FREE PROBABILITY THEORY

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Lecture 4
Applications of Freeness to Operator Algebras

Now we want to see what kind of information the idea can yield that free group factors can be realized by random matrices. I want to present what free probability can tell us about the fundamental group of $L(\mathbb{F}_n)$. Let me first recall that notion.

4.1. **Fundamental group of a II_1-factor.** Let $\mathcal{M}$ be a II_1-factor with trace τ and $0 < \lambda \leq 1$. Then there exists a projection $p \in \mathcal{M}$ with $\tau(p) = \lambda$. Put now

$$p\mathcal{M}p := \{pap \mid a \in \mathcal{M}\}.$$

This is called a *compression of* $\mathcal{M}$; it is again a von Neumann algebra, with p as unit, and represented on the Hilbert space $p\mathcal{H}$ if $\mathcal{M} \subset B(\mathcal{H})$. Furthermore, $p\mathcal{M}p$ is also a factor and

$$\tilde{\tau} : p\mathcal{M}p \to \mathbb{C}$$

with

$$\tilde{\tau}(pap) := \frac{1}{\lambda}\tau(pap)$$

is a trace; thus $p\mathcal{M}p$ is a II_1-factor.

By basic theory for equivalence of projections it follows that $p\mathcal{M}p$ only depends (as isomorphism class of von Neumann algebras) on λ, but not on the specific p. Thus we can denote

$$\mathcal{M}_\lambda := p\mathcal{M}p \qquad \text{for projection } p \in \mathcal{M} \text{ with } \tau(p) = \lambda.$$

One can also define $\mathcal{M}_\lambda$ for $\lambda > 1$; in particular, for $\lambda = 1/n$ this are just the $n \times n$-matrices over $\mathcal{M}$ and one has that $\mathcal{N} = \mathcal{M}_{1/n}$ is equivalent to $\mathcal{M} = M_n(\mathcal{N})$.

Definition 1. The *fundamental group* of $\mathcal{M}$ consists of those $\lambda > 0$ for which $\mathcal{M}_\lambda \cong \mathcal{M}$. This is a multiplicative subgroup of $\mathbb{R}_+$.

Research supported by a Discovery Grant and a Leadership Support Inititative Award from the Natural Sciences and Engineering Research Council of Canada and by a Premier's Research Excellence Award from the Province of Ontario.

Lectures at the "Summer School in Operator Algebras", Ottawa, June 2005.

The fundamental group (introduced by Murray and von Neumann) is one of the few invariants which one has for von Neumann algebras. It is usually quite hard to calculate it.

If one wants to see a concrete example for a compression, here is one in the type I_3 world of 3×3-matrices M_3; of course, there $\tau(p)$ can only take on the values $1/3, 2/3, 3/3$. Let's take

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$pM_3p = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cong M_2.$$

For the hyperfinite factor R, it was shown by Murray and von Neumann that its fundamental group is all of $\mathbb{R}_+$.

4.2. **The fundamental group of $L(\mathbb{F}_n)$.** For the free groups factors nothing was known about its compressions or fundamental group before the work of Voiculescu. Voiculescu showed that compressions of free group factors are again free group factors, more precisely

Theorem 1. *Let k and n be natural numbers with $k, n \geq 2$. Then we have*

$$\big(L(\mathbb{F}_n)\big)_{1/k} \cong L(\mathbb{F}_{k^2(n-1)+1}).$$

As a concrete example take $n = 3$ and $k = 2$, then this theorem claims that

$$\big(L(\mathbb{F}_3)\big)_{1/2} \cong L(\mathbb{F}_9).$$

Before we show the proof of this, I want to indicate its implications.

Firstly, this theorem is also valid for $n = \infty$, in which case $k^2(n - 1) + 1 = \infty$, too, and thus

$$L(\mathbb{F}_\infty)_{1/k} \cong L(\mathbb{F}_\infty).$$

This tells us that the fundamental group of $L(\mathbb{F}_\infty)$ contains the numbers of the form $1/k$, and thus, since it is a group, all positive rational numbers. This argument could be refined by Radulescu to determine the fundamental group in this case.

Theorem 2. *The fundamental group of $L(\mathbb{F}_\infty)$ is $\mathbb{R}_+$.*

How about the case $n < \infty$? Since we do not know whether the free group factors with different number of generators are isomorphic or not, we cannot decide whether the compression $(L(\mathbb{F}_n)_{1/k}$ is isomorphic to

$L(\mathbb{F}_n)$ or not. However, we see that we can connect by compression different $L(\mathbb{F}_n)$ with each other, which means that one isomorphism between free group factors will imply other isomorphisms. For example, the theorem above tells us that

$$\big(L(\mathbb{F}_3)\big)_{1/2} \cong L(\mathbb{F}_9) \qquad \text{and} \qquad \big(L(\mathbb{F}_2)\big)_{1/2} \cong L(\mathbb{F}_5).$$

But this means that

$$L(\mathbb{F}_3) \cong L(\mathbb{F}_2) \qquad \Longrightarrow \qquad L(\mathbb{F}_9) \cong L(\mathbb{F}_5).$$

This analysis was refined by Dykema and Radulescu, resulting in the following dichotomy.

Theorem 3. *We have exactly one of the following two possibilities:*

- *Either all $L(\mathbb{F}_n)$ for $2 \le n < \infty$ are isomorphic; in this case the fundamental group of $L(\mathbb{F}_n)$ is $\mathbb{R}_+$*
- *or the $L(\mathbb{F}_n)$ are pairwise not isomorphic; in this case the fundamental group of each $L(\mathbb{F}_n)$ consists only of $\{1\}$.*

An important ingredient of this analysis was to define $L(\mathbb{F}_r)$ also for non-integer r, in such a way that the formula

$$\big(L(\mathbb{F}_r)\big)_{1/k} \cong L(\mathbb{F}_{k^2(r-1)+1})$$

remains also true in general.

4.3. **Writing semi-circular elements as 2×2-matrices.** I will present in the following the main ideas of the proof for the compression result

$$\big(L(\mathbb{F}_n)\big)_{1/k} \cong L(\mathbb{F}_{k^2(n-1)+1}).$$

For concreteness, I will restrict to the case $n = 3$ and $k = 2$.

How can we understand the compression $\mathcal{M} := \big(L(\mathbb{F}_3)\big)_{1/2}$. Since this is the same as

$$L(\mathbb{F}_3) = M_2(\mathcal{M}),$$

we should try to realize $L(\mathbb{F}_3)$ as 2×2-matrices.

Let us first take a look on the generators. If we realize $L(\mathbb{F}_3)$ in its original form, then the generators are given by the generators f_1, f_2, f_3 of the free group $\mathbb{F}_3$ and it is not clear at all what a compression of such an element by a factor $1/2$ should be or how we should write this as a 2×2-matrix. But let us now shift to the picture that $L(\mathbb{F}_3)$ is generated by 3 free semi-circulars. If we think of semi-circulars as being the sum of creation and annihilation operator then, again, we do not have a good idea what compressing this means. However, we know from the last lecture that a semi-circular can also be realized, asymptotically, by random matrices. And the compression of a matrix is something much more familiar. So think of a semi-circular as a big Gaussian $N \times N$-random matrix

$$s \sim \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \dots & \ddots & \dots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix}$$

and then of course it is clear how to write this as a 2×2-matrix, namely just cut this matrix into 4 $N/2 \times N/2$-matrices,

$$s \sim \left(\begin{array}{ccccccc} a_{11} & \cdots & a_{1,N/2} & | & a_{1,N/2+1} & \cdots & a_{1,N} \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ a_{N/2,1} & \cdots & a_{N/2,N/2} & | & a_{N/2,N/2+1} & \cdots & a_{N/2,N} \\ --- & -- & ---- & -- & ---- & -- & --- \\ a_{N/2+1,1} & \cdots & a_{N/2+1,N/2} & | & a_{N/2+1,N/2+1} & \cdots & a_{N/2+1,N} \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ a_{N,1} & \cdots & a_{N,N/2} & | & a_{N,N/2+1} & \cdots & a_{N,N} \end{array} \right)$$

Compressing this by $1/2$ means to take the upper left corner. But this is just another Gaussian random matrix, thus should asymptotically give another semi-circular element s_1. Thus the compression of a semi-circular should just be another semi-circular. For the compression of the algebra, and not just of the generators, we also have to understand the other three parts in the 2×2-realization of s. But this is also quite clear. The lower right corner is just another Gaussian random matrix which is independent from the upper left corner, thus aymptotically it should give a semi-circular s_2 which is free from s_1.

4.4. **Circular elements.** How about the off-diagonal parts. They are also Gaussian random matrices, however, without being symmetric - all their entries are independent from each other (and of course, the lower left corner is the adjoint of the upper right corner). If one realizes that one can write such a non-symmetric Gaussian random matrix C as

$$C = S_1 + iS_2,$$

with independent symmetric Gaussian random matrices S_1 and S_2 (note that, for x and y complex independent Gaussian random variables, one has that $x + iy$ and $\bar{x} + i\bar{y}$ are also independent), then it is clear that in the limit $N \to \infty$ the upper right corner should converge to a circular element in the following sense.

Definition 2. A *circular element* c is of the form

$$c = \frac{s_1 + is_2}{\sqrt{2}},$$

where s_1 and s_2 are free semi-circular elements. (As before, all our semi-circulars are normalized to variance $\varphi(s^2) = 1$.)

A circular element is the non-normal relative of a semi-circular in the same way as a complex Gaussian is the complex relative of a real Gaussian.

In terms of creation and annihilation operators on the full Fock space, a circular element is given by

$$c = l(g) + l^*(f), \qquad c^* = l^*(g) + l(g),$$

where g and f are orthonormal vectors.

In terms of cumulants, a circular element is characterized by

$$k_n(c^{\varepsilon_1}, \dots, c^{\varepsilon_n}) = 0 \qquad \text{if } n \neq 2$$

for all $\varepsilon_1, \dots, \varepsilon_n \in \{*, 1\}$ and by

$$k_2(c, c) = k_2(c^*, c^*) = 0, \qquad k_2(c, c^*) = k_2(c^*, c) = 1.$$

This follows directly by the corresponding desciption of cumulants for semi-circulars and the fact that mixed cumulants in free variables vanish.

Coming back to our random matrix picture of a semi-circular s, cutting this into a 2×2-matrix should result in the following realization of a semi-circular:

$$s = \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix},$$

where s_1 and s_2 are semi-circulars and c is a circular and furthermore s_1, s_2, c are $*$-free.

4.5. **Writing free group factors as 2×2-matrices.** It is also clear from our random matrix picture that if we want to realize 3 free semi-circulars in this form then all the entries from the various matrices should be free. So we might want to think of our free group factor $L(\mathbb{F}_3)$ as being generated by three 2×2-matrices

$$\begin{pmatrix} s_1 & c_1 \\ c_1^* & s_2 \end{pmatrix}, \quad \begin{pmatrix} s_3 & c_2 \\ c_2^* & s_4 \end{pmatrix}, \quad \begin{pmatrix} s_5 & c_3 \\ c_3^* & s_6 \end{pmatrix},$$

with $s_1, \dots, s_6$ semi-circulars, c_1, c_2, c_3 circulars, and all of them $*$-free. However, if we want to compress our von Neumann algebra by the projection

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then this projection has to belong to the von Neumann algebra, which is not clear in the above representation. Thus we replace the third generator in our realization of $L(\mathbb{F}_3)$ in the following way.

$$\begin{pmatrix} s_1 & c_1 \\ c_1^* & s_2 \end{pmatrix}, \quad \begin{pmatrix} s_3 & c_2 \\ c_2^* & s_4 \end{pmatrix}, \quad \begin{pmatrix} u & 0 \\ 0 & 2u \end{pmatrix},$$

where u is a Haar unitary which is free from the other entries. Note that in the random matrix picture we can approximate this third generator by diagonal matrices where we approximate the uniform distribution on the circle of radius 1 in the upper half of the diagonal and the uniform distribution on the circle of radius 2 in the lower half of the diagonal. The asymptotic freeness between Gaussian random matrices and constant matrices yields then that this third generator is indeed free from the other two, thus we have three free generators. Since each of them generates a non-atomic commutative von Neumann algebra, we know (by the remarks at the end of lecture 1) that all together generate $L(\mathbb{F}_3)$. Of course, we could take any non-atomic distribution as diagonal entries for our third matrix, but choosing it of the form as above will be crucial for some of our coming arguments.

Let us state precisely what our random matrix realization of freeness gives us.

Theorem 4. *Let $(\mathcal{M}, \varphi)$ be a non-commutative W^*-probability space. Let $s_1, s_2, s_3, s_4, c_1, c_2, u \in \mathcal{M}$ be such that*

- *s_1, s_2, s_3, s_4 are semi-circular elements*
- *c_1, c_2 are circular elements*
- *u is a Haar unitary element*
- *$s_1, s_2, s_3, s_4, c_1, c_2, u$ are $*$-free*

Then we consider in the non-commutative W^*-probability space

$$(M_2(\mathcal{M}), tr \otimes \varphi)$$

the three random variables

$$X_1 := \begin{pmatrix} s_1 & c_1 \\ c_1^* & s_2 \end{pmatrix}, \quad X_2 := \begin{pmatrix} s_3 & c_2 \\ c_2^* & s_4 \end{pmatrix}, \quad X_3 := \begin{pmatrix} u & 0 \\ 0 & 2u \end{pmatrix}$$

We have that X_1, X_2, X_3 are free and they generate a von Neumann algebra $\mathcal{N} \subset M_2(\mathcal{M})$ which is isomorphic to the free group factor $L(\mathbb{F}_3)$.

This representation is the crucial starting point for considering the compression $L(\mathbb{F}_3)_{1/2}$. Note that we provided above the main steps for a proof of this theorem with the help of our asymptoptic random matrix pictures; however, there exist by now also purely combinatorial proofs of that theorem without having to rely on random matrices. Thus, the random matrix picture of freeness is not absolutely necessary for proving the compression results, however, without it one would hardly have guessed the above theorem.

Let us now see how we can use the above realization of $L(\mathbb{F}_3) \cong \mathcal{N}$ for getting its compression.

First note that the third generator X_3 provides us with

$$X_3X_3^* = \begin{pmatrix} uu^* & 0 \\ 0 & 4uu^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

as an element in $\mathcal{N}$, thus also its spectral projection

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

belongs to $\mathcal{N}$. With the help of P we can cut out the blocks in the 2×2-matrices, i.e., the following elements are in $\mathcal{N}$:

$$\begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} s_3 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 0 & s_4 \end{pmatrix}, \quad \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}.$$

Since we can recover X_1, X_2, X_3 from these elements, we see that $\mathcal{N}$ is generated by the above 8 elements (which are, of course, not free).

Now we are going to look on the compression

$$\big(L(\mathbb{F}_3)\big)_{1/2} \cong P\mathcal{N}P.$$

Of course, the compressed algebra is not just generated by the compressed generators; nevertheless, we can get a generating system by the following observation.

Proposition 1. *Let $\mathcal{N} \subset M_2(\mathcal{M})$ be a von Neumann algebra which is generated by elements $Y_1, \dots, Y_p \in \mathcal{N}$. Assume that we have elements $P, V \in \mathcal{N}$ of the form*

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix},$$

where $v \in \mathcal{M}$ is a unitary element. Then, $P\mathcal{N}P$ is generated by elements of the form ($i = 1, \dots, p$)

$$PY_iP, \quad PY_iV^*, \quad ,VY_iP, \quad VY_iV^*.$$

Note that all listed elements are of the form

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

The proof of the proposition relies just on the simple fact that

$$PP + V^*V = 1.$$

In order to apply this we need the element V. This we get by polar decomposition of

$$C_1 := \begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix}.$$

If we denote the polar decomposition of c_1 by

$$c_1 = v_1 b_1,$$

then we get the polar decomposition of C_1 in the form

$$\begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}.$$

Since polar decomposition stays within a von Neumann algebra we can replace the generator C_1 of $\mathcal{N}$ by the two elements

$$\begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}.$$

In the same way we replace the generator

$$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$$

by its polar decomposition

$$\begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Thus $\mathcal{N}$ is also generated by the 10 elements

$$\begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix}, \quad \begin{pmatrix} s_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & s_4 \end{pmatrix}, \quad \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix},$$

$$\begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}.$$

4.6. **Polar decomposition of circular elements.** This representation is, of course, only useful if we are able to control the distribution of the v's and b's coming from the polar composition of the c's. This is possible since there is a very nice polar decomposition of circular elements.

Theorem 5. *Let $(\mathcal{M}, \varphi)$ be a W^*-probability space and $c \in \mathcal{M}$ a circular element. Let $c = vb$ be the polar decomposition of c. Then we have:*

- *v and b are $*$-free*

- *v is a Haar unitary element*
- *$b \geq 0$ is a quarter circular element, i.e.,*

$$\varphi(b^n) = \frac{1}{\pi} \int_0^2 t^n \sqrt{4 - t^2} dt \qquad \textit{for all } n \geq 0.$$

For proving this one should note that it is enough to prove the reverse implication, namely that whenever we have a Haar unitary $\tilde{v}$ and a quarter circular $\tilde{b}$ which are $*$-free, then $\tilde{c} = \tilde{v}\tilde{b}$ is circular. (If we have this then we argue as follows: Since the two circulars c and $\tilde{c}$ have the same $*$-moments, there is an isomorphism between their generated von Neumann algebras which preserves the $*$-moments. Furthermore, the polar decomposition is unique and does not lead out of the von Neumann algebra of the circular element, thus the two polar decompositons must be mapped onto each other, i.e., v and b have the same joint $*$-moments as $\tilde{v}, \tilde{b}$.) The fact that $\tilde{c}$ is circular can be proved either by random matrix models or directly with combinatorial methods.

4.7. **Calculation of the free compression.** Having this polar decomposition of circular elements we can continue our analysis of $\mathcal{N}$ with the following information about the random variables showing up in the above 10 generators: s_1, s_2, s_3, s_4 are semi-circulars, u, v_1, v_2 are Haar unitaries, b_1, b_2 are quarter-circulars, and all elements s_1, s_2, s_3, s_4, u, v_1, v_2, b_1, b_2 are $*$-free. Now, finally, we have all ingredients for looking on the compression $P\mathcal{N}P$. For the V from the above proposition about a generating set for $P\mathcal{N}P$ we can use

$$V = \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \in \mathcal{N}$$

and thus we get the following set of generators for $P\mathcal{N}P$:

$$\begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} v_1 s_2 v_1^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} s_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} v_1 s_4 v_1^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} v_1 u v_1^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} v_1 b_1 v_1^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} v_2 v_1^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} v_1 b_2 v_1^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that $PVV^* = P$ does not produce an additional generator.

The von Neumann subalgebra of $M_2(\mathcal{M})$ generated by the above 9 elements is of course isomorphic to the von Neumann subalgebra of $\mathcal{M}$ generated by the 9 elements

$$s_1, \quad v_1 s_2 v_1^*, \quad s_3, \quad v_1 s_4 v_1^*, \quad u, \quad v_1 u v_1^*, \quad v_1 b_1 v_1^*, \quad v_2 v_1^*, \quad v_1 b_2 v_1.$$

Each of these elements is either a semi-circular, quarter-circular, or Haar unitary element, thus each of them generates a non-atomic commutative von Neumann algebra. So it only remains to see that all these 9 elements are $*$-free, in order to conclude that all together generate the free group factor $L(\mathbb{F}_9)$. This freeness property is not directly obvious, but it can be checked without greater difficulties by using the very definition of freeness

So we finally have proved that

$$\big(L(\mathbb{F}_3)\big)_{1/2} \cong L(\mathbb{F}_9).$$

Of course, the general case

$$\big(L(\mathbb{F}_n)\big)_{1/k} \cong L(\mathbb{F}_{k^2(n-1)+1})$$

can be done in exactly the same way.

4.8. **The hyperinvariant subspace problem.** The idea that properties of operator algebras or operators can be understood by modelling them by random matrices is not only useful for investigating the structure of the free group factors, but it has a much wider applicability. Let me say a few words about recent progress on another important operator theoretic problem – the hyperinvariant subspace problem – relying on this idea.

Let me first recall that the invariant subspace problem asks whether every operator on a Hilbert space has a non-trivial closed invariant subspace, i.e., a closed subspace, different from $\{0\}$ and from the whole Hilbert space, which is left invariant by the operator. For finite dimensional Hilbert spaces this is of course always true, because there exists always at least one eigenvalue of a matrix (as zero of the characteristic polynomial). Also for normal operators the answer is affirmative (by the spectral theorem); however, for non-normal operators the situation is not so clear any more and a general solution has eluded the efforts of many. (Note that for the same problem in a Banach space setting there are counter-examples by Enflo and Read.)

There exists als a II_1-version of the invariant subspace problem. Namely, assume that we have a II_1-factor $\mathcal{M}$. We say that an operator a in $\mathcal{M}$ has a *hyperinvariant* subspace, if it has an invariant subspace and the projection onto this subspace belongs to the von Neumann algebra generated by a (this means that the subspace is not just accidentially a subspace, but is really affiliated with the operator). The *hyperinvariant subspace problem* asks whether every operator in a II_1-factor has a non-trivial closed hyperinvariant subspace.

Recently, there has been some impressive progress on this problem by Haagerup, relying on free probability and random matrix approximation techniques. Let me just give you some very basic idea about this work.

Essentially, there are two directions one can follow - either looking for canonical candidates for counter examples for this conjecture or trying to prove it for classes of operators as large as possible. There was work of Haagerup in both directions.

4.9. **Possible counter examples and DT-operators.** Free probability provides some new classes of non-normal operators; the most basic of which is the circular element. It was tempting to hope that a circular element might have no hyperinvariant subspaces. However, Dykema and Haagerup could show that actually there exist a lot of hyperinvariant subspaces of a circular element. However, the proof of this is not obvious and relies on some non-standard realization of a circular element.

Note that asking for a hyperinvariant subspace of an operator a is – by decomposing the Hilbert space into a sum of the invariant subspace and its orthogonal complement – the same as realizing a as a 2×2-matrix in triangular form

$$a = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix},$$

such that the projection onto the subspace belongs to the von Neumann algebra generated by a. (Note that for non-normal operators the orthogonal complement of an invariant subspace does not need to be invariant.)

Think of a circular element as the limit of non-selfadjoint Gaussian random matrices, bring such a Gaussian random matrix into a triangular form (and keep track of what this means for the joint distribution of the entries), cut the triangular matrix into a 2×2-matrix and see what this gives in the limit $N \to \infty$.

Doing all this rigorously is quite a non-trivial task, but finally it shows that one can realize a circular element also in the form

$$c = \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix},$$

where a_1 and b are circular elements and a_2 is a special R-diagonal element (this is a kind of generalization of circular elements, introduced by Nica and myself) and where a_1, b, a_2 are $*$-free.

As for the 2×2-matrix realizations which we used for the compression of the free group factors, one can also prove the above result purely combinatorially (this was done by Sniady and myself); however, this is also quite complicated. The above non-symmetric realization of the circular element is of a much more complicated nature than the symmetric realization of the semi-circular which we used before.

Note that the above realization does not tell directly that we have a hyperinvariant subspace, because one still has to see that the projection onto this space belongs to the von Neumann algebra generated by c. By examing the spectral properties of the elements appearing in the above 2×2-matrix representation more closely, Dykema and Haagerup could finally show that circular operators possess a one-parameter family of hyperinvariant subspaces.

They went even further, showed that a circular operator is strongly decomposable in the sense of local spectral theory and introduced in the course of this a new class of non-normal operators as generalizations of a circular element. These so-called DT-operators are inspired by random matrix models. DT stands for diagonal + (Gaussian upper) triangular, and these operators are defined as limits in $*$-moments of certain triangular random matrices. First, let T_N be an upper triangular Gaussian random matrix, i.e., $T_N = (t_{ij})_{i,j=1}^N$ with $t_{ij} = 0$ whenever $i \geq j$ and the other, non-vanishing entries form an independent family of random variables, each of which is normally distributed with mean 0 and variance $1/N$. Then, as $N \to \infty$, the $*$-moments of T_N converge to a limit distribution, which we will denote by T. Next, one can consider in addition to T_N also a diagonal operator with prescribed eigenvalue distribution. Namely, let μ be a compactly supported Borel probability measure in the complex plane and let D_N be diagonal random matrices whose diagonal entries are independent and identically distributed according to μ. Furthermore, let D_N and T_N be independent. Then the pair D_N, T_N converges jointly in $*$-moments as $N \to \infty$ (however, to establish the existence of these limits and to describe them in combinatorial terms is a quite non-trivial task). A DT-element is now an element Z whose $*$-moments are given as the limit of the corresponding $*$-moments of $Z_N := D_N + cT_N$ (for some $c > 0$). The limiting distribution of Z_N depends (in an intricate way) only on c and the distribution μ.

The importance of the class of DT-operators (even though it did in the end not produce counter examples to the hyperinvariant subspace problem) lies in the fact that they form a beautiful class of non-normal operators which is quite non-trivial, but still accessible. In particular, the quasi-nilpotent operator T (as a quite unconventional generator of the free group factor $L(\mathbb{F}_2)$) has attracted much attention, and quite a lot of interesting mathematics has been created around the understanding of its properties.

4.10. **Direct attack on the hyperinvariant subspace problem.** In the other direction Haagerup could show that large classes of operators possess hyperinvariant subspaces. His first approach to this relied on random matrix approximations. The rough idea is the following. Consider some operator a in a II_1-factor and assume that we can approximate it in $*$-moments by matrices D_N (it is an open problem whether one can always do this – this is Connes's well-known embedding problem of II_1-factors into the ultraproduct of the hyperfinite factor). Then we have non-trivial invariant subspaces for the D_N and one might hope that something of these survives in the limit. Of course, this is not the case in general, there is no reason that the projections onto the subspaces for the D_N should converge in any sense to anything in the limit.

This reflects a general problem with our type of converge in $*$-moments. Most interesting operator theoretic properties are not continuous with respect to this convergence. (Clearly; otherwise the theory of operators would not be so much more complicated – and interesting – than the theory of matrices.) However, what seems to be the case is that if we have ensembles of matrices approximating a, then some properties are transfered to the limit for generic choices of the approximating matrices. To put it another way: If we have only one sequence of matrices approximating a in the limit, then this sequence might be choosen so badly, that it does not tell us much about the limit. However, if we have many sequences approximating a, then we have a much better chance that a randomly choosen sequence from our ensemble of possibilities will tell us something interesting about the limit.

This is of course not a rigorous theorem (not even a precise statement), but it gives some idea what one might try to do. What Haagerup did was the following. Assume we have a sequence of matrices D_N approximating a in $*$-moments. Then there is no reason that subspaces or eigenvalue distributions of D_N converge to corresponding quantities for a. However, one can try to perturb D_N a bit by a small random matrix, to make it more generic and improve on its spectral approximation properties, however, in such a way that we do not destroy the convergence in $*$-moments. Haagerup did this by passing from D_N to

$$D'_N := D_N + \varepsilon_N X_N Y_N^{-1},$$

where X_N, Y_N are independent non-selfadjoint Gaussian random matrices and where $\lim_{N\to\infty} \varepsilon_N = 0$. (Subsequently Sniady proved that one can also use a random distortion by just one Gaussian random matrix instead of the ratio of two.) Haagerup could show that this allows indeed in many cases to pass spectral properties from matrices to operators.

More recently, Haagerup has developed another approach to this (together with Hanne Schultz), which avoids random matrices (and thus also Connes's embedding problem) and works more directly in the limit by developing a theory of integration through the spectrum. Again a perturbation argument as above plays an important role, this time by adding a free perturbation.

The final result is the following: If a is an operator in a general II_1-factor $\mathcal{M}$, then for every Borel set $B \subset \mathbb{C}$, there is a unique closed a-invariant subspace $\mathcal{K}$ affiliated with $\mathcal{M}$, such that with respect to the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$, a has the form

$$a = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix},$$

where the Brown measures (a generalization of eigenvalue distributions for matrices to operators in II_1-factors) of a_{11} and a_{22} are concentrated on B and $\mathbb{C}\backslash B$, respectively. In particular, if the Brown measure of a is not a Dirac measure, then a has a non-trivial hyperinvariant closed subspace.

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