Statistics of blocks in $k$-divisible non-crossing partitions

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Abstract

We derive a formula for the expected number of blocks of a given size from a non-crossing partition chosen uniformly at random. Moreover, we refine this result subject to the restriction of having a number of blocks given. Furthermore, we generalize to $k$-divisible partitions. In particular, we find that in average the number of blocks of a $k$-divisible non-crossing partitions of $nk$ elements is $\frac{kn+1}{k+1}$.

1 Introduction

In this note we study some statistics of the block structure of non-crossing partitions. A first systematic study of non-crossing partitions was done by G. Kreweras [4]. More recently, much more attention has been paid to non-crossing partitions because, among other reasons, they play a central role in the combinatorial approach of Speicher to Voiculescu’s free probability theory, [7]. For an introduction to this combinatorial approach, see [5].

In this direction, a recent paper by Ortmann [6] studies the asymptotic behavior of the sizes of the blocks of a uniformly chosen random partition. This lead him to a formula for the right-edge of the support of a measure in terms of the so called free cumulants, when these are positive. He noticed a very simple picture of this statistic as $n \to \infty$. Roughly speaking, in average, out of the $\frac{n+1}{2}$ blocks of this random partition, half of them are singletons, one fourth of the blocks are pairings, one eighth of the blocks have size 3, and so on.

Trying to understand better this asymptotic behavior, the question of the exact calculation of this statistic arose. In this paper, we answer this question and refine these results by considering the number of blocks given. More precisely, we show the following.

Proposition 1. Let $C_{n,t}$ be the number of blocks of size $t$ of all the non-crossing partitions of $\{1, 2, \ldots, n\}$ then

$$C_{n,t} = \binom{2n-t-1}{n-1}$$

As can be seen in [1] and [2], $k$-divisible partitions play an important role in the calculation of free cumulants and moments of products of $k$ free random variables. Moreover, in the approach given in [2] for studying asymptotic behavior of the size of the support when $k \to \infty$, understanding the asymptotic behavior of the sizes of blocks was a crucial step. It is then natural to try to extend Proposition 1 to $k$-divisible partitions.

A slight modification of the arguments allows us to generalize to $k$-divisible non-crossing partitions and results are similar as for $k = 1$.

Proposition 2. Let $C^k_{n,t}$ be the number of blocks of size $tk$ of all the $k$-divisible non-crossing partitions of $\{1, 2, \ldots, kn\}$ then

$$C^k_{n,t} = \binom{n(k+1)-t-1}{nk-1}$$

In particular, asymptotically, we have a similar phenomena as for the case $k = 1$; about a portion of $\frac{k}{k+1}$ of all the blocks have size $k$, then $\frac{k}{k+1}$ of the remaining blocks are of size $2k$, etc.

Another consequence is that the expected number of blocks of a $k$-divisible non-crossing partition is given by $\frac{kn+1}{k+1}$. It is then a natural question if this simple formula can be derived in a bijective way. We end with a bijective proof of this fact.

Apart from this introduction the paper is organized as follows. In Section 2 we give basic definitions and collect known results on the enumeration of non-crossing partitions. The case $k = 1$, is studied in Section 3. Section 4 treats the general case. Finally in Section 5 we give a bijective proof of the fact that in average the number of blocks is given by $\frac{kn+1}{k+1}$ and some further consequences of this bijection.
2 Preliminaries

Definition 3. (1) We call $\pi = \{V_1, \ldots, V_r\}$ a partition of the set $[n] := \{1, 2, \ldots, n\}$ if and only if $V_i$ $(1 \leq i \leq r)$ are pairwise disjoint, non-void subsets of $S$, such that $V_1 \cup V_2 \cup \cdots \cup V_r = \{1, 2, \ldots, n\}$. We call $V_1, V_2, \ldots, V_r$ the blocks of $\pi$. The number of blocks of $\pi$ is denoted by $|\pi|$.

(2) A partition $\pi = \{V_1, \ldots, V_r\}$ is called non-crossing if for all $1 \leq a < b < c < d \leq n$ if $a, c \in V_i$ then there is no other subset $V_j$ with $j \neq i$ containing $b$ and $d$.

(3) We say that a partition $\pi$ is $k$-divisible if the size of all the blocks is multiple of $k$. If all the block are exactly of size $k$ we say that $\pi$ is $k$-equal.

We will denote the set of non-crossing partitions of $[n]$ by $NC(n)$, the set of $k$-divisible non-crossing partitions of $[kn]$ by $NC^k(n)$ and the set of $k$-equal non-crossing partitions of $[kn]$ by $NC_k(n)$.

Remark 4. (1) $NC(n)$ can be equipped with the partial order $\preceq$ of reverse refinement ($\pi \preceq \sigma$ if and only if every block of $\pi$ is completely contained in a block of $\sigma$).

(2) For a given $\pi \in NC(n) \cong NC(\{1, 3, \ldots, 2n - 1\})$ we define its Kreweras complement

$$Kr(\pi) := \max\{\sigma \in NC(2, 4, \ldots, 2n) : \pi \cup \sigma \in NC(2n)\}.$$  

The map $Kr : NC(n) \to NC(n)$ is an order reversing isomorphism. Furthermore, for all $\pi \in NC(n)$ we have that $|\pi| + |Kr(\pi)| = n + 1$, see [5] for details.

There is a graphical representation of a partition $\pi$ which makes clear the property of being crossing or non-crossing, usually called the circular representation. We think of $[n]$ as labelling the vertices of a regular n-gon, clockwise. If we identify each block of $\pi$ with the convex hull of its corresponding vertices, then we see that $\pi$ is non-crossing precisely when its blocks are pairwise disjoint (that is, they don’t cross).

![Figure 1: Crossing and Non-Crossing Partitions](image)

Figure 1 shows the non-crossing partition $\{\{1, 2, 5, 9\}, \{3, 4\}, \{6\}, \{7, 8\}, \{10, 11, 12\}\}$ of the set $[12]$, and the crossing partition $\{\{1, 4, 7\}, \{2, 9\}, \{3, 11, 12\}, \{5, 6, 8, 10\}\}$ of $[12]$ in their circular representation.

Remark 5. The following characterization of non-crossing partitions is sometimes useful: for any $\pi \in NC(n)$, one can always find a block $V = \{r + 1, \ldots, r + s\}$ containing consecutive numbers. If one removes this block from $\pi$, the partition $\pi \setminus V \in NC(n - s)$ remains non-crossing.

It is well known that the number of non-crossing partition is given by the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. More generally we have the following.

Proposition 6. Let $NC^k(n)$ be the set of non-crossing partitions of $[nk]$ whose sizes of blocks are multiples of $k$. Then

$$\#NC^k(n) = \frac{(k+1)n}{kn + 1}$$

We recall the following result which gives a formula for the number of partitions with a given type.


Proposition 7. Let \( n \) be a positive integer and let \( r_1, r_2, \ldots, r_n \in \mathbb{N} \cup \{0\} \) such that \( r_1 + 2r_2 + \cdots + nr_n = n \). Then the number of partitions of \( n \) in \( NC(n) \) with \( r_1 \) blocks with 1 element, \( r_2 \) blocks with 2 elements, \( r_3 \) blocks with 3 elements, \ldots, \( r_n \) blocks with \( n \) elements equals

\[
\frac{n!}{(n+1 - \sum_{i=0}^{n} r_i)! \prod_{i=0}^{n} r_i!}
\]  

(3)

From the previous proposition after some algebraic manipulations and counting arguments similar to the ones presented in Section 2, one can derive the following.

Proposition 8. For \( 1 < m \leq n \) let \( NC(n, m) := \{ \pi \in NC(n) \mid \pi \text{ has } m \text{ blocks} \} \) denote the non-crossing partitions with exactly \( m \) blocks. Then we have that

\[
\#NC(n, m) = \frac{1}{n} \binom{n}{m} \binom{n}{m-1}
\]

More generally, for \( k \)-divisible partitions, (see, for example, [3]):

Proposition 9. For \( 1 < m \leq n \), let \( NC^k(n, m) := \{ \pi \in NC^k(n) \mid \pi \text{ has } m \text{ blocks} \} \) we have that

\[
\#NC^k(n, m) = \frac{1}{n} \binom{nk}{m-1} \binom{n}{m}.
\]

On the other hand, from Proposition 7, we can easily count \( k \)-equal partitions.

Corollary 10. Let \( NC_k(n) \) be the set of non-crossing partitions of \( nk \) whose blocks are of size of \( k \). Then

\[
\#NC_k(n) = \frac{(kn)!}{(k-1)n+1}
\]

The reader may have noticed from Proposition 6 and Corollary 10 that the number of \((k+1)\)-equal non-crossing partitions of \( n(k+1) \) and the number of \( k \)-divisible non-crossing partitions of \( nk \) coincide. We will derive a bijective proof of this fact and study further consequences in Section 5.

3 The case \( k = 1 \)

First we calculate the expected number of blocks of a given size \( t \), subject to the restriction of having \( m \) blocks from which the main result will follow.

Proposition 11. Let \( C_{n,t}(m) \) be the number of blocks of size \( t \) of all non-crossing partitions in \( NC(n, m) \).

\[
C_{n,t}(m) = \binom{n}{m-1} \binom{n-t-1}{m-2}
\]  

(4)

Proof. In order to count the number of blocks of size \( k \) of a given partition \( \pi \) in \( NC(n, m) \) with \( r_1 \) blocks with 1 element, \( r_2 \) blocks with 2 elements, \ldots, \( r_n \) blocks we need to multiply by \( r_1 \). So we want to calculate the following sum

\[
\sum_{\sum_{i=0}^{n} r_i = m} \frac{n! r_t}{(n+1 - \sum_{i=0}^{n} r_i)! \prod_{i=0}^{n} r_i!} = \sum_{\sum_{i=0}^{n} r_i = m} \frac{n! r_t}{(n+1 - m)! \prod_{i=0}^{n} r_i!}
\]

\[
= \binom{n}{m-1} \sum_{\sum_{i=0}^{n} \tilde{r}_i = m} \frac{(m-1)! \prod_{i=0}^{n} \tilde{r}_i!}{\prod_{i=0}^{n} r_i!}
\]

Now we make the change of variable \( \tilde{r}_t = r_t - 1 \) and \( \tilde{r}_i = r_i \) for \( i \neq t \). Then

\[
\binom{n}{m-1} \sum_{\sum_{i=0}^{n} \tilde{r}_i = m} \frac{(m-1)! \prod_{i=0}^{n} \tilde{r}_i!}{\prod_{i=0}^{n} r_i!} = \binom{n}{m-1} \sum_{\sum_{i=0}^{n} \tilde{r}_i = m-1} \frac{(m-1)!}{\prod_{i=0}^{n} \tilde{r}_i!}
\]

\[
\binom{n}{m-1} \binom{n-t-1}{m-2}.
\]
We used in the last equality the following identity (which can be proved counting in two ways the number of paths from \((0,0)\) to \((n-1,m-1)\) using the steps \((a,b) \to (a,b+1)\) or \((a,b) \to (a+1,b)\))

\[
\sum_{r_1 + r_2 + \cdots + r_n = m} \frac{m!}{\prod_{i=1}^{n} r_i!} = \binom{n-1}{m-1}.
\] (5)

Corollary 12. The expected number of blocks of size \(t\) of a partition chosen uniformly at random in \(NC(m,n)\) is given by

\[
\frac{n \left( \frac{n}{m-1} \right) \left( \frac{n-t}{m-2} \right)}{\binom{n}{m}}.
\]

Now we state the main result.

Proposition 13. Let \(C_{n,t}\) be the number of blocks of size \(t\) of all the non-crossing partitions in \(NC(n)\) then

\[
C_{n,t} = \binom{2n-t-1}{n-1}.
\] (6)

Proof. We use Proposition 11 and sum over all possible number of blocks. That is

\[
\sum_{m=1}^{n} \left( \frac{n}{m-1} \right) \left( \frac{n-t}{m-2} \right)
\]

Letting \(\tilde{m} = m - 1\) we get

\[
\sum_{m=0}^{n} \left( \frac{n}{m-1} \right) \left( \frac{n-t}{m-2} \right) = \sum_{\tilde{m}=0}^{n-1} \left( \frac{n}{\tilde{m}} \right) \left( \frac{n-t}{\tilde{m}-1} \right)
\]

Now, using the so-called Chu-Vandermonde’s identity

\[
\sum_{m=0}^{t} \left( \binom{x}{s-m} \binom{y}{m} \right) = \binom{x+y}{s}
\]

for \(x = n\) and \(y = n-t-1\) and \(s = n-1\) we get the desired result \(\Box\)

The following corollaries are direct consequences of our main result.

Corollary 14. The expected number of blocks of size \(t\) of a non-crossing partition chosen uniformly at random in \(NC(n)\) is given by

\[
\frac{(n+1)(2n-t-1)}{\binom{2n}{n-1}}
\]

Corollary 15. When \(n \to \infty\) the expected number of blocks of size \(t\) of a non-crossing partition chosen uniformly at random in \(NC(n)\) is asymptotically \(\frac{n}{t}\).

4 The general case

We prove a general version of Proposition 11. From this we will generalize the main result of the last section for \(k\)-divisible non-crossing partitions.

Proposition 16. Let \(C_k^n(m,t)\) be the number of blocks of size \(kt\) of all non-crossing partitions in \(NC^k(n,m)\).

\[
C_k^n(m,t) = \binom{nk}{m-1} \binom{n-t}{m-2}
\] (7)
Proof. We follow the same strategy as in the case $k = 1$ with slight modifications. In this case we need $(r_1, \ldots, r_n)$ such that $r_i = 0$ if $k$ does not divide $i$. So the condition $r_1 + r_2 + \cdots + r_n = m$ is really $r_k + 2r_k + \cdots + nr_k = m$. On the other hand the condition $r_1 + 2r_2 + \cdots + nr_n = nk$ is really $kr_k + 2kr_k + \cdots + (nk)r_{nk} = nk$, or equivalently $r_k + 2r_k + \cdots + nr_{nk} = n$. Making the change of variable $r_{mk} = s_i$ we get

$$\sum_{r_k + 2r_k + \cdots + nr_k = m} \frac{(nk)!s_k}{kr_k + 2kr_k + \cdots + (nk)r_{mk} = nk} = \sum_{s_1 + s_2 + \cdots + s_n = m} \frac{(nk)!s_t}{s_1 + 2s_2 + \cdots + ns_n = n} = \sum_{s_1 + s_2 + \cdots + s_n = m} \frac{(nk)!s_t}{s_1 + 2s_2 + \cdots + ns_n = n}$$

$$= \binom{nk}{m - 1} \sum_{s_1 + s_2 + \cdots + s_n = m} \frac{1}{s_1 + 2s_2 + \cdots + ns_n}$$

Now, the last sum can be treated exactly as for the case $k = 1$, yielding the result.

**Proposition 17.** Let $C_{n,t}^k$ be the number of blocks of size $tk$ of all the $k$-divisible non-crossing partitions in $NC(n)$ then

$$C_{n,t}^k = \binom{n(k + 1) - t - 1}{nk - 1}$$

(8)

**Proof.** The proof is exactly the same as in the case $k = 1$. We only use Chu-Vandermonde's identity

$$\sum_{m=0}^s \binom{x}{s-m} \binom{y}{m} = \binom{x+y}{s}$$

for $s = nk - 1$, $x = n(k + 1) - t - 1$ and $y = nk - 1$.

The following is a direct consequence of the last proposition.

**Corollary 18.** The sum of the number of blocks of all the $k$-divisible non-crossing partitions in $NC^k(n)$ is

$$\binom{n(k + 1) - 1}{nk}$$

**Proof.** Summing over $t$, in Equation (8), we easily get the result.

Finally, from the previous corollary one can calculate the expected number of block of $k$-divisible non-crossing partition.

**Corollary 19.** The expected number of block of a $k$-divisible partition of $[kn]$ chosen uniformly at random is given by $\frac{n+1}{k+1}$.

Moreover, similar to the case $k = 1$, asymptotically the picture is very simple, about a $\frac{k}{k+1}$ portion of all the blocks have size $k$, then $\frac{k}{k+1}$ of the remaining blocks are of size $2k$, and so on. This is easily proved using Equation (8).

**Remark 20.** Corollary 19 has a nice proof for $k = 1$. Recall from Remark 4 that the Kreweras complement is an autoisomorphism $Kr : NC(n) \rightarrow NC(n)$ such that $|Kr(\pi)| + |\pi| = n + 1$. Then summing over $\pi \in NC(n)$ and dividing by two we obtain that the expected value is just $\frac{n+1}{2}$. This suggests that there shall be a bijective proof of Corollary 19. This will be done in the next section.

5 The bijection

In this section we give a bijective proof of the fact that $NC^k(n) = NC_{k+1}(n)$. From this bijection we will derive Corollary 19.

**Lemma 21.** For each $n$ and each $k$ let $f : NC_{k+1}(n) \rightarrow NC^k(n)$ be the map induced by the identification of the pairs $\{k + 1, k + 2\}, \{2(k + 1), 2(k + 1) + 1\}, \ldots, \{n(k + 1), 1\}$. Then $f$ is a bijection.
Proof. We see that the image of this map is in $NC^k(n)$. So, let $\pi$ be a $(k + 1)$-equal partition.

i) Every block has one element on each congruence mod $k + 1$. Indeed, because of Remark 5, using the characterization of non-crossing partitions there is at least one interval, which has of course this property. Removing this interval will not affect the congruence in the elements of other blocks. So by induction on $n$ every block has one element of each congruence mod $k + 1$.

ii) Note that for each two elements identified we reduce 1 point. So suppose that $m$ blocks (of size $k$) are identified in this bijection to form a big block $V$. Then the number of vertices in this big block equals $m(k + 1) - \#(\text{identified vertices})/2$. Now, by i) there is exactly two elements in each block to be identified with another element, that is $2m$. So

$$|V| = m(k + 1) - \#(\text{identified vertices})/2 = m(k + 1) - (2m)/2 = mk.$$ 

this proofs that $f(\pi) \in NC^k(n)$.

Now, it is not very hard to see that by splitting the points of $\pi \in NC^k(n)$ we get a unique inverse $f^{-1}(\pi) \in NC_{k+1}(n)$.

Figure 2: Bijections between 3-equal and 2-divisible non-crossing partitions

Now we give a proof of Corollary 19.

Proof. For each $n$ and each $k$ and each $0 < i \leq k + 1$ let $f_i : NC_{k+1}(n) \rightarrow NC^k(n)$ the map induced by the identification of the pairs \{ $k+1+i$, $k+1+i+1$ \} , \{ $2(k+1)+i$, $2(k+1)+i+1$ \} , ... \{ $n(k+1)+i$, $n(k+1)+i+1$ \} (we consider elements mod $nk$). Then by the proof of the previous lemma, each $f_i$ is a bijection. So, let $\pi$ be a fixed $(k + 1)$-equal partition. Considering all the bijections $f_i$ on this fixed partition, we see that every point $j$ will be identified twice (one with $f_{j-1}$ and one with $f_j$). So for each partition $\pi$ in $NC_{k+1}(n)$, the collection $(f_i(\pi))_{i=1}^{k+1}$ consists of $k + 1$ partitions in $NC^k(n)$ whose number of blocks add $kn + 1$.

In the following example we want to illustrate how the bijection given by Lemma 21 allows us to count $k$-divisible partitions with some restrictions by counting the preimage under $f$.

**Example 22.** Let $NC^k_{1 \rightarrow 2}(n)$ be the set of $k$-divisible non-crossing partitions of $[kn]$ such that 1 and 2 are in the same block. It is clear that $\pi \in NC^k_{1 \rightarrow 2}(n)$ if and only if $f^{-1}(\pi)$ satisfies that 1 and 2 are in the same block since $f$ does not change this property.
Now, counting the \((k+1)\)-equal non-crossing partitions of \([(k+1)n]\) such that 1 and 2 are in the same blocks is the same as counting the non-crossing partitions of \([(k+1)n-1]\) with \(n-1\) blocks of size \(k+1\) and 1 block of size \(k\) containing the element 1, since 1 and 2 can be identified. From Proposition 7, the size of this set is easily seen to be

\[
\frac{k}{(k+1)n-1} \binom{(k+1)n-1}{n-1} = \frac{k}{n-1} \binom{(k+1)n-2}{n-2}
\]

where the first factor of the LHS is the probability that the block of size \(k\) contains the element 1.

\[
\text{Figure 3: A 3-equal and its Kreweras complement divided mod 3.}
\]

Let us finally mention that the bijections \(f_i\) are closely related to the Kreweras complement of a \((k+1)\)-equal non-crossing partitions, which was considered in [2]. Indeed \(Kr(\pi)\) can be divided in a canonical way into \(k+1\) partitions of \([n]\), \(\pi_1, \ldots, \pi_{k+1}\), such that \(|\pi_i| = |f_i(\pi)|\). Fig. 2 shows the bijections \(f_1, f_2\) and \(f_3\) for \(k = 3\), \(n = 4\) and \(\pi = \{\{1, 8, 9\}, \{2, 6, 7\}, \{3, 4, 5\}, \{9, 10, 11\}\}\), while Fig. 3 shows the same partition as Fig. 2 with its Kreweras complement divided into the partitions \(\pi_1, \pi_2\) and \(\pi_3\).

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References


