Limit laws of random matrices beyond the Dyson equation

Tobias Mai
(joint work with Sheng Yin)
Saarland University

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- $X^{(N)} = (X_1^{(N)}, \dots, X_n^{(N)})$: n-tuples of independent standard complex Gaussian random matrices of size $N \times N$
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- $\mathcal{L}$ is flat if there is $c > 0$ so that $c^{-1} \operatorname{tr}_k(b) \mathbf{1}_k \geq \mathcal{L}(b) \geq c \operatorname{tr}_k(b) \mathbf{1}_k$ for *every* positive semi-definite matrix $b \in M_k(\mathbb{C})$.

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- $\mathcal{L}$ is nowhere rank-decreasing if there is *no* positive semi-definite matrix $b \in M_k(\mathbb{C})$ such that $\operatorname{rank}(\mathcal{L}(b)) < \operatorname{rank}(b)$.

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- The Cauchy-transform of $\mathbb{X}$, i.e., $\mathbf{G}_{\mathbb{X}} : \mathbb{H}^+(M_k(\mathbb{C})) \to \mathbb{H}^-(M_k(\mathbb{C}))$,

$$\mathbf{G}_{\mathbb{X}}(b) := (\mathrm{id}_{M_k(\mathbb{C})} \otimes \tau)\big((b \otimes \mathbf{1} - \mathbb{X})^{-1}\big),$$

  is *uniquely determined* by the Dyson equation

$$\mathbf{G}_{\mathbb{X}}(b)^{-1} = b - \mathcal{L}(\mathbf{G}_{\mathbb{X}}(b)) \qquad \text{for all } b \in \mathbb{H}^+(M_k(\mathbb{C})).$$

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Flatness of $\mathcal{L}$ allows deep statements about the spectral distribution $\mu_{\mathbb{X}}$ of $\mathbb{X}$ (aka self-consistent density of states), defined by

$$\mathrm{tr}_k(\mathbf{G}_{\mathbb{X}}(z\mathbf{1}_k)) = \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu_{\mathbb{X}}(t) \qquad \text{for all } z \in \mathbb{C}^+,$$

such as absolute continuity and $\frac{1}{3}$-Hölder continuity of the density.

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- $\mathbf{1} \in \mathcal{B} \subseteq \mathcal{N}$
- $\mathcal{L} : \mathcal{B} \to \mathcal{B}$, positive and flat
- $\mathbb{X}$ in $\mathcal{N}_{\text{sa}}$ such that

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  - (semi-)flat
  - nowhere rank-decreasing
- $X_1, \dots, X_n$ in $\mathcal{M}_{\mathrm{sa}}$ satisfies
  - $\Phi^*(X_1, \dots, X_n) < \infty$
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$\mathbf{1} \in \mathcal{B} \subseteq \mathcal{N}$, $\mathbb{X} \in \mathcal{N}_{\text{sa}}$, $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ positive such that: $\Phi^*(\mathbb{X} : \mathcal{B}, \mathcal{L}) < \infty$

[M., Yin, '20]

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Let $(\mathcal{N}, \tau)$ be a tracial W^*-probability space.

Definition (Voiculescu (1998))

Let $X = X^* \in \mathcal{N}$ be given.

- We call $\xi \in L^2(\mathbb{C}\langle X\rangle, \tau)$ the conjugate variable of X if

$$\langle (\partial P)(X), \mathbf{1} \otimes \mathbf{1}\rangle_{\tau\otimes\tau} = \langle P(X), \xi\rangle_\tau \quad \text{for all } P \in \mathbb{C}\langle x\rangle.$$

- The conjugate variable ξ is automatically unique if it exists; it will be denoted by $\mathcal{J}(X : \mathbb{C})$.
- The free Fisher information is $\Phi^*(X : \mathbb{C}) := \|\mathcal{J}(X : \mathbb{C})\|_2^2$.

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Consider the algebra $\mathcal{B}\langle x\rangle$ of polynomials in the formal variable x and $\mathcal{B}$.

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The noncommutative derivative $\partial : \mathcal{B}\langle x\rangle \to \mathcal{B}\langle x\rangle \otimes \mathcal{B}\langle x\rangle$ is the unique derivation satisfying $\partial x = 1 \otimes 1$ and $\partial b = 0$ for every $b \in \mathcal{B}$.

Let $(\mathcal{N}, \tau)$ be a tracial W^*-probability space and $\mathbf{1} \in \mathcal{B} \subseteq \mathcal{N}$, $\mathcal{L} : \mathcal{B} \to \mathcal{B}$.

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Theorem (M., Yin (2020))

Suppose that $\Phi^*(X_1, \dots, X_n) < \infty$ and that $\mathcal{L}$ is semi-flat. Then the cumulative distribution function $\mathcal{F}_{\mathbb{X}}$, i.e., $\mathcal{F}_{\mathbb{X}}(t) := \mu_{\mathbb{X}}((-\infty, t])$, of the spectral measure $\mu_{\mathbb{X}}$ of $\mathbb{X}$ is $\frac{2}{3}$-Hölder continuous.

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