CATEGORY THEORY SEMINAR 2020

- □ Participants at *all* levels of familiarity with category theory and at *all* stages of their mathematical education in general are cordially invited to participate. The talks are meant to cover a wide spectrum of difficulty.
- □ The below list of possible talks is to be regarded as preliminary and very much open to extension or alteration in response to input from the participants.
- □ In particular, by no means need all topics be covered in the seminar.
- □ Conversely, on certain topics having several talks might be feasible.
- □ Please let us know if there is a topic not listed below on which you would like to give (or attend) a talk!
- □ Also, **email us anytime if you would like more information** about the proposed topics.
- □ References given below for individual talks are merely suggestions and only meant to help you find a starting point for your own engagement with the literature.

Part 1. Basic topics

Among the basic topics each talk can be considered a prerequisite to all the ones following it.

1. Objects and morphisms

"Report on the various different kinds of special objects and morphisms and equivalence classes of such which are commonly distinguished in category theory!"

Important fundamental concepts which feature in many higher-level definitions of categoriy theory are the various types of special objects, special morphisms and special equivalence classes of morphisms usually distinguished. In keeping with the principle of duality they all appear in pairs. In some cases a special name is given to entities possessing both properties. E.g., a *terminal object* by definition admits precisely one morphism from any object. The dual notion is that of an *initial object.* If an object is both, it is called a *zero object.* Other important kinds of objects are *separators* and *co-separators.* Among morphisms one distinguishes several kinds of analogues and generalizations of injective, surjective and bijective maps. E.g., an *epimorphism* is a morphism which is right-cancellative and a *retraction* is an epimorphisms which even has a right-inverse. The dual notions are *monomorphism* respectively *section.* Combining both notions yields so-called *bimorphisms* respectively *isomorphisms.* Among monomorphisms and epimorphisms one frequently distinguishes between *regular*, *strong* and *extremal* ones. *Subobjects* and *co-subojects* (sometimes also called *quotient objects*) are certain kinds of equivalence classes of monomorphisms and epimorphisms, respectively, which generalize the notion of subsets and quotient sets to categories other than that of sets. ■□□□□

[AHS09, Chpt. II]

2. Universal property

"Give a precise formal account of what a universal construction is, why it works and how it is important and exhibit examples, both of different kinds of universal constructions and also, for each kind of construction, of particular instances!"

Many central definitions of category theory are *universal constructions*. A main motivation for using these is the desire to be able to specify a special object not by explaining "how it can be found" but by "what it is like" (the so-called *universal property*): Rather than giving an explicit way of constructing the object, one prescribes how morphisms to or from it compose with other morphisms. The *Yoneda lemma* gives a justification for why this works. And it also explains why a universal construction determines an object not uniquely but only *essentially uniquely*, i.e., up to unique isomorphism. Another way of looking at universal constructions is to recognize that they are equivalent to demands that something be an initial or terminal object in a *comma category*. ■■□□□

[Bra16, Kap. 5]

3. Limits, co-limits, sources, sinks, pull-backs, push-outs

"Present the concept of (co-)limits using the language of sources and sinks and illustrate it by means of the example of pull-backs and push-outs!"

In category theory, it is often helpful to consider families of morphisms with common co-domain (so-called *sinks*) or domain (*co-sinks* or *sources*). Particular kinds of co-sinks are *cones* of *diagrams*; particular kinds of sinks are *co-cones* of diagrams. Universal *(co-)cones* are called *(co-)limits* and form a fundamental concept of category theory. Examples of limits are *pull-backs*, examples of co-limits *push-outs*. They are the "most general" ways of completing a two-member sink respectively co-sink to a commutative square. Further examples of (co-)limits include *(co-)products*, *(co-)equalizers* and *(co-)terminal objects*. Particularly favorable categories with respect to the existence of (co-)limits are *(co-)complete* or at least *finitely (co-)complete* ones. Certain (co-)limits can be constructed from others. Functors may or may not *preserve*, *lift*, *reflect* or *create* limits; but for certain functors and certain limits, some general results are available. ■□□□□

[AHS09, Chpt. III], [Bae16, Chpt. 2], [Bra16, Kap. 6]

4. Adjoints

"Give the three equivalent definitions of adjoint functors, present the important uniqueness, existence and limit-preservation results and provide examples of adjunctions!"

Adjunctions or pairs of *adjoint functors* can be seen as relationships between two categories which generalize that of *equivalence*. (The name "adjoint" comes from a formal similarity with Hilbert space operator adjoints.) Of the many examples, the relationship between sets and groups given by the possibility of forming the *free group* over a given set respectively passing to the set of elements of a given group is just one. Another is the possibility of *adjoining a unit* element to a given ring or algebra respectively forgetting that a unit exists. Three definitions of adjunctions can be shown to be equivalent, one via natural isomorphisms of compositions with hom-functors, one via universal morphisms and one via so-called *unit* and *co-unit* natural transformations. Adjoints are *unique*. *Adjoint* *functor theorems* give conditions for when they exist. *Left adjoint* functors preserve co-limits, *right adjoints* preserve limits. ■■□□□

[AHS09, Chpt. V], [Bra16, Kap. 7]

5. Monoidal categories

"Give an introduction to the theory of (strict and non-strict) monoidal categories via the example of the Temperley-Lieb category!"

Some categories can be equipped with a "multiplicative" structure. In the category of sets for example both the operation of forming the Cartesian product of two sets and the one of taking their disjoint union yield such "multiplications". The notion of *monoidal product* in a *monoidal category* formalizes this idea. The general definition of a monoidal category has a simplified special case: *strict* monoidal categories. The *Temperley-Lieb* category of *non-crossing pair partitions* is an example of the latter. (It plays a role in the representation theory of *quantum groups* and much can be said about it. E.g., it is a *universal category* and its *fibre functors* can be classified.) From it one can construct a non-strict monoidal category by taking the 3-*co-cycle twist*. ■■□□□

[Bae16, Chpt. 8], [BC18]

6. Enriched categories

"Present the fundamental concepts and results of enriched category theory!"

A locally small category is one where for any two objects the class of morphisms between them is not just a class but a set. The theory of enriched categories generalizes this by demanding that the morphism class be not a set but an object of some monoidal category. Many definitions and theorems of ordinary category have analogues in enriched category theory. In particular, there are besides *enriched categories* also *enriched functors* and *enriched natural transformations* and one can prove an *enriched Yoneda lemma*. ■□□□□

[Rie14, Chpt. 3]

7. Abelian categories

"Show how suitable categories can be given a notion of *addition* of morphisms and explain what consequences that entails!"

Locally small categories with a *zero-object*, with binary *products* and *co-products*, with *kernels* and *co-kernels* and in which all *epimorphisms* and *monomorphisms* are *normal* can be endowed canonically with the structure of an enriched category over the monoidal category of abelian groups (with a choice of tensor product of abelian groups as monoidal product). Such categories are called *abelian*. They have various special properties like essentially unique *epi-mono factorizations*, existence of *zero morphisms*, of *images* and *co-images*, well-behaved *subobjects* and *co-subobjects* and many more. They are of great importance in homological algebra. By the Freyd-Mitchell embedding theorem each abelian category embeds into a category of modules over a ring. ■■□□□

[Lan13, Chpt. VIII]

Part 2. Special topics

8. Deligne categories

"Show how it is possible to use partitions of finite sets to define a virtual group of 'permutations of a *non-integer* number of points'! "

Deligne categories are tensor categories of a combinatorial type. Their morphisms are given by formal linear combinations of "partitions", certain kinds of diagrams one can draw by hand. Two partitions are composed by a natural graphical procedure applied to the diagrams. More precisely, for each complex number there is a specific procedure giving a categorical structure. For non-negative integer values N of this parameter the resulting tensor category is equivalent to the representation category of the symmetric group S_N. If one defines S_t for other parameters t as the fictitious "group" whose representation theory is given by the corresponding Deligne category, one has *generalized the notion of a permutation group*. More generally, as Knop showed, such a construction can be carried out in a many other ways where the morphisms are not given by partitions but by *relations* inside any *regular category*. ■■■□□

Knowledge of topics 3 and 5 is probably helpful.

[CO14], [Eti14], [Eti16], [Kno07]

9. Generalizations of regular and exact categories

"Illustrate when and how it is possible to have in a category other than that of sets an analogous theory of (not necessarily binary) *relations*!"

The notions of *regular categories* and *exact categories* with their *calculus of relations* can be understood and generalized in a framework called *familial regularity and exactness* developed by Ross Street and Michael Shulman. The suggested objective of the talk is to explain what is meant by that, why it is interesting and, of course, as time allows, how it works. ■■■■□

To a large extent topic no. 3 has to be considered a prerequisite for this one. And it will be most satisfying if one also knows a bit about topic no. 8.

[Shu12], [Str84]

10. Tensor categories and reconstruction

"Give a rough definition and overview of the field of tensor categories under special consideration of its link with the study of Hopf algebras!"

Tensor categories are certain abelian *rigid* monoidal categories which are in addition enriched over the category of finite-dimensional vector spaces over some field and which are subject to certain "simplicity" assumptions; in particular the set of morphisms between two objects is a finite-dimensional vector space and the monoidal product is a bi-linear tensor product. If such a tensor category is paired with a *fiber functor*, i.e., a certain kind of linear additive monoidal functor into the category of finite-dimensional vector spaces, this gives rise to *Hopf algebras*. The converse viewpoint that this Hopf algebra can be recovered from the tensor category motivates the name *reconstruction theory* for this link between the theories of tensor categories and Hopf algebras. ■■■■■

Familiarity with topics no. 5–7 is required, familiarity with topics no. 11 and 12 helpful.

[Eti+16, Chpt. 4 and 5]

11. C*-tensor-categories

"Provide an introduction to the theory of C*-tensor categories and prove how it is possible to reconstruct a compact quantum group from its C*-tensor of representations and how, conversely, each C*-tensor category gives rise to a compact quantum group!"

C^*-categories are topological variants of algebraic *tensor categories.* The morphism spaces are not given by vector spaces and algebras of linear maps but by C^*-modules and C^*-algebras. An important consequence of the $*$-structure is the existence of *complex conjugates* under certain circumstances. The analogue of the reconstruction theory for Hopf-algebras for algebraic tensor categories is of great importance in the theory of C^*-algebraic quantum groups: By a famous theorem of Woronowicz's certain finite-dimensional C^*-categories give rise to *compact quantum groups*, particular types of Hopf-$*$-algebras. ■■■■■

Familiarity with topics no. 5–7 is required, familiarity with topics no. 10 and 12 helpful.
[NT13, Chpt. 3]

12. Tannakian duality exemplified

"Explain roughly how for certain notions of symmetry one can *reconstruct* the abstract embodiment of that symmetry (e.g., a group or quantum group) if one only knows in a certain way sufficiently many systems exhibiting that symmetry (e.g., the group or quantum group representations)."

"Tannaka(-Krein)-type" reconstruction theorems can be given abstractly as applications of the *Yoneda lemma* of enriched category theory to the module categories of a monoid object. However, in important concrete cases peculiar *improvements* thereupon are possible, to the effect that not all modules are necessary. This can be illustrated on various concrete examples. For instance, co-algebras can be reconstructed from the endomorphisms of the functor forgetting the co-action of co-algebra co-modules; but, as shown by André Joyal and Ross Street, the same is possible if one restricts to finite-dimensional co-modules. ■■■■□

Familiarity with topics no. 5–7 is required, familiarity with topics no. 10 and 11 helpful.
[Day96], [DM82], [JS91]

13. Symmetries and braidings in monoidal categories

"Explain how in monoidal categories there is more than one way for the product to be 'commutative', present concepts to 'quantify' how 'symmetric' a given monoidal category is and clarify the relationship this question has to braid theory!"

In general monoidal categories the product of two objects taken in one order need *not* have any relation with the one taken in the other order. However, for many monoidal categories the product does display some *symmetry* in that respect. Of particular interest are *braided monoidal categories* where there exists a natural isomorphism between these two functors, the *braiding*, introduced by Joyal and Street. Depending on what properties this natural isomorphism possesses, it makes sense to say that the monoidal product displays different kinds of "commutativity". The most symmetric case is that of *symmetric braided categories* where the braiding is an involution. More generally, a braided monoidal category can be interpreted as a category with an action of the *braid category*, which encompasses all *braid groups*. ■■□□□

Topic no. 5 is a prerequisite.

[Lan13, Chpt. XI]

14. Stone duality: concrete and abstract

"Prove the classical version of Stone's representation theorem and interpret it in the terms of category theory."

By a famous result of Marshall H. Stone's, totally disconnected compact Hausdorff spaces, so-called *Stone spaces*, are *dual* to *Boolean algebras*, i.e., the first category is categorically equivalent to the opposite category of the second one. The duality between the two categories is realized by means of a *dualizing object*, meaning that the equivalence can be expressed as a restriction of a *representable functor*.

This topic is *not* about so-called "abstract Stone duality (ASD)" developed by Paul Taylor which seeks to give a reaxiomatization of general topology in constructive mathematics. ■■■■□

[Sto36]

15. Isbell duality: The paradigm and a rigorous instantiation

"Make the popular heuristic that geometry and algebra are two sides of the same coin precise in the framework of category theory!"

Following William T. Lawvere, (generalized) spaces can be formalized as *presheaves* and (generalized) algebras of functions over a space as *co-presheaves*. In the special case of enriched (co-)presheaves the left Kan extensions of the covariant and contravariant enriched Yoneda-embeddings along each other then define an enriched adjunction between spaces and function algebras, the *Isbell conjugacy*. More generally, the theory of the *Isbell envelope* allows reasoning about whether two categories satisfy such a notion of *Isbell duality*.

Knowledge of topic no. 4 has to be considered a prerequisite. Though not strictly necessary, understanding a bit of topic no. 16 will be helpful. ■■■■□

[Isb66]

16. All concepts are Kan extensions

"Explain how all other universal constructions, in particular limits and adjunctions, are subsumed by that of Kan extensions and clarify to what extent the reverse is true!"

The notion of *Kan extensions* allows understanding all other fundamental definitions of category theory, (co-)limits, adjunctions, (co-)ends, as special cases of just one universal construction: finding an optimal solution to the problem of extending a functor from a "subcategory" to the whole category. ■■□□□

Prerequisites are topics no. 3 and 4.

[Rie17, Chpt. 6]

17. Homological algebra

see topic no. 21.

18. The homotopy hypothesis: The idea and a proven example

"Provide a brief glimpse into higher category theory by explaining what is meant when it said that '∞-groupoids are equivalent to (weak homotopy classes of) topological spaces.'"

In contrast to ordinary, "first-order" category theory, there is no single commonly accepted approach to *higher category theory*. The central motivation to considering higher categories in the first place is Quillen's famous equivalence between the homotopy theories of topological spaces on the one hand and Kan complexes (of simplicial sets) on the other hand. This theorem suggests a certain notion of *higher groupoids* (in particular *∞-groupoids*) generalizing that of ordinary, "first-order" groupoids. But there are other homotopy theories equivalent to that of topological spaces which yield different notions of higher groupoids. Oversimplifying, higher category theory can be seen as the endeavour to develop a theory which behaves to higher groupoids as ordinary category theory behaves to ordinary groupoids, and a main issue is what notion of higher groupoid to choose as a starting point. For historical reasons this question is often framed as the so-called *homotopy hypothesis*. ■■■■■

[Bae07], [Qui06]

19. Examples from Wengenroth's book

"Demonstrate how methods and results from category theory enable proofs in other fields by showing how *inverse limits* and *derived functors* are used in functional analysis in the theory of locally convex vector spaces!"

Following the work of Palamodov, category-theoretical methods, in particular, homological ideas, have been employed in functional analysis. The Hahn-Banach theorem implies that the category of (not necesarily Hausdorff) locally convex (complex or real vector) spaces (with continuous linear maps) has all *inverse limits*, is *quasi-abelian* and has *enough injective objects*. The operation of taking the inverse limit constitutes a *left-exact functor* from the category of *co-directed diagrams* (of a certain shape) of locally convex spaces (called "projective spectra" in this context) to the category of locally convex spaces. Hence, its *right derivatives* can be defined and measure how far this functor is from being (also *right-*)*exact*. That makes it possible to ask whether, e.g., a global solution to a partial differential equation exists and can be found by aggregating local solutions. Further questions which can be addressed using category-theoretical instruments is whether an operator onto a quotient space of a locally convex space can be lifted to a map onto the whole space or whether the transpose linear map of an operator between locally convex spaces is again continuous with respect to the strong topology. ■■■■■

To a large extent, knowledge of topic no. 17/21 has to be regarded a requirement.

[Wen03]

20. From posets to categories

"Clarify the relationships between posets and categories and between concepts of poset theory like minimal/maximal elements, meets and joins on the one hand and categorical terms like limits and co-limits on the other hand!"

In a certain sense, category theory can be interpreted as a generalization of the theory of *partially ordered sets* (*posets*). For example, a poset is the same as a skeleton of a small category. Moreover, the notions of *meets* and *joins* correspond to certain *limits* and *co-limits*, respectively, namely *products* and *co-products*. ■□□□□

This topic requires knowledge of topics no. 1 and 3.

[Arm16, Chpt. 1]

21. Derived functors/categories (e.g., Tor, Ext, triangulated categories)

The topics no. 17, "Homological algebra", and no. 21, "Derived functors/categories (e.g., Tor, Ext, triangulated categories)", are for all intents and purposes identical. Having multiple talks on it seems very much possible. Up to four are outlined below.

Chain complexes. *Chain complexes*, usually of modules over a ring, are the prominent entities treated in *homological algebra*. Chain complexes form a category with *chain maps* as morphisms. Actually, they form a 2-*category* in the sense of higher order category theory with *chain homotopies* as second-order morphisms. The functor given by the operation of taking the *homology* of given chain complex is what gives the subject of homological algebra its name. It permits measuring to what degree a given chain complex fails to be *exact* (a property inspired by that of the same name for *short* or *long sequences*). ■■■□□

[Wei94, § 1.1–§ 1.5], [Bla11, § 11.1]

Left and right derived functors. A main interest of homological algebra is to study functors between categories of modules (or more generally abelian categories) via the homology of their compositions with other functors. One of the crucial tools for doing so is the concept of *derived functors*. The categories in question must have *projective* or *injective objects*, and *enough* of them at that, for this idea to work. The functor to be studied is then applied to *projective* or *injective resolutions*, chain complexes consisting of projective/injective objects, after which homology is taken. ■■■■□

[Wei94, § 2.1–§ 2.5], [Bla11, § 11.2–§ 11.3]

Tor and Ext. Of particular interest are the derived functors of the tensor product functor of an abelian category, the so-called *Tor*-functor, and of the hom-functor, the so-called *Ext*-functor. They have been used to study algebro-topological invariants of, e.g., groups or Lie algebras. ■■□□□

[Wei94, § 3.1–§ 3.4], [Bla11, § 11.4–§ 11.5]

Triangulated categories. The theory of derived functors can alternatively be expressed through that of *derived categories*, a special case of *triangulated categories* (or more precisely triangulated categories with chosen *t-structures*). Generalizing abelian categories, triangulated categories are merely additive categories but come equipped with a generalized notion of "exact sequences", namely *exact triangles*. Much of classical homological algebra can be carried out on triangulated categories as well. ■■■■■

[Wei94, § 10.1–§ 10.2]

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