Introduction to (co)-homology

References

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1 Rings and modules

1.1 Definition. Let R be a ring. A **(left)** $\boldsymbol{R}$**-module** is an additive abelian group M equipped with scalar multiplication $R \times M \to M$ such that for all $r, r' \in R$ and $m, m' \in M$ we have

1. $r(m + m') = rm + rm'$,
2. $(r + r')m = rm + r'm$,
3. $(rr')m = r(r'm)$,
4. $1m = m$.

1.2 Examples.

a) Every vector space over a field k is a k-module.
b) Every group is a $\mathbb{Z}$-module.
c) Every ring is a module over itself.
d) Every ideal in a ring R is an R-module
e) For A an algebra and $\varphi\colon A \to \mathscr{L}(V)$ a representation of A on a vector space V, we can define the structure of A-module on V as $av := \varphi(a)v$.

1.3 Definition. An R-module F is called **free** if it has a **basis**. That is, if there exists a set $\{e_\alpha\}$ (not necessarily finite), which is R-generating and R-linearly independent, i.e. any element $x \in F$ can be uniquely represented as $x = \sum r_\alpha e_\alpha$ for some coefficients $r_\alpha \in R$.

A free module with a basis of size n can be constructed as a direct sum of n copies of R. That is,

$$R^n := \{(r_1, \dots, r_n) \mid r_1, \dots, r_n \in R\}$$

with component-wise addition and multiplication. The basis is formed by the elements $e_i := (0, \dots, 0, 1, 0, \dots, 0)$ having the identity on the i-th position.

A **free group** is a free $\mathbb{Z}$-module.

1.4 Example. A canonical example of a module: Consider an algebra of polynomials $A := \mathbb{C}[x_1, \dots, x_n]$ as a ring and the vector space of n-vectors with components in $\mathbb{C}[x_1, \dots, x_n]$ as an A-module.

1.5 Example. Not every R-module has a basis. Consider for example $A := \mathbb{C}[x, y]$ as a ring and the ideal $\langle x, y\rangle$ generated by the polynomials x and y as an A-module. Then any generating set must contain both x and y (or any $\mathbb{C}$-linearly independent pair of polynomials of degree one), but those are not A-linearly independent since $y \cdot x + (-x) \cdot y = 0$.

1.6 Definition. Let M and N be R-modules. Then a map $f\colon M \to N$ is called an ***R*-homomorphism** (or an ***R*-map**) if it is R-linear in the sense that for all $r \in R$ and $m, m' \in M$

1. $f(m + m') = f(m) + f(m')$,
2. $f(rm) = rf(m)$.

The set of all R-homomorphisms between M and N will be denoted $\operatorname{Hom}(M, N)$.

1.7 Example.

a) If R is a field, then an R-module is an R-vector space and an R-map is an (R-)linear map.
b) Homomorphisms of abelian groups are $\mathbb{Z}$-maps.

1.8 Remark. Given two R-homomorphisms $f, g \in \operatorname{Hom}(M, N)$, the sum $f + g$ is again an R-homomorphism. This gives $\operatorname{Hom}(M, N)$ the structure of an abelian group. Note however that *it does not hold* that $rf\colon x \mapsto rf(x)$ is a homomorphism for $r \in R$ unless r commutes with everything in R (i.e. r is in the centre). Thus, we cannot define the structure of an R-module on $\operatorname{Hom}(M, N)$ unless R is commutative. We can only define the structure of $Z(R)$-module, where $Z(R)$ is the centre of R.

2 Functors

2.1 Notation. Let R be a ring. The class (category) of all R-modules will be denoted as R-**Mod**. We usually denote **Ab** $:= \mathbb{Z}$-**Mod** the category of all abelian groups.

2.2 Definition. Let R_1 and R_2 be rings. A **functor** $T\colon R_1$-**Mod** $\to R_2$-**Mod** is a collection of the following maps.

a) A map $T\colon R_1$-**Mod** $\to R_2$-**Mod** assigning to a R_1-module M an R_2-module $T(M)$.
b) For all pairs of R_1-modules M, M' a map $T\colon \operatorname{Hom}(M, M') \to \operatorname{Hom}(T(M), T(M'))$.

Those must satisfy:

1. For a triple of R_1-modules M, M', M'' and a pair of R_1-maps $f \in \operatorname{Hom}(M, M')$ and $g \in \operatorname{Hom}(M', M'')$ we have $T(gf) = T(g)T(f)$.
2. For any R_1-module M, the identity homomorphism $1_M\colon M \to M$ is mapped to the identity morphism $1_{T(M)}\colon T(M) \to T(M)$.

2.3 Remark. A functor can be seen as a morphism between sets of (R-)homomorphisms. Such a concept can be introduced in an abstract way within the theory of categories.

2.4 Definition. Let R be a ring and N an R-module. A **Hom-functor** is a functor $\operatorname{Hom}(N, -)\colon R$-**Mod** $\to$ **Ab** defined by the following maps.

a) Each R-module M is assigned the set $\operatorname{Hom}(N, M)$.
b) Consider a pair of R-modules M_1, M_2. Any R-map $f \in M_1 \to M_2$ is assigned an **induced map** $f_*\colon \operatorname{Hom}(N, M_1) \to \operatorname{Hom}(N, M_2)$ defined by $f_* h = f \circ h$ for any $h \in \operatorname{Hom}(N, M_1)$.

2.5 Remarks.

(i) The Hom-functor is indeed a functor R-**Mod** $\to \mathbb{Z}$-**Mod** since we have $1_* = 1$, $(gf)_* = g_* f_*$ and $f_*(h + h') = f_* h + f_* h'$ (so $f_*(\alpha h) = \alpha f_*(h)$ for $\alpha \in \mathbb{Z}$).

(ii) The Hom-functor can be also understood as a functor R-**Mod** $\to Z(R)$-**Mod**. If R is commutative, then it maps R-**Mod** $\to R$-**Mod**.

(iii) The Hom-functor is, in addition, an **additive functor** in the sense that $(f+g)_* = f_* + g_*$.

2.6 Definition. Let R_1 and R_2 be rings. A **contravariant functor** $T\colon R_1$-**Mod** $\to$ R_2-**Mod** is a collection of the following maps.

a) A map $T\colon R_1$-**Mod** $\to R_2$-**Mod** assigning to a R_1-module M an R_2-module $T(M)$.
b) For all pairs of R_1-modules M, M' a map $T\colon \mathrm{Hom}(M, M') \to \mathrm{Hom}(T(M'), T(M))$.

Those must satisfy:

1. For a triple of R_1-modules M, M', M'' and a pair of R_1-maps $f \in \mathrm{Hom}(M, M')$ and $g \in \mathrm{Hom}(M', M'')$ we have $T(gf) = T(f)T(g)$.
2. For any R_1-module M, the identity homomorphism $1_M\colon M \to M$ is mapped to the identity morphism $1_{T(M)}\colon T(M) \to T(M)$.

2.7 Definition. Let R be a ring and N an R-module. A **contravariant Hom-functor** is a functor $\mathrm{Hom}(-, N)\colon R$-**Mod** $\to$ **Ab** defined by the following maps.

a) Each R-module M is assigned the set $\mathrm{Hom}(M, N)$.
b) Consider a pair of R-modules M_1, M_2. Any R-map $f \in M_1 \to M_2$ is assigned an **induced map** $f^*\colon \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N)$ defined by $f^* h = h \circ f$ for any $h \in \mathrm{Hom}(M_2, N)$.

2.8 Example. To every vector space V over $\mathbb{C}$ (i.e. a $\mathbb{C}$-module), we can associate its dual vector space $V^* = \mathrm{Hom}(V, \mathbb{C})$. To every linear map $A\colon V \to W$ we can associate its adjoint $A^*\colon W^* \to V^*$ by $(A^*\varphi)(v) = \varphi(Av)$ for any $\varphi \in W^*$ and $v \in V$. This is exactly the action of the functor $\mathrm{Hom}(-, \mathbb{C})\colon \mathbb{C}$-**Mod** $\to \mathbb{C}$-**Mod**.

3 Tensor products

3.1 Definition. Let S be a ring. A **right *S*-module** is an additive abelian group M equipped with scalar multiplication $M \times S \to M$ such that for all $s, s' \in R$ and $m, m' \in M$ we have

1. $(m + m')s = ms + m's$,
2. $m(s + s') = ms + ms'$,
3. $m(ss') = (ms)s'$,
4. $m = m1$.

A left R-module that is also a right S-module such that $(rm)s = r(ms)$ for all $m \in M$, $r \in R$ and $s \in S$ is called an **(R, S)-bimodule**.

An additive abelian group M equipped with the structure of left R-module is often denoted ${}_R M$. If it is equipped with the structure of right S-module, we denote it M_S and if it is an (R, S)-bimodule, we denote it ${}_R M_S$.

3.2 Definition. Let S be a ring, A_S a right S-module and ${}_SB$ a left S-module. The **tensor product** of A_S and ${}_SB$ is the abelian group $A \otimes_S B$ defined by generators $a \otimes b$, $a \in A$, $b \in B$ subject to relations

$$\begin{aligned} a \otimes (b + b') &= a \otimes b + a \otimes b', \\ (a + a') \otimes b &= a \otimes b + a' \otimes b, \\ (ar) \otimes b &= a \otimes (rb). \end{aligned}$$

3.3 Proposition. Consider two right modules A_R and A'_R and two left modules ${}_RB$ and ${}_RB'$. For every pair of R-maps $f\colon A_R \to A'_R$ and $g\colon {}_RB \to {}_RB'$ there exists a unique $\mathbb{Z}$-map $f \otimes g\colon A \otimes_R B \to A' \otimes_R B'$ mapping $a \otimes b \mapsto f(a) \otimes f(b)$ for any $a \in A$ and $b \in B$.

Proof. See [Rotman, Section 2.2]. □

3.4 Theorem. Let R be a ring and A_R be a right R-module. Then there is a functor $F\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ given by

$$F(B) = A \otimes_R B \quad \text{and} \quad F(g) = 1_A \otimes g$$

for any left R-module B and any homomorphism $g \in \mathrm{Hom}(B_1, B_2)$.

Proof. By definition of homomorphism tensor product formulated in the previous proposition, $F(g)$ is indeed a $\mathbb{Z}$-map. It is easy to see that $1_A \otimes 1_B = 1_{A \otimes B}$. As an exercise, use the uniqueness to show that $(f \otimes g)(f' \otimes g') = (ff') \otimes (gg')$. □

3.5 Proposition. Let R and S be rings. Given an (R, S)-bimodule A and left S-module B, we can define the structure of left R-module on $A \otimes_S B$ as

$$r(a \otimes b) := (ra) \otimes b.$$

Proof. To prove that the left action of R on the tensor product is well and uniquely defined by the above formula, we again use Proposition 3.3. Given $r \in R$ let us define $f_r\colon A \to A$ the homomorphism of left multiplication $a \mapsto ra$. The left multiplication on $A \otimes_S B$ is defined as $rx := (f_r \otimes 1)x$ for any $x \in A \otimes_S B$.

Now, it is straightforward to check the axioms for a module. □

4 Short exact sequences

4.1 Definition. A finite or infinite sequence of R-maps and R-modules

$$\cdots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \cdots$$

is called **exact sequence** if $\operatorname{im} f_{n+1} = \ker f_n$ for all n.

4.2 Proposition.

(i) A sequence $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if f is injective.
(ii) A sequence $B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if g is surjective.
(iii) A sequence $0 \longrightarrow A \xrightarrow{h} B \longrightarrow 0$ is exact if and only if h is bijective.

Proof. Follows directly from the definition of exact modules. □

4.3 Definition. A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

4.4 Proposition. If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a short exact sequence, then

$$A \simeq \operatorname{im} f \quad \text{and} \quad C \simeq B/\operatorname{im} f.$$

Proof. The first part follows from the fact that an injective homomorphism maps isomorphically the domain to its image. The second part follows from the first isomorphism theorem for modules, which states that $\operatorname{im} g = B/\ker g$ for any homomorphism $g\colon B \to C$. □

4.5 Definition. A functor $T\colon R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ is called **left** resp. **right exact** if, for every exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C, \quad \text{resp.} \quad A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the sequence

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \quad \text{resp.} \quad T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \longrightarrow 0$$

is also exact.

It is called **exact** if it is both left and right exact. That is, if, for every exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

the sequence

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \longrightarrow 0$$

is also exact.

4.6 Theorem. Let R be a ring and M an R-module. Then the covariant Hom-functor $\operatorname{Hom}(M, -)\colon R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ is left-exact.

Proof. Consider an exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$, which means that f is injective and $\operatorname{im} f = \ker g$. We claim that it remains exact after applying the Hom-functor. That is, we have to prove that f_* is injective and $\operatorname{im} f_* = \ker g_*$.

Injectivity of f_*: Consider $h \in \ker f_*$, i.e. $h \in \operatorname{Hom}(M, A)$ such that $0 = f_* h = f \circ h$. That is, for every $x \in M$, we have $f(h(x)) = 0$. Since f is injective, it implies $h(x) = 0$, so $h = 0$.

$\operatorname{im} f_* \subset \ker g_*$: For any $h \in \operatorname{Hom}(M, A)$, we have to prove that $0 = g_* f_* h = g \circ f \circ h$. But we know that $g \circ f = 0$.

$\operatorname{im} f_* \supset \ker g_*$: Consider $h \in \ker g_*$, i.e. $g \circ h = 0$. We are looking for $\tilde{h} \in \operatorname{Hom}(M, A)$ such that $h = f_* \tilde{h}$. Since $g \circ h = 0$, we have that $\operatorname{im} h \subset \ker g = \operatorname{im} f$, so we can define take the composition $\tilde{h} := f^{-1} \circ h$. Since f is injective, f^{-1} is an isomorphism $\operatorname{im} f \to A$, so $\tilde{h}\colon M \to A$ is indeed a homomorphism. Obviously, we have $f_* \tilde{h} = f \circ f^{-1} \circ h = h$. □

4.7 Theorem. Let R be a ring and M an R-module. Then the contravariant Hom-functor $\operatorname{Hom}(-, M)\colon R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ is left-exact.

Proof. Exercise. □

4.8 Theorem. Let R be a ring and A be a right R-module. Then the tensor product functor $A \otimes_R -\colon R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ is right exact.

Proof. Exercise. □

5 Projective modules and resolutions

5.1 Definition. Let R be a ring. An R-module P is **projective** if any homomorphism from P can be lifted. That is, for any pair of R-modules A, B, any surjective homomorphism $p: A \to B$ and any homomorphism $h: P \to B$, there is a homomorphism $g: P \to A$ such that $h = p \circ g = p_* g$.

5.2 Proposition. An R-map of a free module is uniquely determined by action on the basis: Let F be a free R-module with basis $\mathcal{E} = \{e_\alpha\}$. Let M be an R-module. Consider an arbitrary map $f_0: \mathcal{E} \to M$. Then there exists a unique R-map $f: F \to M$ such that $f|_{\mathcal{E}} = f_0$.

Proof. Any element $x \in F$ can be uniquely represented as a linear combination of basis elements $x = \sum r_\alpha e_\alpha$. We define $f(x) := \sum r_\alpha f_0(e_\alpha)$. It is clear that any homomorphism extending f_0 must be of this form. As an exercise, check that f indeed is a homomorphism. □

5.3 Theorem. Any free module is projective.

Proof. Consider a free R-module F with a basis $\{e_\alpha\}$. For each basis element e_α, we may consider the preimage $p^{-1}(h(e_\alpha))$ and choose one element x_α here (in general, we need the axiom of choice). Now, we can define the homomorphism $g: F \to A$ by linearly extending $e_\alpha \mapsto x_\alpha$. □

5.4 Proposition. A left R-module P is projective if and only if $\mathrm{Hom}(P, -)$ is an exact functor.

Proof. We know that the Hom-functor is left exact. The definition of projective module indeed precisely says that for any surjective homomorphism $p: A \to B$ the induced map $p_*: \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B)$ is also surjective. That is, for any exact sequence $A \xrightarrow{p} B \longrightarrow 0$ the sequence $\mathrm{Hom}(P, A) \xrightarrow{p_*} \mathrm{Hom}(P, B) \longrightarrow 0$ is also exact. □

5.5 Theorem. An R-module P is projective if and only if P is a direct summand of a free R-module.

Proof. See [Rotman, Theorem 3.5]. □

5.6 Definition. Let M be an R-module. A **projective resolution** of M is an exact sequence

$$\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0,$$

where all P_i are projective. If all P_i are free, it is called a **free resolution**. If there is some $n \in \mathbb{N}$ such that $P_n \neq 0$, but $P_{n+1} = 0$, we say that the resolution is **finite** and the number n is called the **length** of the resolution.

5.7 Definition. Let M be an R-module. The length of the shortest projective resolution of M is called the **projective dimension** of M.

5.8 Proposition. Every R-module M has a free resolution.

Proof. Denote X a set of generators of the module M. Take a free module F_0 with basis indexed $\{e_x\}_{x \in X}$ indexed by the generators. We can define an R-map $\varepsilon: F_0 \to M$ by $e_x \mapsto x$. If we set $K_1 := \ker \varepsilon$, we have a short exact sequence

$$0 \longrightarrow K_1 \xrightarrow{i_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Performing the same construction for the module K_1 instead of M, we obtain the following short exact sequence

$$0 \longrightarrow K_2 \xrightarrow{i_2} F_1 \xrightarrow{\varepsilon_1} K_1 \longrightarrow 0.$$

The map $\varepsilon_1 \colon F_1 \to K_1 \subset F_0$ induces the map $d_1 = i_1 \circ \varepsilon_1 \colon F_1 \to F_0$ with $\operatorname{im} d_1 = \operatorname{im} \varepsilon_1 = K_1 = \ker \varepsilon$.

Iterating this process, we obtain a free resolution of the form

$$\cdots F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \to 0,$$

where $d_j = i_j \circ \varepsilon_j$. □

6 Homology

6.1 Definition. Let R be a ring. A **chain complex** of R-modules is a sequence of R-modules $C_* = (C_i)_{i=0}^{\infty}$ together with homomorphisms d_i of the form

$$\cdots C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \cdots \xrightarrow{d_1} C_0 \longrightarrow 0$$

such that $d_n d_{n+1} = 0$ for all n.

The condition $d_n d_{n+1} = 0$ is equivalent to $\ker d_n \subset \operatorname{im} d_{n+1}$.

The elements of C_n are called ***n*-chains**. We define $Z_n(C_*) := \ker d_n$ the set of ***n*-cycles** and $B_n(C_*) := \operatorname{im} d_{n+1}$ the set of ***n*-boundaries**. The quotient $H_n(C_*) := Z_n(C_*)/B_n(C_*)$ is called the ***n*-th homology**.

6.2 Definition. Let R be a ring. A **cochain complex** of R-modules is a sequence of R-modules $C_* = (C_i)_{i=0}^{\infty}$ together with homomorphisms d_i of the form

$$0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \cdots C_{n-1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \cdots$$

such that $d_{n+1} d_n = 0$ for all n.

The condition $d_{n+1} d_n = 0$ is equivalent to $\ker d_{n+1} \subset \operatorname{im} d_n$.

The elements of C_n are called ***n*-cochains**. We define $Z_n(C_*) := \ker d_n$ the set of ***n*-cocycles** and $B_n(C_*) := \operatorname{im} d_{n+1}$ the set of ***n*-coboundaries**. The quotient $H_n(C_*) := Z_n(C_*)/B_n(C_*)$ is called the ***n*-th cohomology**.

6.3 Remark. A complex with trivial (co)homologies is an exact sequence.

6.4 Remark. Any chain complex induces for arbitrary n **fundamental exact sequences**

$$0 \longrightarrow B_n \xrightarrow{i_n} Z_n \longrightarrow H_n \longrightarrow 0,$$
$$0 \longrightarrow Z_n \xrightarrow{j_n} C_n \xrightarrow{d_n} B_{n-1} \longrightarrow 0,$$

6.5 Example. Homology of cell complexes and simplicial and singular homology. See [Hatcher, Chapter 2].

6.6 Example (De Rham cohomology)**.** Consider a manifold M. Denote $\Omega^n(M)$ the space of differential n-forms on M. Then the de Rham cohomology is defined as the cohomology of the following cochain complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \cdots$$

Note that there is a duality between simplicial homology and de Rham cohomology expressed by the Stokes theorem

$$\int_{\partial N} \omega = \int_N \mathrm{d}\omega.$$

7 Derived functors

The question is now, where to get the chain complex, whose homology would somehow characterize a given module. We can use the following construction.

Let us fix a ring R and a left or right R-module M. Now, choose some projective resolution of M
$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0.$$
Delete the module M from this resolution. That is, consider the following complex
$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0.$$
Now, choose some additive functor T and apply it to this resolution to get
$$\cdots \xrightarrow{T(d_3)} T(P_2) \xrightarrow{T(d_2)} T(P_1) \xrightarrow{T(d_1)} T(P_0) \longrightarrow 0.$$

7.1 Proposition. The homologies $H_n(T(P_*)) = \ker T(d_n)/\operatorname{im} T(d_{n+1})$ are (up to isomorphism) independent of the choice of the projective resolution of M.

7.2 Definition. We denote $(L_nT)M := H_n(T(P_*))$. The map $L_nT\colon R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ is called the **left-derived** functor of T.

In particular suppose M is a right R-module. Then we can consider $T := -\otimes_R N$ for some left R-module N. We construct a chain complex
$$\cdots \xrightarrow{d_3\otimes 1_N} P_2\otimes N \xrightarrow{d_2\otimes 1_N} P_1\otimes N \xrightarrow{d_1\otimes 1_N} P_0\otimes N \longrightarrow 0$$
and compute its homology. The corresponding derived functor is called **Tor**, so we denote
$$\operatorname{Tor}_n^R(M,N) := H_n(P_*\otimes N) = \ker(d_n\otimes 1_N)/\operatorname{im}(d_{n+1}\otimes 1_N).$$
Note that the functor $-\otimes N$ is right exact, so the exact sequence
$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0.$$
is mapped to an exact sequence
$$P_1\otimes N \xrightarrow{d_1\otimes 1_N} P_0\otimes N \xrightarrow{\varepsilon\otimes 1_N} M\otimes N \to 0.$$
So, we have $M\otimes N \simeq P_0\otimes N/\operatorname{im}(d_1\otimes 1_N) = \operatorname{Tor}_0^R(M,N)$.

If we consider a k-algebra A as the ring, a canonical choice for the module N is the field k. Consider for example the algebra $A := \mathbb{C}[x_1,\dots,x_n]$. Then there exists a homomorphism $\varepsilon\colon A \to \mathbb{C}$ mapping $f \mapsto f(0)$, i.e. assigning to every polynomial the absolute coefficient. Then we can define left or right action of A on $\mathbb{C}$ as $f\cdot\alpha = \varepsilon(f)\alpha$, which gives $\mathbb{C}$ the structure of A-module. In general, part of the definition of a Hopf algebra A is such a homomorphism $\varepsilon\colon A \to \mathbb{C}$.

Alternatively, considering a right R-module N, we can consider the contravariant functor $\operatorname{Hom}(-,N)$ and construct the cochain complex
$$0 \longrightarrow \operatorname{Hom}(P_0,N) \xrightarrow{d_1^*} \operatorname{Hom}(P_1,N) \xrightarrow{d_2^*} \operatorname{Hom}(P_2,N) \xrightarrow{d_3^*} \cdots$$
The corresponding derived functor is called **Ext**. We denote
$$\operatorname{Ext}_R^n(M,N) := H^n(\operatorname{Hom}(P_*,N)) = \ker d_{i+1}^*/\operatorname{im} d_i^*.$$