Noncommutative functions and regularity properties of spectral distributions

Tobias Mai

(joint work with M. Banna, R. Speicher, and S. Yin)

Saarland University

Workshop on Applications to Random Matrices and Free Probability of Free Noncommutative Functions

Fields Institute Toronto

June 17, 2019

Supported by the ERC Advanced Grant "Non-commutative distributions in free probability"

Noncommutative probability spaces

Noncommutative probability spaces

Definition

A noncommutative probability space $(\mathcal{A}, \phi)$ consists of

- a complex algebra $\mathcal{A}$ with unit $1_{\mathcal{A}}$ and
- a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ with $\phi(1_{\mathcal{A}}) = 1$ (expectation).

Elements $X \in \mathcal{A}$ are called noncommutative random variables.

Noncommutative probability spaces

Definition

A noncommutative probability space $(\mathcal{A}, \phi)$ consists of

- a complex algebra $\mathcal{A}$ with unit $1_\mathcal{A}$ and
- a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ with $\phi(1_\mathcal{A}) = 1$ (expectation).

Elements $X \in \mathcal{A}$ are called noncommutative random variables.

Definition

A noncommutative probability space $(\mathcal{A}, \phi)$ is called

- C^*-probability space if
  - $\mathcal{A}$ is a unital C^*-algebra and
  - ϕ is a state on $\mathcal{A}$.
- tracial W^*-probability space, if
  - $\mathcal{A}$ is a von Neumann algebra and
  - ϕ is a faithful normal tracial state on $\mathcal{A}$.

Noncommutative distributions

Noncommutative distributions

Definition ("combinatorial distribution")

Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. For any given family $X = (X_i)_{i \in I}$ of noncommutative random variables, we call

$$\mu_X : \mathbb{C}\langle x_i \mid i \in I \rangle \to \mathbb{C}, \quad x_{i_1} \cdots x_{i_k} \mapsto \phi(X_{i_1} \cdots X_{i_k})$$

the (joint) noncommutative distribution of X.

Noncommutative distributions

Definition ("combinatorial distribution")

Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. For any given family $X = (X_i)_{i \in I}$ of noncommutative random variables, we call

$$\mu_X : \mathbb{C}\langle x_i \mid i \in I \rangle \to \mathbb{C}, \quad x_{i_1} \cdots x_{i_k} \mapsto \phi(X_{i_1} \cdots X_{i_k})$$

the (joint) noncommutative distribution of X.

Definition ("analytic distribution")

Let $(\mathcal{A}, \phi)$ be a C^*-probability space. For any given $X = X^* \in \mathcal{A}$, the noncommutative distribution of X can be identified with the unique Borel probability measure μ_X on the real line $\mathbb{R}$ that satisfies

$$\phi(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all integers } k \geq 0.$$

Noncommutative distribution $\mu_{X_1,\dots,X_n}$

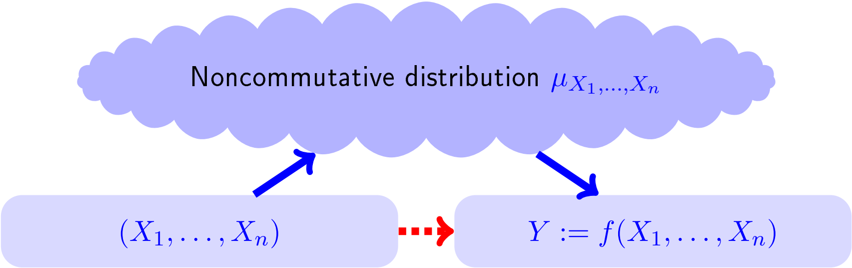
Noncommutative distribution $\mu_{X_1,\dots,X_n}$

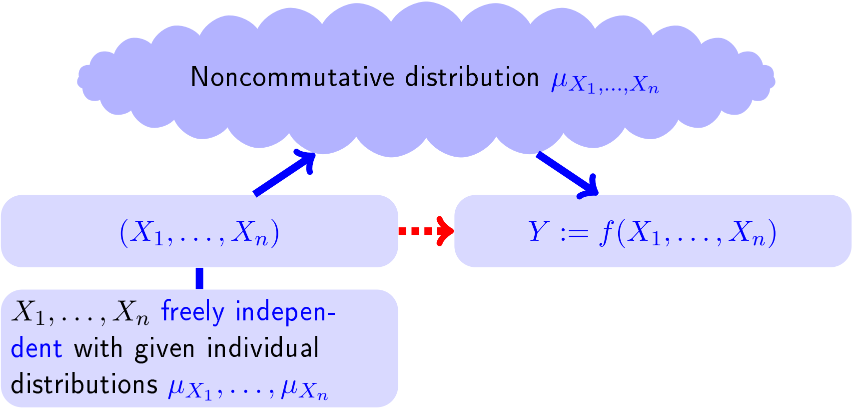
$(X_1, \dots, X_n)$

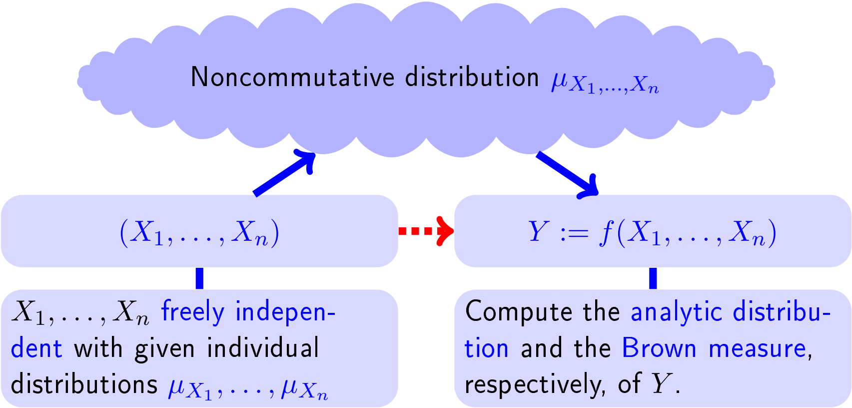
Noncommutative distribution $\mu_{X_1,\dots,X_n}$

$(X_1, \dots, X_n)$

$Y := f(X_1, \dots, X_n)$


Noncommutative distribution $\mu_{X_1,\dots,X_n}$
$(X_1, \dots, X_n)$
$Y := f(X_1, \dots, X_n)$


Noncommutative distribution $\mu_{X_1,\dots,X_n}$
$(X_1, \dots, X_n)$
$Y := f(X_1, \dots, X_n)$
$X_1, \dots, X_n$ freely independent with given individual distributions $\mu_{X_1}, \dots, \mu_{X_n}$


Noncommutative distribution $\mu_{X_1,\dots,X_n}$
$(X_1, \dots, X_n)$
$Y := f(X_1, \dots, X_n)$
$X_1, \dots, X_n$ freely independent with given individual distributions $\mu_{X_1}, \dots, \mu_{X_n}$
Compute the analytic distribution and the Brown measure, respectively, of Y.

Noncommutative distribution $\mu_{X_1,\dots,X_n}$

$(X_1,\dots,X_n)$

$X_1,\dots,X_n$ freely independent with given individual distributions $\mu_{X_1},\dots,\mu_{X_n}$

$Y := f(X_1,\dots,X_n)$

Compute the analytic distribution and the Brown measure, respectively, of Y.

Study regularity properties like

- absolute continuity of μ_Y
- Hölder continuity of $\mathcal{F}_{\mu_Y}$
- absence of atoms for μ_Y

Noncommutative distribution $\mu_{X_1,\dots,X_n}$

$(X_1, \dots, X_n)$

$Y := f(X_1, \dots, X_n)$

$X_1, \dots, X_n$ freely independent with given individual distributions $\mu_{X_1}, \dots, \mu_{X_n}$

Compute the analytic distribution and the Brown measure, respectively, of Y.

Study regularity properties like

- absolute continuity of μ_Y
- Hölder continuity of $\mathcal{F}_{\mu_Y}$
- absence of atoms for μ_Y

[Belinschi-M.-Speicher, '17], [Belinschi-Sniady-Speicher, '18], [Helton-M.-Speicher, '18]

Noncommutative distribution $\mu_{X_1,\dots,X_n}$

$(X_1,\dots,X_n)$

$X_1,\dots,X_n$ freely independent with given individual distributions $\mu_{X_1},\dots,\mu_{X_n}$

$Y := f(X_1,\dots,X_n)$

Compute the analytic distribution and the Brown measure, respectively, of Y.

Study regularity properties like

- absolute continuity of μ_Y
- Hölder continuity of $\mathcal{F}_{\mu_Y}$
- absence of atoms for μ_Y

[Shlyakhtenko-Skoufranis, '15], [Ajanki-Erdös-Krüger, '16], [Alt-Erdös-Krüger, '18]

Noncommutative distribution $\mu_{X_1,\dots,X_n}$

$(X_1, \dots, X_n)$ → $Y := f(X_1, \dots, X_n)$

$X_1, \dots, X_n$ freely independent with given individual distributions $\mu_{X_1}, \dots, \mu_{X_n}$

Regularity conditions such as

- $\Phi^*(X_1, \dots, X_n) < \infty$
- $\chi^*(X_1, \dots, X_n) > -\infty$
- $\delta^*(X_1, \dots, X_n) = n$

Compute the analytic distribution and the Brown measure, respectively, of Y.

Study regularity properties like

- absolute continuity of μ_Y
- Hölder continuity of $\mathcal{F}_{\mu_Y}$
- absence of atoms for μ_Y

Noncommutative distribution $\mu_{X_1,\dots,X_n}$

$(X_1, \dots, X_n)$

$Y := f(X_1, \dots, X_n)$

$X_1, \dots, X_n$ freely independent with given individual distributions $\mu_{X_1}, \dots, \mu_{X_n}$

Compute the analytic distribution and the Brown measure, respectively, of Y.

Regularity conditions such as

- $\Phi^*(X_1, \dots, X_n) < \infty$
- $\chi^*(X_1, \dots, X_n) > -\infty$
- $\delta^*(X_1, \dots, X_n) = n$

Study regularity properties like

- absolute continuity of μ_Y
- Hölder continuity of $\mathcal{F}_{\mu_Y}$
- absence of atoms for μ_Y

[Charlesworth-Shlyakhtenko, '16], [M.-Speicher-Weber, '17], [M.-Speicher-Yin, '18], [Banna-M., '18]

Basic classes of "noncommutative test functions"

Basic classes of "noncommutative test functions"

1. Noncommutative polynomials, i.e., expressions of the form

$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$

in formal non-commuting indeterminates $x_1, \dots, x_n$; we denote the unital complex algebra consisting of all noncommutative polynomials by $\mathbb{C}\langle x_1, \dots, x_n \rangle$.

Basic classes of "noncommutative test functions"

1. Noncommutative polynomials, i.e., expressions of the form
$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$
in formal non-commuting indeterminates $x_1, \dots, x_n$; we denote the unital complex algebra consisting of all noncommutative polynomials by $\mathbb{C}\langle x_1, \dots, x_n\rangle$.
2. Matrices of noncommutative polynomials, i.e., elements $\mathbf{P}$ in $M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle)$ for an arbitrary $N \in \mathbb{N}$.

Basic classes of "noncommutative test functions"

1. Noncommutative polynomials, i.e., expressions of the form

$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$

   in formal non-commuting indeterminates $x_1, \dots, x_n$; we denote the unital complex algebra consisting of all noncommutative polynomials by $\mathbb{C}\langle x_1, \dots, x_n \rangle$.

2. Matrices of noncommutative polynomials, i.e., elements $\mathbf{P}$ in $M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ for an arbitrary $N \in \mathbb{N}$.

3. Affine linear pencils, i.e., matrices of noncommutative polynomials that are of the particular form

$$\mathbf{P} = b_0 + b_1 x_1 + \cdots + b_n x_n$$

   with scalar matrices $b_0, b_1, \dots, b_n$ of appropriate size.

A more advanced class: noncommutative rational functions

A more advanced class: noncommutative rational functions

- By definition, noncommutative rational functions are elements of the free field $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$.

A more advanced class: noncommutative rational functions

- By definition, noncommutative rational functions are elements of the free field $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$.
- Formally, $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ is the universal field of fractions for the ring $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ of noncommutative polynomials.

A more advanced class: noncommutative rational functions

- By definition, noncommutative rational functions are elements of the free field $\mathbb{C}(\!\langle x_1, \ldots, x_n \rangle\!)$.
- Formally, $\mathbb{C}(\!\langle x_1, \ldots, x_n \rangle\!)$ is the universal field of fractions for the ring $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ of noncommutative polynomials.
- Its existence is a highly non-trivial fact: Amitsur, Bergman, Cohn, ...

A more advanced class: noncommutative rational functions

- By definition, noncommutative rational functions are elements of the free field $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$.
- Formally, $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ is the universal field of fractions for the ring $\mathbb{C}\langle x_1, \dots, x_n \rangle$ of noncommutative polynomials.
- Its existence is a highly non-trivial fact: Amitsur, Bergman, Cohn, ...

But what does this actually mean?

A more advanced class: noncommutative rational functions

- By definition, noncommutative rational functions are elements of the free field $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$.
- Formally, $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ is the universal field of fractions for the ring $\mathbb{C}\langle x_1, \dots, x_n \rangle$ of noncommutative polynomials.
- Its existence is a highly non-trivial fact: Amitsur, Bergman, Cohn, ...

But what does this actually mean?

☞ Loosely speaking, noncommutative rational functions are built out of noncommutative polynomials by successive applications of the arithmetic operations addition, multiplication, and inversion.

A more advanced class: noncommutative rational functions

- By definition, noncommutative rational functions are elements of the free field $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$.
- Formally, $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ is the universal field of fractions for the ring $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ of noncommutative polynomials.
- Its existence is a highly non-trivial fact: Amitsur, Bergman, Cohn, ...

But what does this actually mean?

☞ Loosely speaking, noncommutative rational functions are built out of noncommutative polynomials by successive applications of the arithmetic operations addition, multiplication, and inversion.

☞ They can be realized as *equivalence classes* of noncommutative rational expressions which are non-degenerate.

From rational functions to affine linear pencils

From rational functions to affine linear pencils

Definition

Let $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ be given.

- The (inner) rank of $\mathbf{Q}$, denoted by $\rho(\mathbf{Q})$, is the least integer $k \geq 1$ for which $\mathbf{Q}$ can be written as $\mathbf{Q} = \mathbf{R}_1 \mathbf{R}_2$ with some rectangular matrices

$$\mathbf{R}_1 \in M_{N \times k}(\mathbb{C}\langle x_1, \dots, x_n \rangle) \quad \text{and} \quad \mathbf{R}_2 \in M_{k \times N}(\mathbb{C}\langle x_1, \dots, x_n \rangle).$$

From rational functions to affine linear pencils

Definition

Let $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ be given.

- The (inner) rank of $\mathbf{Q}$, denoted by $\rho(\mathbf{Q})$, is the least integer $k \geq 1$ for which $\mathbf{Q}$ can be written as $\mathbf{Q} = \mathbf{R}_1 \mathbf{R}_2$ with some rectangular matrices

$$\mathbf{R}_1 \in M_{N \times k}(\mathbb{C}\langle x_1, \dots, x_n \rangle) \quad \text{and} \quad \mathbf{R}_2 \in M_{k \times N}(\mathbb{C}\langle x_1, \dots, x_n \rangle).$$

- We call $\mathbf{Q}$ full if it has full rank, i.e., if $\rho(\mathbf{Q}) = N$.

From rational functions to affine linear pencils

Definition

Let $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle)$ be given.

- The (inner) rank of $\mathbf{Q}$, denoted by $\rho(\mathbf{Q})$, is the least integer $k \geq 1$ for which $\mathbf{Q}$ can be written as $\mathbf{Q} = \mathbf{R}_1\mathbf{R}_2$ with some rectangular matrices

$$\mathbf{R}_1 \in M_{N\times k}(\mathbb{C}\langle x_1, \dots, x_n\rangle) \quad \text{and} \quad \mathbf{R}_2 \in M_{k\times N}(\mathbb{C}\langle x_1, \dots, x_n\rangle).$$

- We call $\mathbf{Q}$ full if it has full rank, i.e., if $\rho(\mathbf{Q}) = N$.

Facts

- $\mathbf{Q}$ full $\iff$ $\mathbf{Q}$ invertible in $M_N(\mathbb{C}\langle\!\!\langle x_1, \dots, x_n\rangle\!\!\rangle)$

From rational functions to affine linear pencils

Definition

Let $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ be given.

- The (inner) rank of $\mathbf{Q}$, denoted by $\rho(\mathbf{Q})$, is the least integer $k \geq 1$ for which $\mathbf{Q}$ can be written as $\mathbf{Q} = \mathbf{R}_1 \mathbf{R}_2$ with some rectangular matrices

$$\mathbf{R}_1 \in M_{N \times k}(\mathbb{C}\langle x_1, \dots, x_n \rangle) \quad \text{and} \quad \mathbf{R}_2 \in M_{k \times N}(\mathbb{C}\langle x_1, \dots, x_n \rangle).$$

- We call $\mathbf{Q}$ full if it has full rank, i.e., if $\rho(\mathbf{Q}) = N$.

Facts

- $\mathbf{Q}$ full $\iff$ $\mathbf{Q}$ invertible in $M_N(\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle)$
- Every noncommutative rational function $r \in \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ admits a linear representation, i.e., it can be written as $r = u\mathbf{Q}^{-1}v$ with a full affine linear pencil $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ and scalar vectors u and v of appropriate size.

How can we “evaluate” such functions?

How can we "evaluate" such functions?

Trivial facts

Let $\mathcal{A}$ be a unital algebra and consider $X_1, \ldots, X_n \in \mathcal{A}$. There is a unital homomorphism

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathcal{A}$$

that is uniquely determined by the condition that $\mathrm{ev}_X(x_i) = X_i$ for each $i = 1, \ldots, n$. The latter extends naturally to a unital homomorphism

$$\mathrm{ev}_X^{(N)} : M_N(\mathbb{C}\langle x_1, \ldots, x_n \rangle) \to M_N(\mathcal{A}), \ (P_{kl})_{k,l=1}^N \mapsto (\mathrm{ev}_X(P_{kl}))_{k,l=1}^N.$$

How can we "evaluate" such functions?

Trivial facts

Let $\mathcal{A}$ be a unital algebra and consider $X_1, \dots, X_n \in \mathcal{A}$. There is a unital homomorphism

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathcal{A}$$

that is uniquely determined by the condition that $\mathrm{ev}_X(x_i) = X_i$ for each $i = 1, \dots, n$. The latter extends naturally to a unital homomorphism

$$\mathrm{ev}_X^{(N)} : M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle) \to M_N(\mathcal{A}), \ (P_{kl})_{k,l=1}^N \mapsto (\mathrm{ev}_X(P_{kl}))_{k,l=1}^N.$$

This defines evaluation of (matrix-valued) noncommutative polynomials and of affine linear pencils, in particular. Rational functions are more subtle.

How can we “evaluate” such functions?

Trivial facts

Let $\mathcal{A}$ be a unital algebra and consider $X_1, \dots, X_n \in \mathcal{A}$. There is a unital homomorphism

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathcal{A}$$

that is uniquely determined by the condition that $\mathrm{ev}_X(x_i) = X_i$ for each $i = 1, \dots, n$. The latter extends naturally to a unital homomorphism

$$\mathrm{ev}_X^{(N)} : M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle) \to M_N(\mathcal{A}), \ (P_{kl})_{k,l=1}^N \mapsto (\mathrm{ev}_X(P_{kl}))_{k,l=1}^N.$$

This defines evaluation of (matrix-valued) noncommutative polynomials and of affine linear pencils, in particular. Rational functions are more subtle.

Challenging facts

- Not every rational expression can be evaluated everywhere.

How can we "evaluate" such functions?

Trivial facts

Let $\mathcal{A}$ be a unital algebra and consider $X_1, \dots, X_n \in \mathcal{A}$. There is a unital homomorphism

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathcal{A}$$

that is uniquely determined by the condition that $\mathrm{ev}_X(x_i) = X_i$ for each $i = 1, \dots, n$. The latter extends naturally to a unital homomorphism

$$\mathrm{ev}_X^{(N)} : M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle) \to M_N(\mathcal{A}), \ (P_{kl})_{k,l=1}^N \mapsto (\mathrm{ev}_X(P_{kl}))_{k,l=1}^N.$$

This defines evaluation of (matrix-valued) noncommutative polynomials and of affine linear pencils, in particular. Rational functions are more subtle.

Challenging facts

- Not every rational expression can be evaluated everywhere.
- Two rational expressions representing the same rational function do not necessarily give the same value under evaluation.

Noncommutative derivatives

Noncommutative derivatives

Consider again the ($*$-)algebra $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ of noncommutative polynomials in formal (selfadjoint) variables $x_1, \ldots, x_n$.

Noncommutative derivatives

Consider again the ($*$-)algebra $\mathbb{C}\langle x_1,\dots,x_n\rangle$ of noncommutative polynomials in formal (selfadjoint) variables $x_1,\dots,x_n$.

Definition

The noncommutative derivatives are the linear mappings

$$\partial_1,\dots,\partial_n:\ \mathbb{C}\langle x_1,\dots,x_n\rangle \to \mathbb{C}\langle x_1,\dots,x_n\rangle \otimes \mathbb{C}\langle x_1,\dots,x_n\rangle$$

which are uniquely determined by the two conditions

- $\partial_j(P_1P_2) = (\partial_j P_1)\cdot P_2 + P_1 \cdot (\partial_j P_2)$ for all $P_1, P_2 \in \mathbb{C}\langle x_1,\dots,x_n\rangle$,
- $\partial_j x_i = \delta_{i,j} 1 \otimes 1$ for $i,j = 1,\dots,n$.

Noncommutative derivatives

Consider again the ($*$-)algebra $\mathbb{C}\langle x_1, \dots, x_n \rangle$ of noncommutative polynomials in formal (selfadjoint) variables $x_1, \dots, x_n$.

Definition

The noncommutative derivatives are the linear mappings

$$\partial_1, \dots, \partial_n : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathbb{C}\langle x_1, \dots, x_n \rangle \otimes \mathbb{C}\langle x_1, \dots, x_n \rangle$$

which are uniquely determined by the two conditions

- $\partial_j(P_1 P_2) = (\partial_j P_1) \cdot P_2 + P_1 \cdot (\partial_j P_2)$ for all $P_1, P_2 \in \mathbb{C}\langle x_1, \dots, x_n \rangle$,
- $\partial_j x_i = \delta_{i,j} 1 \otimes 1$ for $i, j = 1, \dots, n$.

$\mathbb{C}\langle x_1, \dots, x_n \rangle \otimes \mathbb{C}\langle x_1, \dots, x_n \rangle$ becomes a $\mathbb{C}\langle x_1, \dots, x_n \rangle$-bimodule via

$$P_1 \cdot (Q_1 \otimes Q_2) \cdot P_2 := (P_1 Q_1) \otimes (Q_2 P_2).$$

Conjugate variables and free Fisher information

Conjugate variables and free Fisher information

Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and consider any selfadjoint operators $X_1, \dots, X_n \in \mathcal{M}$; we put $\mathcal{M}_0 := \mathrm{vN}(X_1, \dots, X_n)$.

Conjugate variables and free Fisher information

Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and consider any selfadjoint operators $X_1, \dots, X_n \in \mathcal{M}$; we put $\mathcal{M}_0 := \mathrm{vN}(X_1, \dots, X_n)$.

Definition (Voiculescu (1998))

If $\xi_1, \dots, \xi_n \in L^2(\mathcal{M}_0, \tau)$ are such that for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$

$$(\tau \otimes \tau)((\partial_j P)(X_1, \dots, X_n)) = \tau(\xi_j P(X_1, \dots, X_n)), \quad j = 1, \dots, n,$$

then $(\xi_1, \dots, \xi_n)$ is called the conjugate system for $(X_1, \dots, X_n)$.

Conjugate variables and free Fisher information

Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and consider any selfadjoint operators $X_1, \dots, X_n \in \mathcal{M}$; we put $\mathcal{M}_0 := \mathrm{vN}(X_1, \dots, X_n)$.

Definition (Voiculescu (1998))

If $\xi_1, \dots, \xi_n \in L^2(\mathcal{M}_0, \tau)$ are such that for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$

$$(\tau \otimes \tau)((\partial_j P)(X_1, \dots, X_n)) = \tau(\xi_j P(X_1, \dots, X_n)), \quad j = 1, \dots, n,$$

then $(\xi_1, \dots, \xi_n)$ is called the conjugate system for $(X_1, \dots, X_n)$.

Conjugate variables and free Fisher information

Let $(\mathcal{M},\tau)$ be a tracial W^*-probability space and consider any selfadjoint operators $X_1,\dots,X_n\in\mathcal{M}$; we put $\mathcal{M}_0 := \mathrm{vN}(X_1,\dots,X_n)$.

Definition (Voiculescu (1998))

If $\xi_1,\dots,\xi_n \in L^2(\mathcal{M}_0,\tau)$ are such that for all $P\in\mathbb{C}\langle x_1,\dots,x_n\rangle$

$$(\tau\otimes\tau)((\partial_j P)(X_1,\dots,X_n)) = \tau(\xi_j P(X_1,\dots,X_n)),\quad j=1,\dots,n,$$

then $(\xi_1,\dots,\xi_n)$ is called the conjugate system for $(X_1,\dots,X_n)$.

Definition (Voiculescu (1998))

The (non-microstates) free Fisher information is defined by

$$\Phi^*(X_1,\dots,X_n) := \begin{cases} \sum_{j=1}^n \|\xi_j\|_2^2, & \text{if a conjugate system } (\xi_1,\dots,\xi_n) \text{ for } (X_1,\dots,X_n) \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

Interlude: Gibbs laws and the Schwinger-Dyson equation

Interlude: Gibbs laws and the Schwinger-Dyson equation

Theorem (Guionnet, Shlyakhtenko (2009))

Let $V \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ following the Gibbs law

$$d\Lambda_N^V(X_1^{(N)}, \dots, X_n^{(N)}) = \frac{1}{Z_N^V} e^{-N \operatorname{Tr}(V(X_1^{(N)}, \dots, X_n^{(N)}))} \, dX_1^{(N)} \dots dX_n^{(N)}.$$

Then, for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$,

$$\lim_{N \to \infty} \operatorname{tr}_N(P(X_1^{(N)}, \dots, X_n^{(N)})) = \tau(P(X_1, \dots, X_n)) \quad \text{almost surely}$$

for selfadjoint operators $X_1, \dots, X_n$ in some W^*-probability space $(\mathcal{M}, \tau)$ that satisfy the Schwinger-Dyson equation with potential V.

Interlude: Gibbs laws and the Schwinger-Dyson equation

Theorem (Guionnet, Shlyakhtenko (2009))

Let $V \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ following the Gibbs law

$$d\Lambda_N^V(X_1^{(N)}, \dots, X_n^{(N)}) = \frac{1}{Z_N^V} e^{-N \operatorname{Tr}(V(X_1^{(N)}, \dots, X_n^{(N)}))} \, dX_1^{(N)} \dots dX_n^{(N)}.$$

Then, for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$,

$$\lim_{N \to \infty} \operatorname{tr}_N(P(X_1^{(N)}, \dots, X_n^{(N)})) = \tau(P(X_1, \dots, X_n)) \quad \text{almost surely}$$

for selfadjoint operators $X_1, \dots, X_n$ in some W^*-probability space $(\mathcal{M}, \tau)$ that satisfy the Schwinger-Dyson equation with potential V.

Observation

For such $(X_1, \dots, X_n)$, we always have that $\Phi^*(X_1, \dots, X_n) < \infty$.

Useful variants of the free entropy dimension

Useful variants of the free entropy dimension

Suppose that $S_1, \dots, S_n$ are freely independent semicircular elements that are also free from $\{X_1, \dots, X_n\}$, then $(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n)$ admits a conjugate system for each $t > 0$. More precisely, we have

$$\frac{n^2}{C^2 + nt} \leq \Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n) \leq \frac{n}{t} \quad \text{for all } t > 0,$$

with $C^2 := \tau(X_1^2 + \cdots + X_n^2)$.

Useful variants of the free entropy dimension

Suppose that $S_1, \dots, S_n$ are freely independent semicircular elements that are also free from $\{X_1, \dots, X_n\}$, then $(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n)$ admits a conjugate system for each $t > 0$. More precisely, we have

$$\frac{n^2}{C^2 + nt} \leq \Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n) \leq \frac{n}{t} \quad \text{for all } t > 0,$$

with $C^2 := \tau(X_1^2 + \cdots + X_n^2)$.

Definition (Connes, Shlyakhtenko (2005))

- $\delta^\star(X) := n - \liminf_{t \searrow 0} t\Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n)$

Useful variants of the free entropy dimension

Suppose that $S_1, \dots, S_n$ are freely independent semicircular elements that are also free from $\{X_1, \dots, X_n\}$, then $(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n)$ admits a conjugate system for each $t > 0$. More precisely, we have

$$\frac{n^2}{C^2 + nt} \leq \Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n) \leq \frac{n}{t} \quad \text{for all } t > 0,$$

with $C^2 := \tau(X_1^2 + \dots + X_n^2)$.

Definition (Connes, Shlyakhtenko (2005))

- $\delta^\star(X) := n - \liminf_{t \searrow 0} t\Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n)$
- $\Delta(X) := n - \dim_{\mathcal{M}\overline{\otimes}\mathcal{M}^{\text{op}}} \overline{\left\{T \in \mathcal{F}(L^2(\mathcal{M}))^n \,\middle|\, \sum_{j=1}^{n} [T_j, JX_j^* J] = 0\right\}}$

Useful variants of the free entropy dimension

Suppose that $S_1, \dots, S_n$ are freely independent semicircular elements that are also free from $\{X_1, \dots, X_n\}$, then $(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n)$ admits a conjugate system for each $t > 0$. More precisely, we have

$$\frac{n^2}{C^2 + nt} \leq \Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n) \leq \frac{n}{t} \quad \text{for all } t > 0,$$

with $C^2 := \tau(X_1^2 + \cdots + X_n^2)$.

Definition (Connes, Shlyakhtenko (2005))

- $\delta^\star(X) := n - \liminf_{t \searrow 0} t\Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n)$
- $\Delta(X) := n - \dim_{\mathcal{M}\overline{\otimes}\mathcal{M}^{\text{op}}} \overline{\left\{T \in \mathcal{F}(L^2(\mathcal{M}))^n \;\middle|\; \sum_{j=1}^{n} [T_j, JX_j^* J] = 0\right\}}$

Theorem (Connes, Shlyakhtenko (2005))

$$0 \leq \delta^*(X_1, \dots, X_n) \leq \Delta(X_1, \dots, X_n) \leq n$$

Atoms of (matrices of) polynomials and rational functions

Atoms of (matrices of) polynomials and rational functions

Theorem (M., Speicher, Yin (2019))

The condition $\Delta(X_1, \dots, X_n) = n$ is satisfied if and only if

> $(X_1, \dots, X_n)$ *induces a "faithful model" of the free field inside the $*$-algebra $\mathcal{A}$ of all unbounded linear operators affiliated to $\mathcal{M}$.*

Atoms of (matrices of) polynomials and rational functions

Theorem (M., Speicher, Yin (2019))

The condition $\Delta(X_1, \dots, X_n) = n$ is satisfied if and only if

> $(X_1, \dots, X_n)$ *induces a "faithful model" of the free field inside the $*$-algebra $\mathcal{A}$ of all unbounded linear operators affiliated to $\mathcal{M}$.*

In particular, if $\Delta(X_1, \dots, X_n) = n$, then the following holds:

1. $X = (X_1, \dots, X_n)$ has the strong Atiyah property.

Atoms of (matrices of) polynomials and rational functions

Theorem (M., Speicher, Yin (2019))

The condition $\Delta(X_1,\dots,X_n) = n$ is satisfied if and only if

$(X_1,\dots,X_n)$ induces a "faithful model" of the free field inside the $*$-algebra $\mathcal{A}$ of all unbounded linear operators affiliated to $\mathcal{M}$.

In particular, if $\Delta(X_1,\dots,X_n) = n$, then the following holds:

1. $X = (X_1,\dots,X_n)$ has the strong Atiyah property.
2. For every selfadjoint $\mathbf{P} \in M_N(\mathbb{C}\langle x_1,\dots,x_n\rangle)$, the operator $\mathbf{Y} := \mathbf{P}(X_1,\dots,X_n)$ has atoms precisely at the points in the set

$$\{\lambda \in \mathbb{C} \mid \mathbf{P} - \lambda 1_N \in M_N(\mathbb{C}\langle x_1,\dots,x_n\rangle) \text{ is not full}\}$$

with size $\mu_{\mathbf{Y}}(\{\lambda\}) = 1 - \frac{1}{N}\rho(\mathbf{P} - \lambda 1_N)$.

Atoms of (matrices of) polynomials and rational functions

Theorem (M., Speicher, Yin (2019))

The condition $\Delta(X_1, \dots, X_n) = n$ is satisfied if and only if

> *$(X_1, \dots, X_n)$ induces a "faithful model" of the free field inside the $*$-algebra $\mathcal{A}$ of all unbounded linear operators affiliated to $\mathcal{M}$.*

In particular, if $\Delta(X_1, \dots, X_n) = n$, then the following holds:

1. $X = (X_1, \dots, X_n)$ has the strong Atiyah property.
2. For every selfadjoint $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle)$, the operator $\mathbf{Y} := \mathbf{P}(X_1, \dots, X_n)$ has atoms precisely at the points in the set
$$\left\{\lambda \in \mathbb{C} \mid \mathbf{P} - \lambda 1_N \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle) \text{ is not full}\right\}$$
with size $\mu_{\mathbf{Y}}(\{\lambda\}) = 1 - \frac{1}{N}\rho(\mathbf{P} - \lambda 1_N)$.
3. Every $r \in \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ admits a well-defined evaluation $r(X) \in \mathcal{A}$. If r is non-constant and selfadjoint, then the analytic distribution of $r(X)$ has no atoms.

Hölder continuity: a criterion

Hölder continuity: a criterion

Consider $Y = Y^*$ in $(\mathcal{M}, \tau)$. Let μ_Y be the analytic distribution of Y and let $\mathcal{F}_Y$ be its cumulative distribution function, i.e., $\mathcal{F}_Y(t) := \mu_Y((-\infty, t])$.

Hölder continuity: a criterion

Consider $Y = Y^*$ in $(\mathcal{M}, \tau)$. Let μ_Y be the analytic distribution of Y and let $\mathcal{F}_Y$ be its cumulative distribution function, i.e., $\mathcal{F}_Y(t) := \mu_Y((-\infty, t])$.

Lemma (M., Speicher, Yin (2018))

If there exist $c > 0$ and $\alpha > 1$ such that

$$c\|(Y - s)p\|_2 \geq \|p\|_2^\alpha$$

for all $s \in \mathbb{R}$ and each spectral projection p of Y, then $\mathcal{F}_Y$ is Hölder continuous with exponent $\beta := \frac{2}{\alpha - 1}$; more precisely, we have that

$$|\mathcal{F}_Y(t) - \mathcal{F}_Y(s)| \leq c^\beta |t - s|^\beta \qquad \text{for all } s, t \in \mathbb{R}.$$

Hölder continuity: a criterion

Consider $Y = Y^*$ in $(\mathcal{M}, \tau)$. Let μ_Y be the analytic distribution of Y and let $\mathcal{F}_Y$ be its cumulative distribution function, i.e., $\mathcal{F}_Y(t) := \mu_Y((-\infty, t])$.

Lemma (M., Speicher, Yin (2018))

If there exist $c > 0$ and $\alpha > 1$ such that

$$c\|(Y-s)p\|_2 \geq \|p\|_2^\alpha$$

for all $s \in \mathbb{R}$ and each spectral projection p of Y, then $\mathcal{F}_Y$ is Hölder continuous with exponent $\beta := \frac{2}{\alpha-1}$; more precisely, we have that

$$|\mathcal{F}_Y(t) - \mathcal{F}_Y(s)| \leq c^\beta |t-s|^\beta \qquad \text{for all } s,t \in \mathbb{R}.$$

Proof – following ideas of [Charlesworth, Shlyakhtenko (2016)].

Take $p = E_Y((s,t])$ for the spectral measure E_Y of Y and observe that

$$\|p\|_2 = \mu_Y((s,t])^{1/2} \qquad \text{and} \qquad \|(Y-s)p\|_2 \leq |t-s|\mu_Y((s,t])^{1/2}.$$

□

Hölder continuity of polynomials

Hölder continuity of polynomials

Suppose that $\Phi^*(X_1, \ldots, X_n) < \infty$.

Hölder continuity of polynomials

Suppose that $\Phi^*(X_1, \dots, X_n) < \infty$.

Theorem (Banna, M. (2018))

Let $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be selfadjoint with degree $d \geq 1$ and consider

$$Y := P(X_1, \dots, X_n).$$

Then there exists some constant $C > 0$ such that

$$|\mathcal{F}_Y(t) - \mathcal{F}_Y(s)| \leq C|t-s|^{\frac{2}{3(2^d-1)}} \qquad \text{for all } s, t \in \mathbb{R}.$$

Hölder continuity of polynomials

Suppose that $\Phi^*(X_1, \ldots, X_n) < \infty$.

Theorem (Banna, M. (2018))

Let $P \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ be selfadjoint with degree $d \geq 1$ and consider

$$Y := P(X_1, \ldots, X_n).$$

Then there exists some constant $C > 0$ such that

$$|\mathcal{F}_Y(t) - \mathcal{F}_Y(s)| \leq C|t-s|^{\frac{2}{3(2^d-1)}} \qquad \text{for all } s, t \in \mathbb{R}.$$

In fact, for every $R > \max_{i=1,\ldots,n} \|X_i\|$, we can take

$$C = \big(8\Phi^*(X)^{1/2} R\big)^{\frac{2}{3}} \rho_R(P)^{-\frac{2^d}{3(2^d-1)}} \|P\|_R^{-\frac{2}{3(2^d-1)}} \prod_{k=1}^{d-1} \Big(\frac{d!}{(d-k)!}\Big)^{\frac{2^k}{3(2^d-1)}},$$

where $\|P\|_R$ and $\rho_R(P)$ are quantities that depend only on P and R.

Hölder continuity and finiteness of free entropy

Hölder continuity and finiteness of free entropy

Suppose that $\Phi^*(X_1, \ldots, X_n) < \infty$.

Hölder continuity and finiteness of free entropy

Suppose that $\Phi^*(X_1, \dots, X_n) < \infty$.

Corollary (Banna, M. (2018))

Let $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be selfadjoint with degree $d \geq 1$; consider

$$Y := P(X_1, \dots, X_n).$$

Then, the logarithmic energy (and thus also the free entropy $\chi^*(Y)$)

$$I(\mu_Y) := \int_{\mathbb{R}} \int_{\mathbb{R}} \log \frac{1}{|s-t|} \, d\mu_Y(s) \, d\mu_Y(t)$$

is finite; in fact, there is an explicit bound in terms of the input data.

Hölder continuity and finiteness of free entropy

Suppose that $\Phi^*(X_1,\dots,X_n) < \infty$.

Corollary (Banna, M. (2018))

Let $P \in \mathbb{C}\langle x_1,\dots,x_n\rangle$ be selfadjoint with degree $d \geq 1$; consider

$$Y := P(X_1,\dots,X_n).$$

Then, the logarithmic energy (and thus also the free entropy $\chi^*(Y)$)

$$I(\mu_Y) := \int_{\mathbb{R}}\int_{\mathbb{R}} \log\frac{1}{|s-t|}\,d\mu_Y(s)\,d\mu_Y(t)$$

is finite; in fact, there is an explicit bound in terms of the input data.

Remark

This is a first step towards a conjecture of Charlesworth and Shlyakhtenko (2016) saying that this should remain valid under the weaker condition

$$\chi^*(X_1,\dots,X_n) > -\infty.$$

Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$

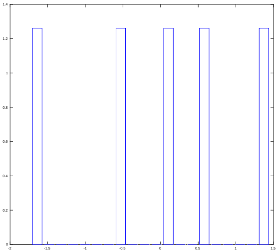
Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$



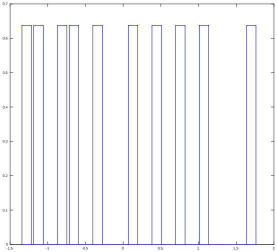
Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$



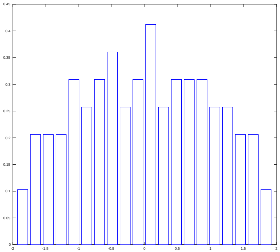
Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$



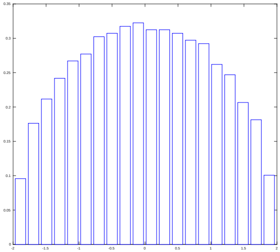
Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$



Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

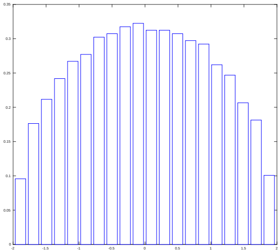
The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$

Definition

The mean eigenvalue distribution of X is the probability measure $\overline{\mu}_X$ on $\mathbb{C}$ that is given by

$$\overline{\mu}_X := \mathbb{E}[\mu_X].$$



Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

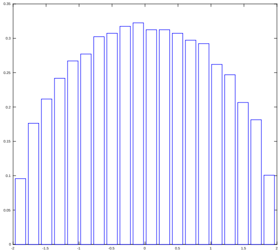
The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

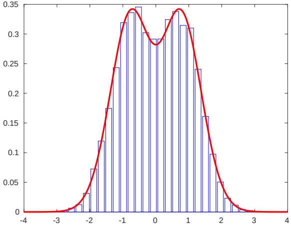
$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$

Definition

The mean eigenvalue distribution of X is the probability measure $\overline{\mu}_X$ on $\mathbb{C}$ that is given by

$$\overline{\mu}_X := \mathbb{E}[\mu_X].$$





Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

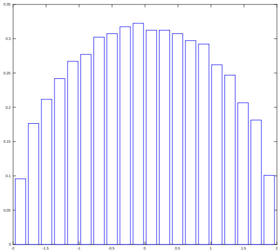
The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

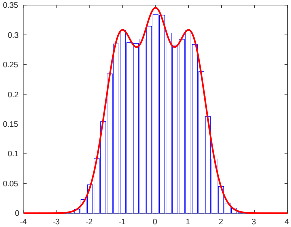
$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$

Definition

The mean eigenvalue distribution of X is the probability measure $\overline{\mu}_X$ on $\mathbb{C}$ that is given by

$$\overline{\mu}_X := \mathbb{E}[\mu_X].$$





Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

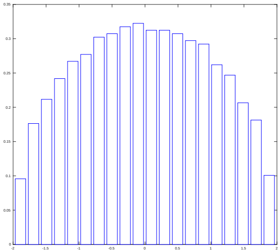
The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

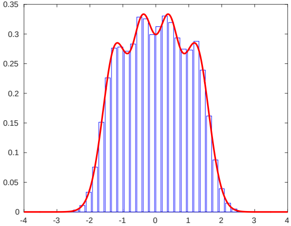
$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$

Definition

The mean eigenvalue distribution of X is the probability measure $\overline{\mu}_X$ on $\mathbb{C}$ that is given by

$$\overline{\mu}_X := \mathbb{E}[\mu_X].$$





Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Definition

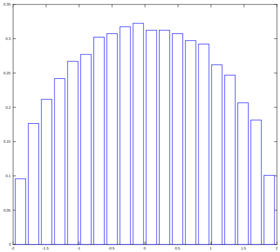
The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

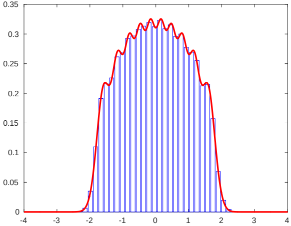
$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$

Definition

The mean eigenvalue distribution of X is the probability measure $\overline{\mu}_X$ on $\mathbb{C}$ that is given by

$$\overline{\mu}_X := \mathbb{E}[\mu_X].$$





Eigenvalue distributions

Consider a random matrix X of size $N \times N$.

Gaussian random matrices

$\rightsquigarrow$ Wigner's semicircle theorem

Definition

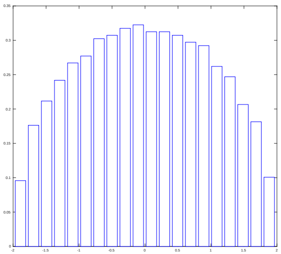
The empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

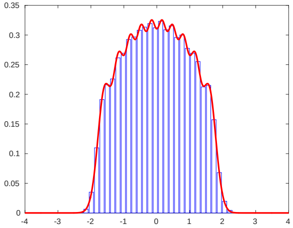
$$\omega \mapsto \mu_{X(\omega)} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)}.$$

Definition

The mean eigenvalue distribution of X is the probability measure $\overline{\mu}_X$ on $\mathbb{C}$ that is given by

$$\overline{\mu}_X := \mathbb{E}[\mu_X].$$





Gaussian block random matrices

Gaussian block random matrices

Corollary (M., Speicher, Yin (2018); Banna, M. (2018))

Let $b_0, b_1, \ldots, b_n \in M_d(\mathbb{C})$ be selfadjoint such that the quantum operator

$$\mathcal{L}: \ M_d(\mathbb{C}) \to M_d(\mathbb{C}), \quad b \mapsto b_1 b b_1 + \cdots + b_n b b_n$$

satisfies $\mathcal{L}(b) \geq c \operatorname{tr}_d(b) 1_d$ for all positive $b \in M_d(\mathbb{C})$ for some $c > 0$. Put

$$\mathbf{X}^{(N)} := b_0 \otimes 1_N + b_1 \otimes X_1^{(N)} + \cdots + b_n \otimes X_n^{(N)}$$

for independent standard Gaussian random matrices $(X_1^{(N)}, \ldots, X_n^{(N)})$ and

$$\mathbf{S} := b_0 \otimes 1 + b_1 \otimes S_1 + \cdots + b_n \otimes S_n$$

for freely independent semicircular elements $S_1, \ldots, S_n$.

Gaussian block random matrices

Corollary (M., Speicher, Yin (2018); Banna, M. (2018))

Let $b_0, b_1, \dots, b_n \in M_d(\mathbb{C})$ be selfadjoint such that the quantum operator

$$\mathcal{L}: \ M_d(\mathbb{C}) \to M_d(\mathbb{C}), \quad b \mapsto b_1 b b_1 + \cdots + b_n b b_n$$

satisfies $\mathcal{L}(b) \geq c \operatorname{tr}_d(b) 1_d$ for all positive $b \in M_d(\mathbb{C})$ for some $c > 0$. Put

$$\mathbf{X}^{(N)} := b_0 \otimes 1_N + b_1 \otimes X_1^{(N)} + \cdots + b_n \otimes X_n^{(N)}$$

for independent standard Gaussian random matrices $(X_1^{(N)}, \dots, X_n^{(N)})$ and

$$\mathbf{S} := b_0 \otimes 1 + b_1 \otimes S_1 + \cdots + b_n \otimes S_n$$

for freely independent semicircular elements $S_1, \dots, S_n$. Then we have:

(i) $\mathcal{F}_{\mathbf{S}}$ is Hölder continuous with exponent $\frac{2}{3}$.

Gaussian block random matrices

Corollary (M., Speicher, Yin (2018); Banna, M. (2018))

Let $b_0, b_1, \dots, b_n \in M_d(\mathbb{C})$ be selfadjoint such that the quantum operator

$$\mathcal{L}: \ M_d(\mathbb{C}) \to M_d(\mathbb{C}), \quad b \mapsto b_1 b b_1 + \cdots + b_n b b_n$$

satisfies $\mathcal{L}(b) \geq c \operatorname{tr}_d(b) 1_d$ for all positive $b \in M_d(\mathbb{C})$ for some $c > 0$. Put

$$\mathbf{X}^{(N)} := b_0 \otimes 1_N + b_1 \otimes X_1^{(N)} + \cdots + b_n \otimes X_n^{(N)}$$

for independent standard Gaussian random matrices $(X_1^{(N)}, \dots, X_n^{(N)})$ and

$$\mathbf{S} := b_0 \otimes 1 + b_1 \otimes S_1 + \cdots + b_n \otimes S_n$$

for freely independent semicircular elements $S_1, \dots, S_n$. Then we have:

(i) $\mathcal{F}_{\mathbf{S}}$ is Hölder continuous with exponent $\frac{2}{3}$.

☞ [Ajanki-Erdös-Krüger, '16], [Alt-Erdös-Krüger, '18]

Gaussian block random matrices

Corollary (M., Speicher, Yin (2018); Banna, M. (2018))

Let $b_0, b_1, \dots, b_n \in M_d(\mathbb{C})$ be selfadjoint such that the quantum operator

$$\mathcal{L}: \ M_d(\mathbb{C}) \to M_d(\mathbb{C}), \quad b \mapsto b_1 b b_1 + \cdots + b_n b b_n$$

satisfies $\mathcal{L}(b) \geq c \operatorname{tr}_d(b) 1_d$ for all positive $b \in M_d(\mathbb{C})$ for some $c > 0$. Put

$$\mathbf{X}^{(N)} := b_0 \otimes 1_N + b_1 \otimes X_1^{(N)} + \cdots + b_n \otimes X_n^{(N)}$$

for independent standard Gaussian random matrices $(X_1^{(N)}, \dots, X_n^{(N)})$ and

$$\mathbf{S} := b_0 \otimes 1 + b_1 \otimes S_1 + \cdots + b_n \otimes S_n$$

for freely independent semicircular elements $S_1, \dots, S_n$. Then we have:

(i) $\mathcal{F}_{\mathbf{S}}$ is Hölder continuous with exponent $\frac{2}{3}$.

(ii) There is some $C > 0$ such that, with respect to the Kolmogorov distance d_{Kol}, we have that $d_{\mathrm{Kol}}(\overline{\mu}_{\mathbf{X}^{(N)}}, \mu_{\mathbf{S}}) \leq C N^{-2/11}$.

☞ [Ajanki-Erdös-Krüger, '16], [Alt-Erdös-Krüger, '18]

Polynomial evaluations for Gibbs laws

Polynomial evaluations for Gibbs laws

Corollary (Banna, M. (2018))

Let $V \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ be "nice" and let $(X_1^{(N)}, \ldots, X_n^{(N)})$ be random matrices of size $N \times N$ distributed according to the Gibbs law

$$d\Lambda_N^V(X_1^{(N)}, \ldots, X_n^{(N)}) = \frac{1}{Z_N^V} e^{-N \operatorname{Tr}(V(X_1^{(N)}, \ldots, X_n^{(N)}))} \, dX_1^{(N)} \ldots dX_n^{(N)}.$$

Polynomial evaluations for Gibbs laws

Corollary (Banna, M. (2018))

Let $V \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ distributed according to the Gibbs law

$$d\Lambda_N^V(X_1^{(N)}, \dots, X_n^{(N)}) = \frac{1}{Z_N^V} e^{-N \operatorname{Tr}(V(X_1^{(N)}, \dots, X_n^{(N)}))} \, dX_1^{(N)} \dots dX_n^{(N)}.$$

Then, for each non-constant selfadjoint $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$, we have that:

(i) The empirical eigenvalue distribution $\mu_{Y^{(N)}}$ of

$$Y^{(N)} := P(X_1^{(N)}, \dots, X_n^{(N)})$$

converges in distribution almost surely to a compactly supported Borel probability measure μ on $\mathbb{R}$ with a cumulative distribution function that is Hölder continuous with exponent $\frac{2}{3(2^d - 1)}$.

Polynomial evaluations for Gibbs laws

Corollary (Banna, M. (2018))

Let $V \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ distributed according to the Gibbs law

$$d\Lambda_N^V(X_1^{(N)}, \dots, X_n^{(N)}) = \frac{1}{Z_N^V} e^{-N\,\mathrm{Tr}(V(X_1^{(N)}, \dots, X_n^{(N)}))}\, dX_1^{(N)} \dots dX_n^{(N)}.$$

Then, for each non-constant selfadjoint $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$, we have that:

(i) The empirical eigenvalue distribution $\mu_{Y^{(N)}}$ of

$$Y^{(N)} := P(X_1^{(N)}, \dots, X_n^{(N)})$$

converges in distribution almost surely to a compactly supported Borel probability measure μ on $\mathbb{R}$ with a cumulative distribution function that is Hölder continuous with exponent $\frac{2}{3(2^d-1)}$.

(ii) We have that

$$d_{\mathrm{Kol}}(\overline{\mu}_{Y^{(N)}}, \mu) \to 0 \qquad \text{as } N \to \infty.$$

Polynomial evaluations for GUEs

Polynomial evaluations for GUEs

Corollary (Banna, M. (2018))

Let $(X_1^{(N)}, \dots, X_n^{(N)})$ be independent Gaussian random matrices of size $N \times N$.

Polynomial evaluations for GUEs

Corollary (Banna, M. (2018))

Let $(X_1^{(N)}, \dots, X_n^{(N)})$ be independent Gaussian random matrices of size $N \times N$. For each non-constant selfadjoint $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$, we have:

(i) The empirical eigenvalue distribution $\mu_{Y^{(N)}}$ of

$$Y^{(N)} := P(X_1^{(N)}, \dots, X_n^{(N)})$$

converges in distribution almost surely to a compactly supported Borel probability measure μ on $\mathbb{R}$ with a cumulative distribution function that is Hölder continuous with exponent $\frac{1}{2^d - 1}$.

Polynomial evaluations for GUEs

Corollary (Banna, M. (2018))

Let $(X_1^{(N)}, \dots, X_n^{(N)})$ be independent Gaussian random matrices of size $N \times N$. For each non-constant selfadjoint $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$, we have:

(i) The empirical eigenvalue distribution $\mu_{Y^{(N)}}$ of

$$Y^{(N)} := P(X_1^{(N)}, \dots, X_n^{(N)})$$

converges in distribution almost surely to a compactly supported Borel probability measure μ on $\mathbb{R}$ with a cumulative distribution function that is Hölder continuous with exponent $\frac{1}{2^d - 1}$.

(ii) There is a constant $C > 0$ such that

$$d_{\mathrm{Kol}}(\overline{\mu}_{Y^{(N)}}, \mu) \leq C N^{-\frac{1}{13 \cdot 2^{d+2} - 60}} \qquad \text{for all } N \in \mathbb{N}.$$

Polynomial evaluations for GUEs

Corollary (Banna, M. (2018))

Let $(X_1^{(N)}, \dots, X_n^{(N)})$ be independent Gaussian random matrices of size $N \times N$. For each non-constant selfadjoint $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$, we have:

(i) The empirical eigenvalue distribution $\mu_{Y^{(N)}}$ of

$$Y^{(N)} := P(X_1^{(N)}, \dots, X_n^{(N)})$$

converges in distribution almost surely to a compactly supported Borel probability measure μ on $\mathbb{R}$ with a cumulative distribution function that is Hölder continuous with exponent $\frac{1}{2^d - 1}$.

(ii) There is a constant $C > 0$ such that

$$d_{\mathrm{Kol}}(\overline{\mu}_{Y^{(N)}}, \mu) \le C N^{-\frac{1}{13 \cdot 2^{d+2} - 60}} \qquad \text{for all } N \in \mathbb{N}.$$

Thank you!