Regularity properties of noncommutative distributions

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Noncommutative probability spaces

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Definition

A noncommutative probability space $(\mathcal{A}, \phi)$ consists of

- a complex algebra $\mathcal{A}$ with unit $1_{\mathcal{A}}$ and
- a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ with $\phi(1_{\mathcal{A}}) = 1$ (expectation).

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Definition

A noncommutative probability space $(\mathcal{A}, \phi)$ is called

- C^*-probability space if
  - $\mathcal{A}$ is a unital C^*-algebra and
  - ϕ is a state on $\mathcal{A}$.
- tracial W^*-probability space, if
  - $\mathcal{A}$ is a von Neumann algebra and
  - ϕ is a faithful normal tracial state on $\mathcal{A}$.

Noncommutative distributions

Noncommutative distributions

Definition ("combinatorial distribution")

Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. For any given family $X = (X_i)_{i \in I}$ of noncommutative random variables, we call

$$\mu_X : \mathbb{C}\langle x_i \mid i \in I \rangle \to \mathbb{C}, \quad x_{i_1} \cdots x_{i_k} \mapsto \phi(X_{i_1} \cdots X_{i_k})$$

the (joint) noncommutative distribution of X.

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Definition ("analytic distribution")

Let $(\mathcal{A}, \phi)$ be a C^*-probability space. For any given $X = X^* \in \mathcal{A}$, the noncommutative distribution of X can be identified with the unique Borel probability measure μ_X on the real line $\mathbb{R}$ that satisfies

$$\phi(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all integers } k \geq 0.$$

Noncommutative distribution $\mu_{X_1,\dots,X_n}$

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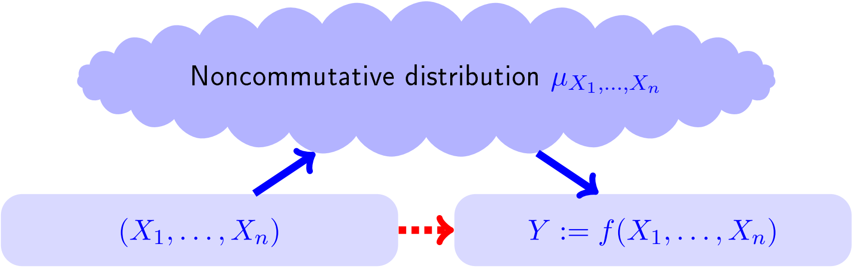
$Y := f(X_1, \ldots, X_n)$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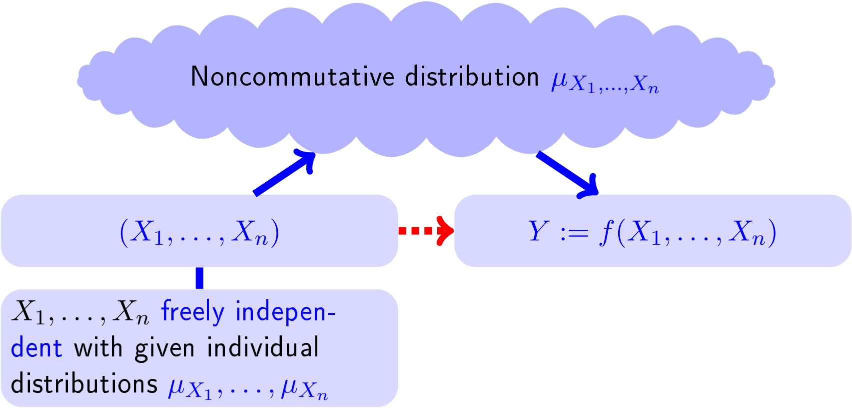
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[Belinschi-M.-Speicher, 17], [Belinschi-Sniady-Speicher, 18], [Helton-M.-Speicher, 18]

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[Skoufranis-Shlyakhtenko, 15], [Ajanki-Erdös-Krüger, 16], [Alt-Erdös-Krüger, 18]

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[Charlesworth-Shlyakhtenko, 16], [M.-Speicher-Weber, 17], [M.-Speicher-Yin, 18]

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1. Noncommutative polynomials, i.e., expressions of the form

$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$

in (formal) non-commuting indeterminates $x_1, \dots, x_n$; we denote the (unital) complex $*$-algebra that consists of all noncommutative polynomials by $\mathbb{C}\langle x_1, \dots, x_n \rangle$.

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2. Affine linear pencils, i.e., elements $\mathbf{P}$ in $M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ for an arbitrary $N \in \mathbb{N}$ that are of the particular form

$$\mathbf{P} = b_0 + b_1 x_1 + \cdots + b_n x_n$$

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3. Noncommutative rational functions, i.e., elements in the universal field of fractions for $\mathbb{C}\langle x_1, \dots, x_n \rangle$; for the latter, we write $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$.

A glimpse on noncommutative rational functions

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Definition

Let $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ be given.

- The (inner) rank of $\mathbf{Q}$, denoted by $\rho(\mathbf{Q})$, is the least integer $k \geq 1$ for which $\mathbf{Q}$ can be written as $\mathbf{Q} = \mathbf{R}_1 \mathbf{R}_2$ with some rectangular matrices

$$\mathbf{R}_1 \in M_{N \times k}(\mathbb{C}\langle x_1, \dots, x_n \rangle) \quad \text{and} \quad \mathbf{R}_2 \in M_{k \times N}(\mathbb{C}\langle x_1, \dots, x_n \rangle).$$

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Facts

- $\mathbf{Q}$ full $\quad \Longleftrightarrow \quad$ $\mathbf{Q}$ invertible in $M_N(\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle)$
- Every noncommutative rational function $r \in \mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ admits a linear representation, i.e., it can be written as $r = -u\mathbf{Q}^{-1}v$ with an affine linear pencil $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ and scalar vectors u and v of appropriate size.

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Trivial facts

Let $\mathcal{A}$ be a unital algebra and consider $X_1, \dots, X_n \in \mathcal{A}$. There is a unital homomorphism

$$\mathrm{ev}_X : \ \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathcal{A}$$

that is uniquely determined by the condition that $\mathrm{ev}_X(x_i) = X_i$ for $i = 1, \dots, n$. The latter extends naturally to a unital homomorphism

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Challenging facts

- Not every rational expression can be evaluated everywhere.
- Two rational expressions representing the same rational function need not to give the same value under evaluation.

Noncommutative and cyclic derivatives

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Definition

(i) The noncommutative derivatives are the linear mappings

$$\partial_1, \dots, \partial_n : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle$$

which are uniquely determined by the two conditions

- $\partial_j(P_1 P_2) = (\partial_j P_1) \cdot P_2 + P_1 \cdot (\partial_j P_2)$ for all $P_1, P_2 \in \mathbb{C}\langle x_1, \dots, x_n\rangle$,
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(ii) The cyclic derivatives are the linear mappings

$$D_1, \dots, D_n : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle$$

that are defined by $D_j := \tilde{m} \circ \partial_j$, where

$$\tilde{m} : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle$$

denotes the flipped multiplication defined as $\tilde{m}(P_1 \otimes P_2) := P_2 P_1$.

Conjugate variables and free Fisher information

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Let $(\mathcal{M},\tau)$ be a tracial W^*-probability space and consider any selfadjoint operators $X_1,\dots,X_n \in \mathcal{M}$; we put $\mathcal{M}_0 := \mathrm{vN}(X_1,\dots,X_n)$.

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Definition (Voiculescu (1998))

If $\xi_1,\dots,\xi_n \in L^2(\mathcal{M}_0,\tau)$ are such that for all $P \in \mathbb{C}\langle x_1,\dots,x_n\rangle$

$$(\tau\otimes\tau)((\partial_j P)(X_1,\dots,X_n)) = \tau(\xi_j P(X_1,\dots,X_n)), \quad j=1,\dots,n,$$

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Definition (Voiculescu (1998))

The (non-microstates) free Fisher information is defined by

$$\Phi^*(X_1, \dots, X_n) := \begin{cases} \sum_{j=1}^{n} \|\xi_j\|_2^2, & \text{if a conjugate system } (\xi_1, \dots, \xi_n) \text{ for } (X_1, \dots, X_n) \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

Gibbs laws and the Schwinger-Dyson equation I

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Theorem (Guionnet, Shlyakhtenko (2009))

Let $V \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ following the Gibbs law

$$d\Lambda_N^V(X_1^{(N)}, \dots, X_n^{(N)}) = \frac{1}{Z_N^V} e^{-N\,\mathrm{Tr}(V(X_1^{(N)}, \dots, X_n^{(N)}))}\, dX_1^{(N)} \dots dX_n^{(N)}.$$

Then, for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$,

$$\lim_{N \to \infty} \mathrm{tr}_N(P(X_1^{(N)}, \dots, X_n^{(N)})) = \tau(P(X_1, \dots, X_n)) \quad \text{almost surely}$$

for selfadjoint operators $X_1, \dots, X_n$ in some W^*-probability space $(\mathcal{M}, \tau)$ that satisfy the Schwinger-Dyson equation with potential V, i.e.

$$(\tau \otimes \tau)\big((\partial_j P)(X_1, \dots, X_n)\big) = \tau\big((D_j V)(X_1, \dots, X_n) P(X_1, \dots, X_n)\big)$$

Gibbs laws and the Schwinger-Dyson equation I

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Gibbs laws and the Schwinger-Dyson equation II

Definition (Schwinger-Dyson equation)

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for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ and for every $j = 1, \dots, n$.

Observation

If $(X_1, \dots, X_n)$ satisfies the Schwinger-Dyson equation with potential V, then $(\xi_1, \dots, \xi_n)$ defined by

$$\xi_j := (D_j V)(X_1, \dots, X_n) \qquad \text{for } j = 1, \dots, n$$

is the conjugate system of $(X_1, \dots, X_n)$. Hence:

$$\Phi^*(X_1, \dots, X_n) < \infty$$

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Suppose that $S_1, \dots, S_n$ are freely independent semicircular elements that are also free from $\{X_1, \dots, X_n\}$, then $(X_1 + \sqrt{t}S_n, \dots, X_n + \sqrt{t}S_n)$ admits a conjugate system for each $t > 0$. More precisely, we have

$$\frac{n^2}{C^2 + nt} \leq \Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n) \leq \frac{n}{t} \quad \text{for all } t > 0,$$

with $C^2 := \tau(X_1^2 + \cdots + X_n^2)$.

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Philosophy

If $\delta^\star(X_1, \dots, X_n) = n$, then $\mu_{X_1,\dots,X_n}$ has no "atomic part".

Absence of atoms

Suppose that $\delta^{\star}(X_1, \dots, X_n) = n$.

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Theorem (CS16, MSW17)

Let $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be selfadjoint and non-constant. Then $\mu_{P(X_1,\dots,X_n)}$ has no atoms.

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Take selfadjoint matrices $b_0,b_1,\dots,b_n\in M_N(\mathbb{C})$, then the matrix-valued element $\mathbf{Y}:=\mathbf{P}(X_1,\dots,X_n)$ for

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Consider $Y = Y^*$ in $(\mathcal{M}, \tau)$. Let μ_Y be the analytic distribution of Y and let $\mathcal{F}_Y$ be its cumulative distribution function, i.e., $\mathcal{F}_Y(t) := \mu_Y((-\infty, t])$.

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Lemma (M., Speicher, Yin (2018))

If there exist $c > 0$ and $\alpha > 1$ such that

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for all $s \in \mathbb{R}$ and each spectral projection p of Y, then $\mathcal{F}_Y$ is Hölder continuous with exponent $\beta := \frac{2}{\alpha-1}$; more precisely, we have that

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Proof – following ideas of [Charlesworth, Shlyakhtenko (2016)].

Take $p = E_Y((s, t])$ for the spectral measure E_Y of Y and observe that

$$\|p\|_2 = \mu_Y((s, t])^{1/2} \qquad \text{and} \qquad \|(Y - s)p\|_2 \leq |t - s|\mu_Y((s, t])^{1/2}.$$

□

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Let $P \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ be selfadjoint with degree $d \geq 1$ and consider

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Then there exists some constant $C > 0$ such that

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In fact, for every $R > \max_{i=1,\dots,n} \|X_i\|$, we can take

$$C = \big(8\Phi^*(X)^{1/2}R\big)^{\frac{2}{3}} \rho_R(P)^{-\frac{2^d}{3(2^d-1)}} \|P\|_R^{-\frac{2}{3(2^d-1)}} \prod_{k=1}^{d-1} \Big(\frac{d!}{(d-k)!}\Big)^{\frac{2^k}{3(2^d-1)}},$$

where $\|P\|_R$ and $\rho_R(P)$ are quantities that depend only on P and R.

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Let $P \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ be selfadjoint with degree $d \geq 1$; consider

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Then, the logarithmic energy (and thus also the free entropy $\chi^*(Y)$)

$$I(\mu_Y) := \int_{\mathbb{R}} \int_{\mathbb{R}} \log \frac{1}{|s-t|} \, d\mu_Y(s) \, d\mu_Y(t)$$

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Remark

This is a first step towards a conjecture of Charlesworth and Shlyakhtenko (2016) saying that this should remain valid under the weaker condition

$$\chi^*(X_1, \ldots, X_n) > -\infty.$$

Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

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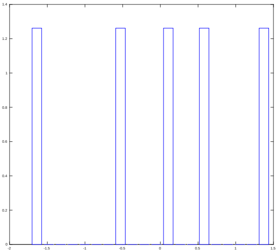

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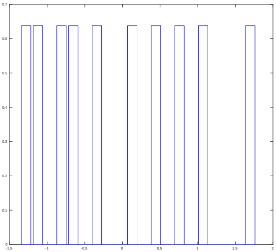

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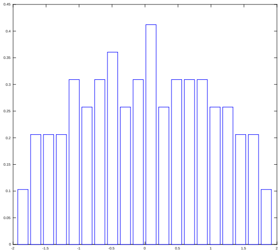

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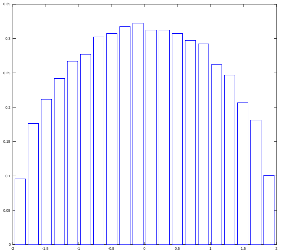

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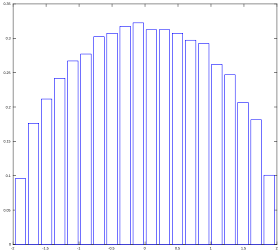

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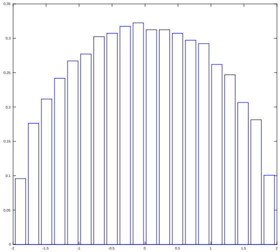

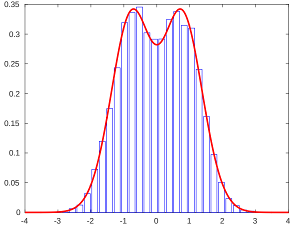

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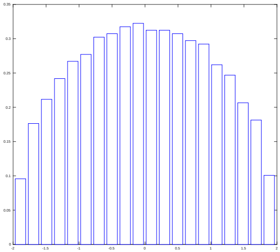

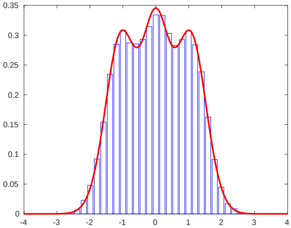

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$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$

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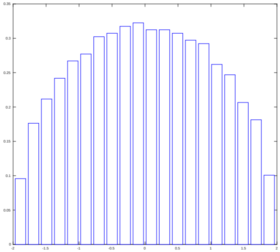
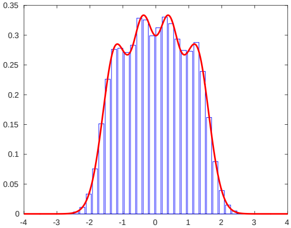
The mean eigenvalue distribution of X is the probability measure $\overline{\mu}_X$ on $\mathbb{C}$ that is given by

$$\overline{\mu}_X := \mathbb{E}[\mu_X].$$





Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

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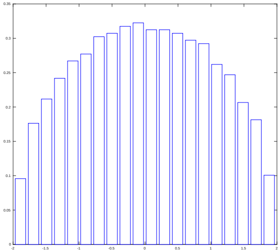

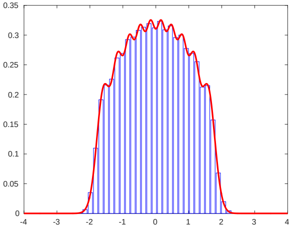

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Gaussian random matrices

$\rightsquigarrow$ Wigner's semicircle theorem

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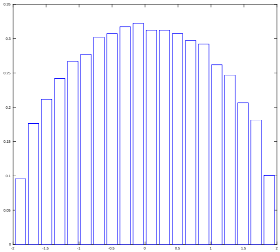

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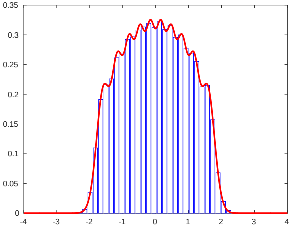

Hölder continuity consequences: Gibbs laws

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Corollary (Banna, M. (2018))

Let $V \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ distributed according to the Gibbs law

$$d\Lambda_N^V(X_1^{(N)}, \dots, X_n^{(N)}) = \frac{1}{Z_N^V} e^{-N \operatorname{Tr}(V(X_1^{(N)}, \dots, X_n^{(N)}))} \, dX_1^{(N)} \dots dX_n^{(N)}.$$

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Then, for each non-constant selfadjoint $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$, we have that:

(i) The empirical eigenvalue distribution $\mu_{Y^{(N)}}$ of

$$Y^{(N)} := P(X_1^{(N)}, \dots, X_n^{(N)})$$

converges in distribution almost surely to a compactly supported Borel probability measure μ on $\mathbb{R}$ with a cumulative distribution function that is Hölder continuous with exponent $\frac{2}{3(2^d-1)}$.

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(ii) With respect to the Kolmogorov distance Δ, we have that

$$\Delta(\overline{\mu}_{Y^{(N)}}, \mu) \to 0 \qquad \text{as } N \to \infty.$$

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(ii) There is a constant $C > 0$ such that

$$\Delta(\overline{\mu}_{Y^{(N)}}, \mu) \leq C N^{-\frac{1}{13 \cdot 2^{d+2} - 60}} \qquad \text{for all } N \in \mathbb{N}.$$

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Thank you!