Limit laws of random matrices beyond the Dyson equation

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Some basic random matrix model

\[ \text{Definition (self-energy operator / quantum operator)} \]

To the matrices \( b_1, \ldots, b_n \), we associate the positive linear map \( L : M_k(\mathbb{C}) \to M_k(\mathbb{C}) \), \( b \mapsto b_1 b \oplus \cdots \oplus b_n b_n \).

\( L \) is tight if there is \( c > 0 \) so that
\[ c - 1 \text{tr}_k(b) \geq L(b) \geq c \text{tr}_k(b) \]
for every positive semi-definite matrix \( b \in M_k(\mathbb{C}) \).

\( L \) is nowhere rank-decreasing if there is no positive semi-definite matrix \( b \in M_k(\mathbb{C}) \) such that \( \text{rank}(L(b)) < \text{rank}(b) \).
Some basic random matrix model

Question

- $X^{(N)} = (X_1^{(N)}, \ldots, X_n^{(N)})$: $n$-tuples of independent standard complex Gaussian random matrices of size $N \times N$
- $b_1, \ldots, b_n$: hermitian complex deterministic matrices of size $k \times k$
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What can be said about the asymptotic eigenvalue distribution of

$$X^{(N)} := b_1 \otimes X_1^{(N)} + \cdots + b_n \otimes X_n^{(N)}$$
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Definition (self-energy operator / quantum operator)

To the matrices \( b_1, \ldots, b_n \), we associate the positive linear map

\[
\mathcal{L} : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C}), \quad b \mapsto b_1 bb_1 + \cdots + b_n bb_n.
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\[ \mathcal{L} : M_k(\mathbb{C}) \to M_k(\mathbb{C}), \quad b \mapsto b_1 bb_1 + \cdots + b_n bb_n. \]

- \( \mathcal{L} \) is flat if there is \( c > 0 \) so that \( c^{-1} \text{tr}_k(b) \mathbf{1}_k \geq \mathcal{L}(b) \geq c \text{tr}_k(b) \mathbf{1}_k \) for every positive semi-definite matrix \( b \in M_k(\mathbb{C}) \).
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- \( \mathcal{L} \) is nowhere rank-decreasing if there is no positive semi-definite matrix \( b \in M_k(\mathbb{C}) \) such that \( \text{rank}(\mathcal{L}(b)) < \text{rank}(b) \).
The Dyson equation

[Erdős, Knowles, Yau, Yin, ’13], [Ajanki, Erdős, Krüger, ’16], [Alt, Erdős, Krüger, ’18]
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- \( X^{(N)} \xrightarrow{\text{dist}} X \), where \( X = (X_1, \ldots, X_n) \) is an \( n \)-tuple of freely independent semicircular elements in a \( W^* \)-probability space \((\mathcal{M}, \tau)\).
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- \( \overline{X^{(N)}} \xrightarrow{\text{dist}} \overline{X} \), where \( \overline{X} := b_1 \otimes X_1 + \cdots + b_n \otimes X_n \) is a matrix-valued semicircular element with covariance map \( \mathcal{L} \).
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- $\mathbf{X}^{(N)} \xrightarrow{\text{dist}} \mathbf{X}$, where $\mathbf{X} := b_1 \otimes X_1 + \cdots + b_n \otimes X_n$ is a matrix-valued semicircular element with covariance map $\mathcal{L}$.
- The Cauchy-transform of $\mathbf{X}$, i.e., $\mathbf{G}_X : \mathbb{H}^+(M_k(\mathbb{C})) \rightarrow \mathbb{H}^-(M_k(\mathbb{C}))$,

$$
\mathbf{G}_X(b) := (\text{id}_{M_k(\mathbb{C})} \otimes \tau)((b \otimes 1 - \mathbf{X})^{-1}),
$$

is uniquely determined by the Dyson equation

$$
\mathbf{G}_X(b)^{-1} = b - \mathcal{L}(\mathbf{G}_X(b)) \quad \text{for all } b \in \mathbb{H}^+(M_k(\mathbb{C})).
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The Dyson equation

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- $X^{(N)} \xrightarrow{\text{dist}} X$, where $X = (X_1, \ldots, X_n)$ is an $n$-tuple of freely independent semicircular elements in a $W^*$-probability space $(\mathcal{M}, \tau)$.
- $X^{(N)} \xrightarrow{\text{dist}} \mathbb{X}$, where $\mathbb{X} := b_1 \otimes X_1 + \cdots + b_n \otimes X_n$ is a matrix-valued semicircular element with covariance map $\mathcal{L}$.
- The Cauchy-transform of $\mathbb{X}$, i.e., $G_{\mathbb{X}} : \mathbb{H}^+(M_k(\mathbb{C})) \to \mathbb{H}^-(M_k(\mathbb{C}))$,

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Flatness of $\mathcal{L}$ allows deep statements about the spectral distribution $\mu_{\mathbb{X}}$ of $\mathbf{X}$ (aka self-consistent density of states), defined by

$$\text{tr}_k(G_{\mathbb{X}}(z 1_k)) = \int_{\mathbb{R}} \frac{1}{z - t} d\mu_{\mathbb{X}}(t) \quad \text{for all } z \in \mathbb{C}^+,$$

such as absolute continuity and $\frac{1}{3}$-Hölder continuity of the density.
More generally ...
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For $X_1, \ldots, X_n \in \mathcal{M}_{\text{sa}}$ and $b_1, \ldots, b_n \in \mathcal{M}_k(\mathbb{C})_{\text{sa}}$ consider

$$X := b_1 \otimes X_1 + \cdots + b_n \otimes X_n.$$
More generally ...

For $X_1, \ldots, X_n \in \mathcal{M}_{sa}$ and $b_1, \ldots, b_n \in M_k(\mathbb{C})_{sa}$ consider

$$X := b_1 \otimes X_1 + \cdots + b_n \otimes X_n.$$ 

- $1 \in B \subseteq \mathcal{N}$
- $\mathcal{L} : B \to B$, positive and flat
- $X$ in $\mathcal{N}_{sa}$ such that

$$G_X : \mathbb{H}^+(B) \to \mathbb{H}^-(B)$$

satisfies the Dyson equation

$$G_X(b)^{-1} = b - \mathcal{L}(G_X(b))$$

[Alt, Erdős, Krüger, ’18]
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- $\mathcal{L} : \mathcal{B} \to \mathcal{B}$, positive and flat
- $X$ in $\mathcal{N}_{\text{sa}}$ such that

$$
G_X : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^-(\mathcal{B})
$$

satisfies the Dyson equation

$$
G_X(b)^{-1} = b - \mathcal{L}(G_X(b))
$$

[Alt, Erdős, Krüger, ’18]

- $1_k \in M_k(\mathbb{C}) \subseteq M_k(\mathcal{M})$
- $\mathcal{L} : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ for $b_1, \ldots, b_n \in M_k(\mathbb{C})_{\text{sa}}$ is
  - (semi-)flat
  - nowhere rank-decreasing
- $X_1, \ldots, X_n$ in $\mathcal{M}_{\text{sa}}$ satisfies
  - $\Phi^*(X_1, \ldots, X_n) < \infty$
  - ...

[M., Speicher, Yin ’19], [M., Yin, ’20]
More generally ...

For $X_1, \ldots, X_n \in \mathcal{M}_{sa}$ and $b_1, \ldots, b_n \in M_k(\mathbb{C})_{sa}$ consider

$$X := b_1 \otimes X_1 + \cdots + b_n \otimes X_n.$$ 

1. $1 \in \mathcal{B} \subseteq \mathcal{N}$
2. $\mathcal{L} : \mathcal{B} \to \mathcal{B}$, positive and flat
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   $$G_X : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^-(\mathcal{B})$$
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[Alt, Erdős, Krüger, ’18]

1. $1_k \in M_k(\mathcal{C}) \subseteq M_k(\mathcal{M})$
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   - $\Phi^*(X_1, \ldots, X_n) < \infty$
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[M., Speicher, Yin ’19], [M., Yin, ’20]
Consider the algebra $C\langle x \rangle$ of polynomials in the formal variable $x$.

**Definition**
The noncommutative derivative $\partial: C\langle x \rangle \to C\langle x \rangle \otimes C\langle x \rangle$ is the unique derivation satisfying $\partial x = 1 \otimes 1$.

Let $(N,\tau)$ be a tracial $W^*$-probability space.

**Definition (Voiculescu (1998))**
Let $X = X^* \in N$ be given. We call $\xi \in L^2(C\langle X \rangle,\tau)$ the conjugate variable of $X$ if

$$\langle (\partial P)(X), \tau \otimes \tau \rangle = \langle P(X), \xi \rangle$$

for all $P \in C\langle x \rangle$.

The conjugate variable $\xi$ is automatically unique if it exists; it will be denoted by $J(X:C)$.

The free Fisher information is $\Phi^*(X:C) := \|J(X:C)\|_2^2$. 

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A glimpse at free Fisher information

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A glimpse at free Fisher information

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Tobias Mai (Saarland University)  Beyond the Dyson equation  December 11, 2019  5 / 6
A glimpse at free Fisher information

Consider the algebra $\mathcal{B}\langle x \rangle$ of polynomials in the formal variable $x$ and $\mathcal{B}$.

**Definition**

The noncommutative derivative $\partial : \mathcal{B}\langle x \rangle \rightarrow \mathcal{B}\langle x \rangle \otimes \mathcal{B}\langle x \rangle$ is the unique derivation satisfying $\partial x = 1 \otimes 1$ and $\partial b = 0$ for every $b \in \mathcal{B}$.

Let $(\mathcal{N}, \tau)$ be a tracial $W^*$-probability space and $1 \in \mathcal{B} \subseteq \mathcal{N}$, $\mathcal{L} : \mathcal{B} \rightarrow \mathcal{B}$.

**Definition (Voiculescu (1998), Shlyakhtenko (2000))**

Let $X = X^* \in \mathcal{N}$ be given.

- We call $\xi \in L^2(\mathcal{B}\langle X \rangle, \tau)$ the conjugate variable of $X$ if

  $$\langle (\partial P)(X), 1 \otimes 1 \rangle_{\tau, \mathcal{L}} = \langle P(X), \xi \rangle_{\tau} \text{ for all } P \in \mathcal{B}\langle x \rangle.$$ 

- The conjugate variable $\xi$ is automatically unique if it exists; it will be denoted by $\mathcal{J}(X : \mathcal{B}, \mathcal{L})$.

- The free Fisher information is $\Phi^*(X : \mathcal{B}, \mathcal{L}) := \| \mathcal{J}(X : \mathcal{B}, \mathcal{L}) \|_2^2$. 

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Beyond the Dyson equation  
December 11, 2019
Back to the matricial case ...

$X_1, \ldots, X_n$ selfadjoint elements in $(M, \tau)$

$b_1, \ldots, b_n$ selfadjoint matrices in $M_k(\mathbb{C})$

$L : M_k(\mathbb{C}) \to M_k(\mathbb{C})$

$X := b_1 \otimes X_1 + \cdots + b_n \otimes X_n$

**Theorem (M., Yin (2020))**

Suppose that $\Phi^*(X_1, \ldots, X_n) < \infty$ and that $L$ is semi-at. Then the cumulative distribution function $F_{X}$, i.e., $F_X(t) := \mu_X((-\infty, t])$, of the spectral measure $\mu_X$ of $X$ is $2$-Hölder continuous.

**Theorem (M., Yin (2020))**

Suppose that $(X_1, \ldots, X_n)$ admits a dual system and that $L$ is nowhere rank-decreasing. Then the spectral measure $\mu_X$ of $X$ is absolutely continuous with respect to the Lebesgue measure.
Back to the matricial case ...

- $X_1, \ldots, X_n$ selfadjoint elements in $(\mathcal{M}, \tau)$
- $b_1, \ldots, b_n$ selfadjoint matrices in $M_k(\mathbb{C})$, $\mathcal{L} : M_k(\mathbb{C}) \to M_k(\mathbb{C})$
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Suppose that $\Phi^*(X_1, \ldots, X_n) < \infty$ and that $\mathcal{L}$ is semi-flat. Then the cumulative distribution function $\mathcal{F}_X$, i.e., $\mathcal{F}_X(t) := \mu_X((-\infty, t])$, of the spectral measure $\mu_X$ of $X$ is $\frac{2}{3}$-Hölder continuous.
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*Banna, M., 2019*
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**[Banna, M., 2019]**

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[![Banna, M., 2019](image)](image)

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Suppose that $(X_1, \ldots, X_n)$ admits a dual system and that $\mathcal{L}$ is nowhere rank-decreasing. Then the spectral measure $\mu_X$ of $X$ is absolutely continuous with respect to the Lebesgue measure.

Thank you!