FREE PROBABILITY THEORY

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Lecture 1
Free Group Factors and Freeness

1.1. **Motivation: the free group factors $L(\mathbb{F}_n)$.** Let G be a discrete group, then one can construct out of G canonical C^*- and von Neumann algebras by completing the left regular representation of the group G on its group algebra $\mathbb{C}G$ in appropriate topologies; more precisely: let

$$\mathbb{C}(G) := \{\sum_{\text{finite}} \alpha_g \delta_g \mid \alpha_g \in \mathbb{C}\}$$

be the group algebra of G, consisting of finite linear combinations of group elements – a group element g is here identified with the formal symbol δ_g in the group algebra. If one prefers, one can identify the group algebra $\mathbb{C}G$ with finitely supported functions $G \to \mathbb{C}$, and δ_g is then what it ought to be, namely

$$\delta_g(h) = \begin{cases} 1, & g = h \\ 0, & g \neq h \end{cases}.$$

We can introduce an inner product on $\mathbb{C}G$ by declaring any two different group elements as orthonormal

$$\langle \delta_g, \delta_h \rangle := \begin{cases} 1, & g = h \\ 0, & g \neq h. \end{cases}$$

and extend this sesquilinearly. This gives us a corresponding l_2-norm on the group algebra

$$\|a\|_2 := \sqrt{\langle a, a\rangle}, \qquad a \in \mathbb{C}G,$$

and completing the group algebra with respect to this norm gives us a Hilbert space

$$l_2(G) := \overline{\mathbb{C}G}^{\|\cdot\|_2}.$$

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In the function picture this is just the usual L_2-norm for complex-valued functions on G with respect to the discrete counting measure on G.

We can now let the group G or its group algebra $\mathbb{C}G$ act on this Hilbert space $l_2(G)$ by left multiplication, i.e., this action λ is defined through linear extension and continuity by

$$\lambda(g)\delta_h := \delta_{gh} \qquad (g, h \in G).$$

One sees quite easily that the linear mapping

$$\lambda(a) : l_2(G) \to l_2(G)$$

is, for any $a \in \mathbb{C}G$, bounded, thus our left regular representation λ represents the group algebra of G by bounded operators on $l_2(G)$,

$$\lambda : \mathbb{C}G \to B(l_2(G)).$$

(In the function picture, our operators act of course as convolution operators.) Note that we have

$$\lambda(e) = 1$$

and, for any $g \in G$, that

$$\lambda(g^{-1}) = \lambda(g)^*$$

thus

$$\lambda(g)\lambda(g)^* = 1 = \lambda(g)^*\lambda(g)$$

and our group elements are represented on $l_2(G)$ by unitary operators.

If we now take the closure of $\lambda(\mathbb{C}G)$ in $B(l_2(G))$ with respect to the operator norm or with respect to the strong operator topology then we get the reduced C^*-algebra $C^*_{\text{red}}(G)$ or the von Neumann algebra $L(G)$,

$$C^*_{\text{red}}(G) := \overline{\mathbb{C}G}^{\|\cdot\|} \subset B(l_2(G))$$

and

$$L(G) := \overline{\mathbb{C}G}^{\text{strong operator topology}} \subset B(l_2(G)).$$

(As usual, one identifies $\mathbb{C}G$ with $\lambda(\mathbb{C}G)$, so that the formulas don't look to overloaden.)

(On the C^*-algebra level there is also a full C^*-algebra $C^*(G)$, which is not built from the left regular representation of G, but from all representations and which is in general different from the reduced one. However, in our context, $C^*_{\text{red}}(G)$ is the more interesting object.)

$L(G)$ is a type II_1 von Neumann algebra, which means that it has a trace, i.e., a linear functional $\tau : L(G) \to \mathbb{C}$ which is tracial in the sense that

$$\tau(ab) = \tau(ba) \qquad \text{for all } a, b \in L(G).$$

This trace τ encodes the information about the neutral element $e \in G$. It is given as the vector state associated with δ_e (which is defined on all of $B(l_2(G))$), i.e.,

$$\tau(a) := \langle \delta_e, a\delta_e \rangle.$$

It is easy to see that restricted to $\mathbb{C}G$ this is a trace (which just states that in a group being a left-inverse is equivalent to being a right-inverse), and by continuity this trace property also goes over to the C^*-algebra and the von Neumann algebra.

If G is i.c.c. (which means that for each non-trivial $g \in G$, $g \neq e$ its conjugacy class $\{hgh^{-1} \mid h \in G\}$ has infinitely many elements), then $L(G)$ is also a factor.

$L(G)$ for amenable i.c.c groups G is well-understood, namely, it always gives the so-called hyperfinite II_1-factor R. Recent interest in von Neumann algebra theory lies in going beyond the hyperfinite situation.

Of particular interest is the case of the free group factors $L(\mathbb{F}_n)$. $G = \mathbb{F}_n$ is here the free group on n generators; for $n \geq 2$ this is an i.c.c. non-amenable group, and it was proved by Murray and von Neumann that $L(\mathbb{F}_n)$ is not isomorphic to the hyperfinite factor R. However, not much more was known about these free group factors, in the mid 1980's when Voiculescu started to study their structure and for this purpose introduced free probability theory. Of particular interest in this context is the *Free Group Isomorphism Problem:* Are $L(\mathbb{F}_n)$ and $L(\mathbb{F}_m)$ isomorphic as von Neumann algebras for $n \neq m$ $(n, m \geq 2)$?

Note in this respect, that $\mathbb{F}_n$ and $\mathbb{F}_m$ are clearly not isomorphic as groups, and, by results of Voiculescu and Pimsner using K-theory, the same remains true on the level of the reduced C^*-algebras. Going over to von Neumann algebras, however, is a much bigger step and the answer to this question is still open. Since von Neumann algebras are very big objects, it is very hard to distinguish them, and there are not many invariants for von Neumann algebras. The operator algebraic side of free probability theory can be seen as trying to find new tools (and in particular, invariants) for dealing with classes of von Neumann algebras which are related to free group factors, or more general, von Neumann algebras coming from free products of groups.

1.2. **Going over to moments.** Concretely the isomorphism problem asks whether we can realize in an invertible way the generators of the free group $\mathbb{F}_n$ as function in the generators of the free group $\mathbb{F}_m$. Clearly one cannot do this on an algebraic level (i.e., in terms of finite sums over group elements); however, if one allows infinite sums this is not so clear any more. Since it is quite hard to deal with infinite sums (there is no direct way of deciding whether an infinite sum converges in the strong operator topology or not) it is of advantage to shift our emphasis from the algebraic properties of operators to their moments under the canoncial trace τ.

Definition 1. Let $a_1, \dots, a_k$ be operators in our von Neumann algebra $L(\mathbb{F}_n)$.
1) We call the numbers

$$\tau(a_{i(1)} \cdots a_{i(l)}),$$

for any possible choice

$$l \in \mathbb{N}, \quad 1 \leq i(1), \dots, i(l) \leq k$$

joint moments of $a_1, \dots, a_k$. The collection of all possible joint moments constitutes the *distribution* of $a_1, \dots, a_k$.
1) We call the numbers

$$\tau(a_{i(1)}^{\varepsilon(1)} \cdots a_{i(l)}^{\varepsilon(l)}),$$

for any possible choice

$$l \in \mathbb{N}, \quad 1 \leq i(1), \dots, i(l) \leq k, \quad \varepsilon(1), \dots, \varepsilon(l) \in \{*, 1\}$$

(a^ε just means we are looking either on $a^1 = a$ or on a^*), *joint $*$-moments* of $a_1, \dots, a_k$. The collection of all possible joint $*$-moments constitutes the *$*$-distribution* of $a_1, \dots, a_k$.

If our operators are selfadjoint, then of course $*$-moments and moments are the same. In the general case, we can restrict to talking about moments by going over to the real and imaginary part of our operators (and thus doubling the number of considered operators): the joint moments of the selfadjoint operators $(a + a^*)/2$ and $(a - a^*)/(2i)$ contain the same information as the $*$-moments of the operator a.

It is on first sight of course not clear whether we gain any advantage by looking on moments instead of dealing with the operators more directly in the usual way; however, at least one should notice that we do not have any disadvantage. It might seem that moments contain less information than knowing how the operator acts on the Hilbert space, but the GNS-construction allows us to reconstruct at least some Hilbert space whenever we feel like doing so, and for all questions which have an answer affiliated to the von Neumann algebra the concrete form of the Hilbert space and the action of our operators there are not really important. This is a quite simple observation, but also very fundamental from a more philosophical point of view, since it justifies our shift to the moments, so let us state this a bit more explicitely.

Theorem 1. *Let $\mathcal{M}$ be a von Neumann algebra which is generated by generators $a_1, \dots, a_k$ and let $\mathcal{N}$ be another von Neumann algebra which is generated by generators $b_1, \dots, b_k$. Let φ be a faithful normal state on $\mathcal{M}$ and let ψ be a faithful normal state on $\mathcal{N}$ and assume that the $*$-distribution of the $a_1, \dots, a_k$ with respect to φ is the same as the $*$-distribution of the $b_1, \dots, b_k$ with respect to ψ – which means that*

$$\varphi(a_{i(1)}^{\varepsilon(1)} \cdots a_{i(l)}^{\varepsilon(l)}) = \psi(b_{i(1)}^{\varepsilon(1)} \cdots b_{i(l)}^{\varepsilon(l)})$$

for all choices of $l \in \mathbb{N}$, $1 \leq i(1), \dots, i(l) \leq k$, and $\varepsilon(1), \dots, \varepsilon(l) \in \{*, 1\}$. Then the mapping

$$a_i \mapsto b_i \qquad (i = 1, \dots, k)$$

extends to a $*$-isomorphism from $\mathcal{M}$ to $\mathcal{N}$. In particular, $\mathcal{M}$ and $\mathcal{N}$ are isomorphic von Neumann algebras.

Recall that *faithful* for a state φ on a $*$-algebra means that $\varphi(aa^*) = 0$ implies that $a = 0$.

The main observation to be made for the proof of this theorem is that faithfulness of the states implies that we cannot have relations between elements on one side which we do not have on the other side. Namely, assume p is a polynomial in $2k$ non-commuting variables and that we have $q := p(a_1, a_1^*, \dots, a_k, a_k^*) = 0$. But then also $qq^* = 0$ and hence $\varphi(qq^*) = 0$. Since $\varphi(qq^*)$ is a $*$-moment of $a_1, \dots, a_k$, it must agree with the corresponding $*$-moment of $b_1, \dots, b_k$, hence this must also be zero,

$$\psi\big(p(b_1, b_1^*, \dots, b_k, b_k^*) \cdot p(b_1, b_1^*, \dots, b_k, b_k^*)^*\big) = 0,$$

which implies, by the faithfulness of ψ, that also $p(b_1, b_1^*, \dots, b_k, b_k^*) = 0$. This shows that we get a $*$-isomorphism on the level of $*$-algebras, and by approximating and normality we can extend this to the level of von Neumann algebras.

Of course, if we are using a state which is not faithful then we will loose some information about our operators by just looking on moments. However, for a faithful normal state, each question about a von Neumann algebra can in principle be decided by the knowledge of the joint $*$-moments of generators of the von Neumann algebra. Note that in many situations there are canonical faithful states around. In particular, in the group von Neumann algebra case $L(G)$, the canonical trace τ is always faithful (and normal). The big question of course is whether the fact that moments determine in principle the structure of von Neumann algebras can be translated into concrete statements. In general, it is probably quite hard to get some concrete information out of the moments, however, in the context of free group factors this approach will turn out to be quite succesful.

1.3. **Free products on the level of moments.** We want to understand the structure of $L(\mathbb{F}_n)$. It is tempting to do this by reduction, in a similar way as one can understand the group $\mathbb{F}_n$ as a free product of n copies of $\mathbb{F}_1 = \mathbb{Z}$. Let us concentrate on the reduction step, i.e., consider the free group $G = \mathbb{F}_{m+n}$, as generated by $m+n$ free generators $f_1, \dots, f_{m+n}$. Then G contains in a canonical way $G_1 := \mathbb{F}_m$ (generated by the first m generators $f_1, \dots, f_m$) and $G_2 := \mathbb{F}_n$ (generated by the last n generators $f_{m+1}, \dots, f_{m+n}$) as subgroups and G is built out of G_1 and G_2 as a free product. This means that we can write any element from G as a product of factors coming alternatingly from G_1 and from G_2 and furthermore, we have no non-trivial relations betweens elements from G_1 and elements from G_2. (A trivial relation comes from the fact that both G_1 and G_2 contain the neutral element e of G.) To put it more formally, the fact that G_1 and G_2 are free as subgroups in G means that

$$g_1 \in G_{i(1)}, \dots, g_k \in G_{i(k)}$$

with

$$i(j) \neq i(j+1) \qquad \text{for all } j = 1, \dots, k-1$$

and

$$g_j \neq e \qquad \text{for all } j = 1, \dots, k$$

implies that

$$g_1 \cdots g_k \neq e.$$

We can extend the same algebraic description to the group algebras: we have that $\mathbb{C}G_1$ and $\mathbb{C}G_2$ are free as subalgebras in $\mathbb{C}G$ in the sense that

$$a_1 \in \mathbb{C}G_{i(1)}, \dots, a_k \in \mathbb{C}G_{i(k)}$$

with

$$i(j) \neq i(j+1) \qquad \text{for all } j = 1, \dots, k-1$$

and

$$a_j \text{ does not contain } e \qquad (j = 1, \dots, k)$$

implies that

$$a_1 \cdots a_k \text{ does not contain } e.$$

Here it is of course clear, what it means that $a \in \mathbb{C}G$ does not contain the identity e, namely such an element is a finite sum

$$a = \sum_{\text{finitely many } g \in G} \alpha_g \delta_g \qquad (\alpha_g \in \mathbb{C})$$

and not containing e just means that $\alpha_e = 0$.

Next we would like to extend the same freeness property to the object in which we are really interested, namely to the von Neumann algebra $L(G)$. We should have the same kind of description as above, however, now we have to deal with infinite sums, so that the statement "containing e" is getting problematic. In particular, for the product $a_1 \cdots a_k$ multiplying infinite sums and rearranging terms is not so straighforward. However, let us notice that we can use our canonical trace τ on $\mathbb{C}G$ for checking whether an element contains the identity or not; namely for

$$a = \sum_g \alpha_g \delta_g \in \mathbb{C}G$$

we have

$$\tau(a) = \alpha_e \begin{cases} = 0, & \text{if } a \text{ does not contain } e \\ \neq 0, & \text{if } a \text{ contains } e \end{cases}.$$

Since the trace is continuous with respect to the strong operator topology, it is clear how we should rewrite the above freeness characterization in a form which is also valid for the von Neumann algebras:

$$a_1 \in L(G_{i(1)}), \dots, a_k \in L(G_{i(k)})$$

with

$$i(1) \neq i(2), i(2) \neq i(3), \dots, i(k-1) \neq i(k)$$

and

$$\tau(a_j) = 0 \qquad \text{for all } j = 1, \dots, k$$

implies that

$$\tau(a_1 \dots a_k) = 0.$$

1.4. **Voiculescu's concept of "freeness".** What the above tells us is that in the case of a free group factor there are quite a lot of very specific relations between the joint moments of the generators of the algebra (just reflecting the fact that we are dealing with free products of groups). If we take the idea serious to understand a von Neumann algebra by understanding moments of its generators, then it is surely important to understand these relations better, hoping that this will in the end yield a better understanding of the free group factors themselves. That was exactly the starting point of Voiculescu. The above observation was his motivation for defining the concept of "freeness", formalizing abstractly the specific relations which we have observed to be satisfied for moments in the free group factors.

However, in order to understand these relations better, it is not really important (and maybe even distracting) that these moments are coming from a trace in a von Neumann algebra, so let us forget about this specific frame for the moment and make the definiton in the more general frame of unital algebras and unital linear functionals.

Definition 2. Let $\mathcal{A}$ be a unital algebra and

$$\varphi : \mathcal{A} \to \mathbb{C} \qquad \text{with} \qquad \varphi(1) = 1$$

a unital linear functional on $\mathcal{A}$. We say that unital subalgebras

$$\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{A}$$

are *free* if we have that

$$\varphi(a_1 \cdots a_k) = 0$$

whenever we have

$$a_1 \in \mathcal{A}_{i(1)}, \dots, a_k \in \mathcal{A}_{i(k)}$$

such that neighboring elements in $a_1 \cdots a_k$ are from different subalgebras, i.e.,

$$i(j) \neq i(j+1) \qquad \text{for all } j = 1, \dots, k-1$$

and such that all a_j are centered under φ, i.e.,

$$\varphi(a_j) = 0 \qquad \text{for all } j = 1, \dots, k.$$

Clearly, this freeness concept is not purely algebraic anymore, but it depends on the choosen φ, thus to be precise we should talk about *freeness with respect to* φ. Subalgebras which are free with respect so one functional, are usually not free with respect to some other functional. However, in most cases there is a fixed canonical φ and all our statements will then be with respect to this fixed functional.

Note that we have defined freeness for subalgebras of $\mathcal{A}$, but it will also be useful to be able to talk about freeness of elements of $\mathcal{A}$ (as e.g. freeness of the generators of our free group factors) or, more general, freeness of subsets of $\mathcal{A}$. This can easily be achieved by going over to the generated algebras.

Definition 3. Let $\mathcal{A}$ be a unital algebra and

$$\varphi : \mathcal{A} \to \mathbb{C} \qquad \text{with} \qquad \varphi(1) = 1$$

a unital linear functional on $\mathcal{A}$. We say that subsets

$$\mathcal{X}_1, \dots, \mathcal{X}_n \subset \mathcal{A}$$

are *free* if we have that $\mathcal{A}_1, \dots, \mathcal{A}_n$ are free, where $\mathcal{A}_i$ is the unital subalgebra generated by all elements from $\mathcal{X}_i$. If instead of the generated algebras we take the generated $*$-algebras (in the case where $\mathcal{A}$ is a $*$-algebra), then we say that $\mathcal{X}_1, \dots, \mathcal{X}_n$ are $*$-*free*.

In the case that a set consists only of one element we will also speak about freeness of that element instead of freeness of the set. Thus, e.g., $a_1, a_2, \{b_1, b_2\}$ free means that the three sets $\{a_1\}, \{a_2\}, \{b_1, b_2\}$ are free

Of course, for selfadjoint elements $*$-free means the same as just free. For general non-selfadjoint elements, however, being $*$-free is a stronger condition than just being free. Again we can reduce $*$-freeness to freeness by taking the real and imaginary part. Note, however, that this forces us to go over from elements to sets; namely we have, e.g., that

$$a \text{ and } b \text{ are } *\text{-free} \Leftrightarrow \{\frac{a+a^*}{2}, \frac{a-a^*}{2i}\} \text{ and } \{\frac{b+b^*}{2}, \frac{b-b^*}{2i}\} \text{ are free}$$

In the context of our interest in von Neumann algebras it might look more meaningful to define freeness of elements or subsets by freeness of their generated von Neumann algebras. However, the following statement shows that this does not make a difference.

Proposition 1. *Let $\mathcal{M}$ be a von Neumann algebra with a normal state φ. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be unital $*$-subalgebras of $\mathcal{M}$ and denote by $\mathcal{M}_i := vN(\mathcal{A}_i)$ the generated von Neumann subalgebras of $\mathcal{M}$. Then, freeness of $\mathcal{A}_1, \dots, \mathcal{A}_n$ is equivalent to freeness of $\mathcal{M}_1, \dots, \mathcal{M}_n$.*

1.5. **The probabilistic perspective.** *Free probability* deals with understanding the concept of freeness as defined above, and it should by now be clear what "free" means in this context. But what about "probability". We mainly want to understand special operators on Hilbert spaces and not some kind of random variables. However, we have shifted our emphasis to values of products of our operators under some state and it is no accident that we call these numbers "moments", in analogy with the concept of moments of random variables (which are expectation values of products of the random variables). We are not doing some kind of probability theory in the genuine classical sense, but we see our main objects in analogy with corresponding objects from classical probability theory. In particular, the notion of freeness has some kind of probabilistic flavour. Namely, what freeness comes down in the end are prescriptions how to calculate joint moments of our operators out of the knowledge of the moments of the single operators. But that is comparable to the fundamendal notion of "independence" from classical probability theory - this is defined by the requirement that expectations of products of independent random variables factorize into the product of the expectations of those variables.

Note that our freeness for subalgebras corresponds to the independence of σ-algebras, whereas the freeness of elements corresponds to the independence of random variables. Going a step further in this direction, one usually makes the following definitions.

Definition 4. A pair $(\mathcal{A}, \varphi)$ consisting of a unital algebra and a unital linear functional is called a *non-commutative probability space* and elements from $\mathcal{A}$ are addressed as *non-commutative random variables.* If $\mathcal{M}$ is a von Neumann algebra and φ is a faithful normal state then we call the pair $(\mathcal{M}, \varphi)$ a *W^*-probability space.*

Note that "non-commutative" here is used in the sense "not necessarily commutative", thus allows also commutative situations as special cases. Often, we will just drop the adjective "non-commutative" and talk about probability spaces and random variables. If a probability space $(\mathcal{A}, \varphi)$ is fixed, then freeness of random variables is of course with respect to this φ.

It might not be clear at this point whether this analogy with notions from classical probability theory is just a superficial similarity or whether it will lead to some deeper insight to consider our freeness concept in analogy with the probabilistic independence concept. In any case it was the point of view which Voiculescu took when he set out to investigate the notion of freeness. And as it has turned out since then, the analogy really goes very deep.

1.6. **First properties of freeness.** We will now forget for a while that our motivation comes from special von Neumann algebras and we just want to see whether there is something interesting to say about our notion of freeness.

We mentioned above that freeness is a rule for calculating mixed moments of random variables out of the moments of single random variables. Let us elaborate a bit on this.

Proposition 2. *Let $(\mathcal{B}, \varphi)$ be a non-commutative probability space and consider unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{B}$ which are free. Denote by $\mathcal{A}$ the unital subalgebra of $\mathcal{B}$ which is generated by all $\mathcal{A}_1, \dots, \mathcal{A}_n$. Then the restriction of φ to $\mathcal{A}$ is determined by the restrictions of φ to each of the $\mathcal{A}_i$ (for $i = 1, \dots, n$) and by the freeness condition.*

The proof of this proposition goes by reduction of the length of considered words and by going over to centered elements. Namely, what one has to show is how to calculate φ on words of the form $a_1 \cdots a_k$ where the a_j come alternatingly from different subalgebras $\mathcal{A}_{i(j)}$. In order to be able to use the freeness condition we go over to the corresponding product in centered variables, on which φ is zero by the definiton of freeness, i.e.,

$$\varphi\big((a_1 - \varphi(a_1)1) \cdots (a_k - \varphi(a_k)1)\big) = 0.$$

(Note that here it is important that $\varphi(1) = 1$.) However, if we multiply out the terms on the left hand side we get the term $\varphi(a_1 \cdots a_k)$, in which we are interested, and many other terms, which are of smaller length and thus can be treated by induction hypothesis. This gives a recursive way of reducing $\varphi(a_1 \cdots a_k)$ to the knowledge of φ restricted to the subalgebras.

In order to get a feeling for this it is instructive to look at some simple examples.

Assume that a and b are free random variables. Then the above says that the freeness condition allows us to express any mixed moment in a's and b's in terms of moments of a and moments of b.

The simplest case of a mixed moment is $\varphi(ab)$. The above proof tells us that we should look at

$$\varphi\big((a - \varphi(a)1)(b - \varphi(b)1)\big) = 0,$$

which we can multiply out to

$$\varphi(ab) - \varphi(a)\varphi(b) - \varphi(a)\varphi(b) + \varphi(a)\varphi(b) = 0,$$

and thus

$$\varphi(ab) = \varphi(a)\varphi(b).$$

In the same way we get

$$\varphi(aba) = \varphi(aa)\varphi(b).$$

This does not look too exciting and in particular is the same as for classical random variables. However, looking on $\varphi(abab)$ is getting more interesting. Multiplying out

$$\varphi\big((a - \varphi(a)1)(b - \varphi(b)1)(a - \varphi(a)1)(b - \varphi(b)1)\big) = 0$$

and using the above formulas for $\varphi(ab)$ and $\varphi(aba)$ yields after some work and some cancellations that

$$\varphi(abab) = \varphi(aa)\varphi(b)\varphi(b) + \varphi(a)\varphi(a)\varphi(bb) - \varphi(a)\varphi(b)\varphi(a)\varphi(b).$$

This shows us a few things. First of all, this is different from the result for classical independent random variables (where a and b would commute and the expectation would just factorize into $\varphi(abab) = \varphi(aa)\varphi(bb)$). Thus, freeness might be analogous to independence, but it is really different and not some kind of non-commutative generalization.

Even worse (or better), it is a structure which is genuinely non-commutative and only trivial shadows of it can be seen in the classical commutative world. Namely, assume that we have classical (let's say, real) random variables a and b which are free (with respect to the functional given by taking the expectation). Then on one hand we have the above formula for $\varphi(abab)$, but, since a and b commute, we also have

$$\varphi(abab) = \varphi(aabb) = \varphi(aa)\varphi(bb).$$

(The later is the formula $\varphi(\tilde{a}\tilde{b}) = \varphi(\tilde{a})\varphi(\tilde{b})$ for free variables applied to $\tilde{a} = aa$ and $\tilde{b} = bb$.) Taken together this yields

$$\varphi(aa)\varphi(bb) = \varphi(aa)\varphi(b)\varphi(b) + \varphi(a)\varphi(a)\varphi(bb) - \varphi(a)\varphi(b)\varphi(a)\varphi(b)$$

or

$$\varphi\big((a-\varphi(a)1)^2\big)\cdot\varphi\big((b-\varphi(b)1)^2\big) = \big(\varphi(a^2)-\varphi(a)^2\big)\cdot\big(\varphi(b^2)-\varphi(b^2)\big) = 0.$$

Thus one of the factors must vanish, let's say

$$\varphi\big((a - \varphi(a)1)^2\big) = 0.$$

However, this says that the variance of a is zero, which means that a is almost surely a constant. Thus classical random variables can only be free if at least one of them is a constant. This is not a very interesting situation and thus non-trivial features of freeness cannot be seen in the classical world. In particular, it is a wrong idea to think of freeness as some special kind of dependence between classical random variables.

Another thing we see from the above calculation of $\varphi(abab)$ is that the complexity of the result and even more the complexity of the calculation grows very fast with the length of the considered word. The expression for $\varphi(ababab)$ in terms of moments of a and moments of b consists of 12, the expression for $\varphi(abababab)$ of 55 terms; we have an exponential growth in the length. Just based on the recursive procedure as outlined above one does not see a clear structure for the final formulas. Thus, freeness allows in principle to calculate all mixed moments, but the concrete structure of these formulas is not obvious at all and one of the basic tasks of free probability theory is to reveal this. More about this in the second lecture.

One fundamental simple statement about free variables which we can make right now is the fact that constants are free from everything. The above considerations showed this essentially on the level of moments of short length, but the full proof for all moments is also very easy.

Proposition 3. *Let $(\mathcal{A}, \varphi)$ be a probability space. Then the unital subalgebra $\mathbb{C}1$ of "constants" is free from any unital subalgebra $\mathcal{B} \subset \mathcal{A}$.*

The proof of this is easy. Namely consider words $a_1 \cdots a_k$ as in the defintion of freeness; we have to show that φ applied to them is zero.

However, $k = 1$ is trivial, and if $k \geq 2$ then at least one of the a_j must be from $\mathbb{C}1$, i.e., it must be of the form $a_j = \alpha 1$ for some $\alpha \in \mathbb{C}$. Since the a_j are centered, we necessarily have $\alpha = 0$, and then of course $a_1 \cdots a_k = 0$.

1.7. **How can freeness help to investigate von Neumann algebras.** Before we start in the next lecture a more systematic investigation of the structure of freeness, I want to come back for the moment to the question what freeness can offer for the investigation of von Neumann algebras. Typically we are looking on von Neumann algebras generated by some generators; these are in general not selfadjoint (e.g., in the group case they are unitary operators), thus all relevant information is contained in their $*$-moments. Let us recall the following

- von Neumann algebras with the same joint $*$-moments for the generators are isomorphic
- if the generators are $*$-free then mixed $*$-moments are determined by $*$-moments of each generator

Thus if we have some operators $a_1, \dots, a_n$ on one side and some other operators $b_1, \dots, b_n$ on the other side such that

- $a_1, \dots, a_n$ are $*$-free and $b_1, \dots, b_n$ are $*$-free
- the $*$-moments of a_i are the same as the $*$-moments of b_i, for each $i = 1, \dots, n$

then we know that all mixed $*$-moments in the a's are the same as the corresponding mixed $*$-moments in the b's and thus (provided our states are faithful and normal, as they usually are) the von Neumann algebra generated by the a's is isomorphic to the von Neumann algebra generated by the b's. Let's take for the a's the generators of the free goup factor $L(\mathbb{F}_n)$. Can we find any b's with the same $*$-moments, but different from the usual representation, so that they can tell us something about $L(\mathbb{F}_n)$ which we do not see so easily in the representation given by the a's?

We want to relax a bit the above question by noticing that we can give up on the quite restrictive condition that the $*$-moments of a_i must be the same as the $*$-moments of b_i. Namely, it is enough that the von Neumann algebra $\mathcal{M}_i$ generated by a_i is isomorphic (via an isomorphism which preserves the state) to the von Neumann algebra $\mathcal{N}_i$ generated by b_i. If we have this, then this isomorphism gives us an element $\tilde{b}_i$ in $\mathcal{N}_i$ which has the same moments as a_i and which generates $\mathcal{N}_i$. If we have this for each i, then the von Neumann algebra generated by all a_i is isomorphic to the von Neumann algebra generated by all $\tilde{b}_i$, because we still have freeness between $\tilde{b}_1, \dots, \tilde{b}_n$. However, the latter algebra is isomorphic to the von Neumann algebra generated by the $b_1, \dots, b_n$.

Theorem 2. *Consider a W^*-probability space $(\mathcal{M}, \varphi)$ and a W^*-probability space $(\mathcal{N}, \psi)$. Assume that we have vN-subalgebras $\mathcal{M}_1, \dots, \mathcal{M}_n \subset \mathcal{M}$ which generate $\mathcal{M}$ and vN-subalgebras $\mathcal{N}_1, \dots, \mathcal{N}_n \subset \mathcal{N}$ which generate $\mathcal{N}$. Assume that $\mathcal{M}_1, \dots, \mathcal{M}_n$ are free with respect to φ and that $\mathcal{N}_1, \dots, \mathcal{N}_n$ are free with respect to ψ. Furthermore, we assume that, for each $i = 1, \dots, n$, we have a $*$-isomorphism*

$$\kappa_i : \mathcal{M}_i \to \mathcal{N}_i \qquad \text{with } \psi \circ \kappa_i = \varphi.$$

Then the von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

In many cases the generator of $\mathcal{M}_i$ or $\mathcal{N}_i$ is either unitary or selfadjoint, in which case it is quite easy to find out whether we have a κ_i as asked for in the theorem. More general, if $\mathcal{M}_i$ is generated by a normal operator, then it is commutative, thus of the form $L^\infty(\mu)$ for some measure μ and all such $L^\infty(\mu)$ for which the measure μ has no atoms can be transformed into each other by a κ_i as above. Essentially this tells us that the exact form of each generator is not so important as long as they are non-atomic and we have freeness between them. An example of this will be presented in the next section.

1.8. **Fock space models for freeness and free group factors.** We present now another situation where we also have in a canonical way free operators and which plays quite an important role in many investigations in free probability theory. This situation is quite close to the original definition of the free group factors, however, it realizes these von Neumann algebras in a bit different way - by replacing the unitary generators of $L(\mathbb{F}_n)$ by sums of creation and annihilation operators on full Fock spaces (which are selfadjoint operators).

Definition 5. Let $\mathcal{H}$ be a Hilbert space.
1) The *full Fock space over $\mathcal{H}$* is the Hilbert space

$$\mathcal{F}(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n},$$

where Ω is a distinguished unit vector, called *vacuum.*
2) The *vacuum expectation* is the state

$$B(\mathcal{F}(\mathcal{H})) \to \mathbb{C}$$
$$a \mapsto \langle \Omega, a\Omega \rangle.$$

3) For each $f \in \mathcal{H}$ we define the *(left) annihilation operator* $l^*(f)$ and

the *(left) creation operator* $l(f)$ by

$$l^*(f)\Omega = 0$$
$$l^*(f)f_1 \otimes \cdots \otimes f_n = \langle f, f_1\rangle f_2 \otimes \cdots \otimes f_n$$

and

$$l(f)f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n.$$

One should note that in different communities there are different conventions in regard of whether we denote by $l(f)$ the annihilation or the creation operator. We follow here the usual operator theoretic convention which is also used by Voiculescu; namely the creation operator is an isometry, thus it is the basic object denoted by l whereas the annihilation operator is a co-isometry, getting the *. (Let me point out that in most of my papers I use the opposite convention, following the physical or quantum probabilistic dictum that a creation operator should get a *.) In any case, $l^*(f)$ is the adjoint of $l(f)$ for any $f \in \mathcal{H}$.

Direct checking shows now that orthogonality of vectors in $\mathcal{H}$ translates into freeness between the corresponding creation and annihilation operators.

Proposition 4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and put $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. Consider the full Fock space over $\mathcal{H}$ and the corresponding creation and annihilation operators $l(f)$ and $l^*(f)$ for $f \in \mathcal{H}$. Put*

$$\mathcal{A}_1 := *\text{-algebra generated by } l(f) \text{ for all } f \in \mathcal{H}_1$$

and

$$\mathcal{A}_2 := *\text{-algebra generated by } l(f) \text{ for all } f \in \mathcal{H}_2$$

Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are free with respect to the vacuum expectation.

The key observation for the proof of this is that for $a_1, a_2, \dots, a_n$ which come alternatingly from $\mathcal{A}_1$ and $\mathcal{A}_2$ and for which $\langle \Omega, a_i\Omega\rangle = 0$ for all $i = 1, \dots, n$, we have

$$a_1a_2 \cdots a_n\Omega = a_1\Omega \otimes a_2\Omega \otimes \cdots \otimes a_n\Omega.$$

Note that the vacuum expectation state is not faithful for these algebras, as we have, e.g., that

$$\langle \Omega, l(f)l^*(f)\Omega\rangle = 0.$$

So this realization of freeness might look not so useful for finding realizations of the free group factors. However, if we go over to sums of creation and annihilation operators, things are getting much nicer. Namely, for $f \in \mathcal{H}$, let us denote

$$\omega(f) := l(f) + l^*(f).$$

Then, as above, $\omega(f)$ and $\omega(g)$ are free with respect to the vacuum expectation if f and g are orthogonal. But in this case the vacuum expectation state is also faithful on the von Neumann algebra generated by $\omega(f)$ and $\omega(g)$. Furthermore, each $\omega(f)$ has with respect to the vacuum expectation a very nice distribution (which is one of the basic distributions in free probability theory and about which we will say more in Lecture 2); in particular, the von Neumann algebra generated by a single $\omega(f)$ is non-atomic and thus isomorphic to the von Neumann algebra generated by a canonical generator of a free group factor. Thus the considerations from the last section apply and we get the following realization of free group factors.

Proposition 5. *Let $\mathcal{H}$ be an n-dimensional Hilbert space with orthonormal basis $f_1, \dots, f_n$. Then the von Neumann algebra generated by $\omega(f_1), \dots, \omega(f_n)$ in $B(\mathcal{F}(\mathcal{H}))$ is isomorphic to the free group factor $L(\mathbb{F}_n)$.*

Creation and annihilation operators provide a very important tool for dealing with many aspects of freeness and Fock space constructions were used by Voiculescu for proving many of the basic properties of freeness. I will concentrate more on combinatorial approaches to freeness in the next lecture, but at least I want to point out that in many cases there are alternate routes relying on creation and annihilation operators.

Let me finally remark that the above realization of the free group factors in terms of sums of creation and annihilation operators instead of the free group generators is, though quite nice, still quite close to the original one and does not provide any direct new insight into the structure of the free group factors. In order to gain more insight, we first have to understand freeness better, allowing us to obtain really non-standard realizations.

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