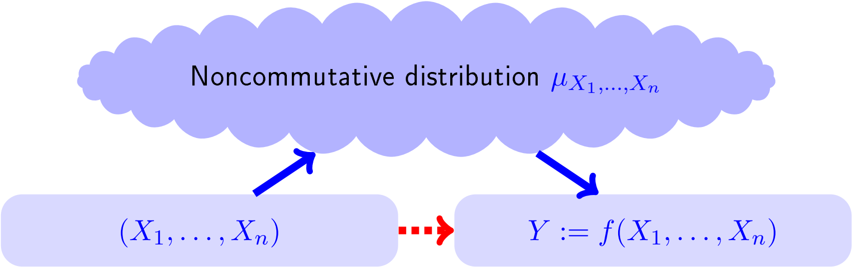

A generic regularity question

Consider the analytic distribution μ_Y of $Y = f(X_1, \dots, X_n)$ for a given a selfadjoint "noncommutative test function" f. Under what conditions on $(X_1, \dots, X_n)$ can we understand the atomic part of μ_Y?

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1. Noncommutative polynomials, i.e., expressions of the form

$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$

in formal non-commuting indeterminates $x_1, \dots, x_n$; we denote the unital complex algebra consisting of all noncommutative polynomials by $\mathbb{C}\langle x_1, \dots, x_n \rangle$.

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3. Affine linear pencils, i.e., matrices of noncommutative polynomials that are of the particular form

$$\mathbf{P} = b_0 + b_1 x_1 + \cdots + b_n x_n$$

   with scalar matrices $b_0, b_1, \dots, b_n$ of appropriate size.

The strong Atiyah property

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Definition (Shlyakhtenko, Skoufranis (2015))

We say that $X = (X_1, \dots, X_n)$ has the strong Atiyah property if

$$\mathrm{rank}(\mathbf{P}(X)) \in [0, \infty) \cap \mathbb{Z} \qquad \text{for each } \mathbf{P} \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle).$$

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Reminder

$$\operatorname{rank}(\mathbf{P}(X)) := N - (\operatorname{Tr}_N \circ \tau^{(N)})(p_{\ker(\mathbf{P}(X))}),$$

where

- $\operatorname{Tr}_N : \ M_N(\mathbb{C}) \to \mathbb{C}, \quad (a_{ij})_{i,j=1}^N \mapsto \sum_{i=1}^N a_{ii},$
- $\tau^{(N)} : \ M_N(\mathcal{M}) \to M_N(\mathbb{C}), \quad (Y_{ij})_{i,j=1}^N \mapsto \big(\tau(Y_{ij})\big)_{i,j=1}^N,$
- and $p_{\ker(\mathbf{P}(X))} \in M_N(\mathcal{M})$ is the orthogonal projection onto the kernel $\ker(\mathbf{P}(X))$ of the operator $\mathbf{P}(X)$.

Results about atoms I

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Suppose that

- the operators $X_1, \dots, X_n$ are freely independent and
- the individual analytic distributions $\mu_{X_1}, \dots, \mu_{X_n}$ are all non-atomic,

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Facts

If $(X_1, \dots, X_n)$ has the strong Atiyah property, then the following holds:

- For every selfadjoint $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$, the measure of each atom in the analytic distribution $\mu_{\mathbf{Y}}$ of the selfadjoint operator $\mathbf{Y} = \mathbf{P}(X_1, \dots, X_n)$ is an integer multiple of $\frac{1}{N}$.

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- In particular, if $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ is a non-constant selfadjoint polynomial, then the analytic distribution μ_Y of the selfadjoint operator $Y = P(X_1, \dots, X_n)$ cannot have atoms.

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The noncommutative derivatives are the linear mappings

$$\partial_1, \dots, \partial_n : \ \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle$$

which are uniquely determined by the two conditions

- $\partial_j(P_1 P_2) = (\partial_j P_1) \cdot P_2 + P_1 \cdot (\partial_j P_2)$ for all $P_1, P_2 \in \mathbb{C}\langle x_1, \dots, x_n\rangle$,
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- $\partial_j x_i = \delta_{i,j} 1 \otimes 1$ for $i, j = 1, \ldots, n$.

$\mathbb{C}\langle x_1, \ldots, x_n\rangle \otimes \mathbb{C}\langle x_1, \ldots, x_n\rangle$ becomes a $\mathbb{C}\langle x_1, \ldots, x_n\rangle$-bimodule via

$$P_1 \cdot (Q_1 \otimes Q_2) \cdot P_2 := (P_1 Q_1) \otimes (Q_2 P_2).$$

Conjugate variables and free Fisher information

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Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and consider any selfadjoint operators $X_1, \dots, X_n \in \mathcal{M}$; we put $\mathcal{M}_0 := \mathrm{vN}(X_1, \dots, X_n)$.

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Definition (Voiculescu (1998))

If $\xi_1, \dots, \xi_n \in L^2(\mathcal{M}_0, \tau)$ are such that for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$

$$(\tau \otimes \tau)((\partial_j P)(X_1, \dots, X_n)) = \tau(\xi_j P(X_1, \dots, X_n)), \quad j = 1, \dots, n,$$

then $(\xi_1, \dots, \xi_n)$ is called the conjugate system for $(X_1, \dots, X_n)$.

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The (non-microstates) free Fisher information is defined by

$$\Phi^*(X_1,\dots,X_n) := \begin{cases} \sum_{j=1}^n \|\xi_j\|_2^2, & \text{if a conjugate system } (\xi_1,\dots,\xi_n) \text{ for } (X_1,\dots,X_n) \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

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Suppose that $S_1, \dots, S_n$ are freely independent semicircular elements that are also free from $\{X_1, \dots, X_n\}$, then $(X_1 + \sqrt{t} S_n, \dots, X_n + \sqrt{t} S_n)$ admits a conjugate system for each $t > 0$. More precisely, we have

$$\frac{n^2}{C^2 + nt} \leq \Phi^*(X_1 + \sqrt{t} S_1, \dots, X_n + \sqrt{t} S_n) \leq \frac{n}{t} \quad \text{for all } t > 0,$$

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Philosophy

If $\delta^\star(X_1, \dots, X_n) = n$, then $\mu_{X_1, \dots, X_n}$ has no "atomic part".

Results about atoms II

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Theorem (Charlesworth, Shlyakhtenko, '16; M., Speicher, Weber, '17)

Suppose that $\delta^\star(X_1,\dots,X_n) = n$. Let $P \in \mathbb{C}\langle x_1,\dots,x_n\rangle$ be a selfadjoint non-constant noncommutative polynomial and consider the selfadjoint operator

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1. Does $\delta^\star(X_1, \dots, X_n) = n$ imply that $(X_1, \dots, X_n)$ has the strong Atiyah property?
2. Can we exclude atoms not only for non-constant noncommutative polynomials but also for operators that are noncommutative rational expressions in $X_1, \dots, X_n$?
   What actually does "non-constant" mean in this case?

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☞ They can be realized as *equivalence classes* of noncommutative rational expressions which are non-degenerate.

From rational functions to matrices of polynomials

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Definition

Let $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ be given.

- The (inner) rank of $\mathbf{Q}$, denoted by $\rho(\mathbf{Q})$, is the least integer $k \geq 1$ for which $\mathbf{Q}$ can be written as $\mathbf{Q} = \mathbf{R}_1 \mathbf{R}_2$ with some rectangular matrices

$$\mathbf{R}_1 \in M_{N \times k}(\mathbb{C}\langle x_1, \dots, x_n \rangle) \quad \text{and} \quad \mathbf{R}_2 \in M_{k \times N}(\mathbb{C}\langle x_1, \dots, x_n \rangle).$$

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Facts

- $\mathbf{Q}$ full $\quad \Longleftrightarrow \quad$ $\mathbf{Q}$ invertible in $M_N(\mathbb{C}\not<x_1, \dots, x_n\not>)$

From rational functions to matrices of polynomials

Definition

Let $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ be given.

- The (inner) rank of $\mathbf{Q}$, denoted by $\rho(\mathbf{Q})$, is the least integer $k \geq 1$ for which $\mathbf{Q}$ can be written as $\mathbf{Q} = \mathbf{R}_1 \mathbf{R}_2$ with some rectangular matrices

$$\mathbf{R}_1 \in M_{N \times k}(\mathbb{C}\langle x_1, \dots, x_n \rangle) \quad \text{and} \quad \mathbf{R}_2 \in M_{k \times N}(\mathbb{C}\langle x_1, \dots, x_n \rangle).$$

- We call $\mathbf{Q}$ full if it has full rank, i.e., if $\rho(\mathbf{Q}) = N$.

Facts

- $\mathbf{Q}$ full $\quad \Longleftrightarrow \quad$ $\mathbf{Q}$ invertible in $M_N(\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle)$
- Every noncommutative rational function $r \in \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ admits a linear representation, i.e., it can be written as $r = u\mathbf{Q}^{-1}v$ with a full affine linear pencil $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$, and scalar vectors u and v of appropriate size.

Evaluations of noncommutative rational functions I

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Trivial facts

Let $\mathcal{A}$ be a unital algebra and consider $X_1, \dots, X_n \in \mathcal{A}$. There is a unital homomorphism

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathcal{A}$$

that is uniquely determined by the condition that $\mathrm{ev}_X(x_i) = X_i$ for $i = 1, \dots, n$. The latter extends naturally to a unital homomorphism

$$\mathrm{ev}_X^{(N)} : M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle) \to M_N(\mathcal{A}),\ (P_{ij})_{i,j=1}^N \mapsto (\mathrm{ev}_X(P_{ij}))_{i,j=1}^N.$$

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This defines evaluation of (matrix-valued) noncommutative polynomials and of affine linear pencils, in particular. Rational functions are more subtle.

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Challenging facts

- Not every rational expression can be evaluated everywhere.
- Two rational expressions representing the same rational function need not to give the same value under evaluation.

Evaluations of noncommutative rational functions II

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Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and let $\mathcal{A}$ be the algebra of unbounded operators affiliated to $\mathcal{M}$.

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For a linear representation $\rho = (u, \mathbf{Q}, v)$ (of any rational function), we put

$$\operatorname{dom}_{\mathcal{A}}(\rho) := \{X \in \mathcal{A}^n \mid \mathbf{Q}(X) \text{ invertible in } M_N(\mathcal{A})\}.$$

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Theorem

1. Suppose that $\rho_1 = (u_1, \mathbf{Q}_1, v_1)$ and $\rho_2 = (u_2, \mathbf{Q}_2, v_2)$ are two linear representations of the same noncommutative rational function. Then

$$X \in \mathrm{dom}_{\mathcal{A}}(\rho_1) \cap \mathrm{dom}_{\mathcal{A}}(\rho_2) \quad \Longrightarrow \quad u_1 \mathbf{Q}_1(X)^{-1} v_1 = u_2 \mathbf{Q}_2(X)^{-1} v_2.$$

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2. Every $r \in \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ admits an evaluation $r(X) = \operatorname{Ev}_X(r)$ on

$$\operatorname{dom}_{\mathcal{A}}(r) := \bigcup \big\{ \operatorname{dom}_{\mathcal{A}}(\rho) \bigm| \rho \text{ linear representation of } r \big\},$$

which is given by $\operatorname{Ev}_X(r) := u\mathbf{Q}(X)^{-1} v$ for any $\rho = (u, \mathbf{Q}, v)$.

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Theorem (M., Speicher, Yin (2018))

For every given $X = (X_1, \ldots, X_n)$ in $\mathcal{M}^n$, the following statements are equivalent:

1. For any $N \in \mathbb{N}$ and every affine linear pencil $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ we have: if $\mathbf{P}$ is full, then $\mathbf{P}(X) \in M_N(\mathcal{A})$ is invertible.

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4. We have $X \in \operatorname{dom}_{\mathcal{A}}(r)$ for each $r \in \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ and $r \mapsto \operatorname{Ev}_X(r)$ induces an injective homomorphism $\operatorname{Ev}_X : \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle \to \mathcal{A}$ that extends the evaluation map $\operatorname{ev}_X : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathcal{A}$.

Results about atoms III

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Theorem (M., Speicher, Yin (2018))

If $\delta^\star(X_1, \dots, X_n) = n$, then the equivalent statements of the previous theorem hold. In particular, we have the following:

1. $X = (X_1, \dots, X_n)$ has the strong Atiyah property. In fact,

$$\operatorname{rank}(\mathbf{P}(X)) = \rho(\mathbf{P}) \qquad \text{for all } \mathbf{P} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle).$$

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2. For every selfadjoint $\mathbf{P}\in M_N(\mathbb{C}\langle x_1,\dots,x_n\rangle)$, the operator $\mathbf{Y}:=\mathbf{P}(X_1,\dots,X_n)$ has atoms precisely at the points in the set
$$\{\lambda\in\mathbb{C}\mid \mathbf{P}-\lambda 1_N\in M_N(\mathbb{C}\langle x_1,\dots,x_n\rangle) \text{ is not full}\}$$
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with size $\mu_{\mathbf{Y}}(\{\lambda\}) = 1 - \frac{1}{N}\rho(\mathbf{P} - \lambda 1_N)$.
3. For every non-constant selfadjoint $r \in \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$, the evaluation $r(X)$ is an affiliated unbounded operator whose analytic distribution has no atoms.

Sketch of proof

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The main step is to prove by induction on N that

$$A(N) \quad \begin{cases} \text{If } b_0, b_1, \dots, b_n \text{ are matrices in } M_N(\mathbb{C}) \text{ for which} \\ \quad \mathbf{P} = b_0 + b_1 x_1 + \cdots + b_n x_n \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle) \\ \text{is full over } \mathbb{C}\langle x_1, \dots, x_n \rangle, \text{ then the only projection} \\ p \in M_N(\mathcal{M}_0) \text{ that satisfies } P(X)p = 0 \text{ is } p = 0. \end{cases}.$$

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and such that, if we put $\tilde{p} := \tau^{(N)}(p)$ and $\tilde{q} := \tau^{(N)}(q)$,
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