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Random Matrices and Combinatorics

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1. Introduction

The notion of free independence was introduced by Voiculescu in 1983 in the context of operator algebras, giving rise to free probability theory. In 1991, Voiculescu discovered that this notion of freeness also appeared in the context of random matrices. The latter had already been a subject of investigation in statistics (Wishart, 1928) and physics (Wigner, 1955) for quite some time. One of the basic results in random matrix theory was Wigner's discovery that the eigenvalue distribution of a *Gaussian unitary ensemble* is asymptotically given by the semicircular law. Since the semicircle law is also the limit in the free version of a central limit theorem, this pointed to a connection between free probability theory and random matrices. We will first present a short introduction to random matrices and show Wigner's semicircle law, and then switch to the free probability side and show that the semicircle shows also up as the limit in a free central limit theorem. This motivated Voiculescu to look for a deeper relation between random matrices and asymptotic freeness. We will present a few examples of this connection in the final Section 6. However, before coming to this, we will give a more thorough treatment of the combinatorial structure of free probability theory, based on the lattice of non-crossing partitions and the notion of free cumulants.

2. Gaussian Random Matrices and Wigner's Semicircle Law

Definition 2.1. Let $(\Omega, \mathbb{P})$ be a classical probability space. A *random matrix* is a matrix $A = (a_{ij})_{i,j=1}^N$ where the entries $a_{ij} : \Omega \to \mathbb{C},\ i, j = 1, \ldots, N$ are classical random variables. The corresponding non-commutative probability space $(\mathcal{A}, \varphi)$ of $N \times N$ - random matrices is given by

$$\mathcal{A} = \mathcal{M}_N(L^{\infty-}(\Omega, \mathbb{P})) = \mathcal{M}_N(\mathbb{C}) \otimes L^{\infty-}(\Omega, \mathbb{P})$$

and

$$\varphi = \operatorname{tr} \otimes \mathbb{E},$$

where

$$L^{\infty-}(\Omega, \mathbb{P}) = \bigcap_{1 \le p < \infty} L^p(\Omega, \mathbb{P})$$

denotes the space of random variables for which all moments exist. Moreover, tr denotes the normalized trace on $\mathcal{M}_N(\mathbb{C})$ and $\mathbb{E}$ denotes the expectation on $(\Omega, \mathbb{P})$. Hence we have $A = (a_{ij})_{i,j=1}^N \in \mathcal{A}$ if and only if $a_{ij} \in L^{\infty-}(\Omega, \mathbb{P})$ for all $i, j = 1, \dots, N$ and

$$\varphi(A) = \mathrm{tr} \otimes \mathbb{E}(A) = \mathbb{E}[\mathrm{tr}(A)] = \frac{1}{N} \sum_{i=1}^N \mathbb{E}[a_{ii}].$$

Remark 2.2. Consider a selfadjoint random matrix $A = A^* \in \mathcal{A}$, i.e. $a_{ij} = \overline{a}_{ji}$ for all $i, j = 1, \dots, N$. Let $\lambda_1, \dots, \lambda_N$ denote the eigenvalues of A. Then

$$\varphi(A^k) = \mathbb{E}[\mathrm{tr}(A^k)] = \frac{1}{N} \mathbb{E}\Big[\sum_{i=1}^N \lambda_i^k\Big] = \int t^k d\mu_A(t),$$

where

$$\mu_A = \frac{1}{N} \int_\Omega \sum_{i=1}^N \delta_{\lambda_i(\omega)} \; d\mathbb{P}(\omega)$$

denotes the *averaged eigenvalue distribution* of A. In other words, the set of moments of A w.r.t. φ corresponds to the analytic object μ_A.

Definition 2.3. A *(selfadjoint) Gaussian random matrix* is a random matrix $A = (a_{ij})_{i,j=1}^N$ where

- $A = A^*$, i.e. $a_{ij} = \overline{a}_{ji}$ for all $i, j = 1, \dots, N$,
- a_{ij} $(1 \le i \le j \le N)$ are independent complex Gaussian random variables with

$$\begin{aligned} \mathbb{E}[a_{ij}] &= 0, \\ \mathbb{E}[a_{ij}^2] &= 0 \quad (i \ne j), \\ \mathbb{E}[a_{ij} a_{ji}] = \mathbb{E}[a_{ij} \overline{a}_{ij}] &= \frac{1}{N}. \end{aligned}$$

Remark 2.4. Such random matrices are addressed as Gaussian unitary ensemble (GUE). "Unitary" refers to the fact that the distribution of the entries of A is invariant under unitary conjugations.

We want now to calculate $\varphi(A^m)$ for a GUE. For this we need expectations of products of entries, which form a Gaussian family in the following sense:

Definition 2.5. Random variables $x_1, \dots, x_n$ form a *Gaussian family*, if for all $m \in \mathbb{N}$ and for all $1 \le i(1), \dots, i(m) \le n$:

$$\mathbb{E}[x_{i(1)} \dots x_{i(m)}] = \sum_{\pi \in \mathcal{P}_2(m)} \prod_{(r,s) \in \pi} \mathbb{E}[x_{i(r)} x_{i(s)}],$$

where $\mathcal{P}_2(m)$ denotes the set of pair-partitions of m elements (i.e., the decomposition of the set $\{1, \dots, m\}$ into disjoint pairs). This combinatorial formula, which expresses all higher moments of a Gaussian family in terms of second moments, is usually called the *Wick formula*.

It might be interesting to note that, while the work of Wick (in physics) is from 1950, the same formula was also shown by Isserlis in 1918 in probability theory - thus the name “Isserlis formula” might also be appropriate.

Example 2.6. Let $x_1, \dots, x_n$ be a Gaussian family. Then for odd m, it follows that $\mathbb{E}[x_{i(1)} \dots x_{i(m)}] = 0$ and for $m = 2$ the Wick-formula yields the trivial identity $\mathbb{E}[x_{i(1)} x_{i(2)}] = \mathbb{E}[x_{i(1)} x_{i(2)}]$. However, for $m = 4$ the set $\mathcal{P}_2(4)$ consists of 3 elements and we have that

$$\begin{aligned}\mathbb{E}[x_{i(1)} x_{i(2)} x_{i(3)} x_{i(4)}] &= \mathbb{E}[x_{i(1)} x_{i(2)}]\mathbb{E}[x_{i(3)} x_{i(4)}] \\ &\quad + \mathbb{E}[x_{i(1)} x_{i(3)}]\mathbb{E}[x_{i(2)} x_{i(4)}] \\ &\quad + \mathbb{E}[x_{i(1)} x_{i(4)}]\mathbb{E}[x_{i(2)} x_{i(3)}].\end{aligned}$$

We note that real i.i.d. Gaussian random variables $x_1, \dots, x_n$ form a Gaussian family with $\mathbb{E}[x_i x_j] = \delta_{ij}\sigma^2$: (i) first note that x_i Gaussian implies

$$\begin{aligned}\mathbb{E}[x_i^m] = \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} t^m e^{-\frac{t^2}{2\sigma^2}}\,dt &= \begin{cases} 0 & m \text{ odd,} \\ \sigma^m (m-1)(m-3)\cdot \dots \cdot 1 & m \text{ even} \end{cases} \\ &= \begin{cases} 0 & m \text{ odd,} \\ \sigma^m \ \#\mathcal{P}_2(m) & m \text{ even;} \end{cases}\end{aligned}$$

(ii) and since $\mathbb{E}[x_i x_j] = \delta_{ij}\sigma^2$, the Wick formula only counts pairings of the same x_i's, hence it factorizes for different ones, corresponding to the independence of different x_i's.

Thus, the entries a_{ij} $(i, j = 1, \dots, N)$ of a GUE $A = (a_{ij})_{i,j=1}^N$ form a Gaussian family, where the second moments are given by

$$\mathbb{E}[a_{ij} a_{kl}] = \frac{1}{N}\delta_{il}\delta_{jk} \qquad \text{for } i, j, k, l = 1, \dots, N.$$

Now we can calculate the moments of the GUE $A = (a_{ij})_{i,j=1}^N$:

$$\begin{aligned}\varphi(A^m) &= \frac{1}{N}\sum_{i(1),\dots,i(m)=1}^{N} \mathbb{E}[a_{i(1)i(2)} a_{i(2)i(3)} \cdots a_{i(m)i(1)}] \\ &= \frac{1}{N}\sum_{i(1),\dots,i(m)=1}^{N} \sum_{\pi\in\mathcal{P}_2(m)} \prod_{(r,s)\in\pi} \mathbb{E}[a_{i(r)i(r+1)} a_{i(s)i(s+1)}] \\ &= \frac{1}{N}\sum_{i(1),\dots,i(m)=1}^{N} \sum_{\pi\in\mathcal{P}_2(m)} \prod_{(r,s)\in\pi} \frac{1}{N}\delta_{i(r)i(s+1)}\delta_{i(r+1)i(s)} \\ &= \frac{1}{N^{1+\frac{m}{2}}}\sum_{\pi\in\mathcal{P}_2(m)} \sum_{i(1),\dots,i(m)=1}^{N} \prod_{r=1}^{m} \delta_{i(r)i(\pi(r)+1)} \\ &= \frac{1}{N^{1+\frac{m}{2}}}\sum_{\pi\in\mathcal{P}_2(m)} \sum_{i(1),\dots,i(m)=1}^{N} \prod_{r=1}^{m} \delta_{i(r)i(\gamma\pi(r))},\end{aligned}$$

where we identify $\pi \in \mathcal{P}_2(m)$ with the permutation $\pi \in S_m$ that switches the places of r and s for $(r,s) \in \pi$ and where we denote the long cycle permutation $(1,2,3,\dots,m-1,m) \in S_m$ by γ. By noting that

$$\sum_{i(1),\dots,i(m)=1}^{N} \prod_{r=1}^{m} \delta_{i(r)i(\gamma\pi(r))} = N^{\#(\gamma\pi)},$$

where $\#(\gamma\pi)$ denotes the number of cycles of $\gamma\pi$, we get the following theorem:

Theorem 2.7. *For an $N \times N$ - GUE random matrix $A = (a_{ij})_{i,j=1}^N$ we have the genus expansion*

$$\varphi(A^m) = \sum_{\pi \in \mathcal{P}_2(m)} N^{\#(\gamma\pi)-1-\frac{m}{2}}.$$

Example 2.8. For $m = 2$, $\mathcal{P}_2(m)$ only consists of the element $\pi = (1,2)$ and we have $\gamma = (1,2)$. Hence $\gamma\pi = e$ and $\#\gamma\pi = 2$ which yields the second moment

$$\varphi(A^2) = N^{2-1-1} = 1.$$

In the case $m = 4$, $\mathcal{P}_2(m)$ contains the elements

$$\pi_1 = (1,2)(3,4), \qquad \pi_2 = (1,3)(2,4), \qquad \pi_3 = (1,4)(2,3)$$

and we have $\gamma = (1,2,3,4)$. Hence

$$\#(\gamma\pi_1) - 3 = 0, \qquad \#(\gamma\pi_2) - 3 = -2, \qquad \#(\gamma\pi_3) - 3 = 0,$$

which yields

$$\varphi(A^4) = 2 + \frac{1}{N^2} \overset{N\to\infty}{\longrightarrow} 2.$$

In the same way we obtain

$$\varphi(A^6) = 5 + \frac{10}{N^2} \overset{N\to\infty}{\longrightarrow} 5$$
$$\varphi(A^8) = 14 + \frac{70}{N^2} + \frac{21}{N^4} \overset{N\to\infty}{\longrightarrow} 14.$$

More general, one has $\#(\gamma\pi) - 1 - \frac{m}{2} \leq 0$ for all $\pi \in \mathcal{P}_2(m)$ and equality holds exactly for the so-called non-crossing π. A pair-partition $\pi \in \mathcal{P}_2(m)$ is crossing if we can find two blocks (r_1, s_1) and (r_2, s_2) of π which cross, i.e., with $r_1 < r_2 < s_1 < s_2$.

Thus, for $N \to \infty$, the moments of a GUE $A = (a_{ij})_{i,j=1}^N$ are given by

$$\lim_{N\to\infty} \varphi(A^m) = \#\mathcal{NC}_2(m),$$

where $\mathcal{NC}_2(m)$ denotes the set of non-crossing pair-partitions.

If we put $c_m = \#\mathcal{NC}_2(2m)$ for $m \in \mathbb{N}$, one can show that

$$c_m = \sum_{k=0}^{m-1} c_k c_{m-k-1},$$

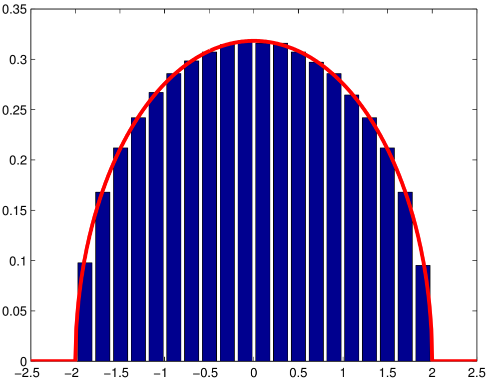


Figure 1. Comparision between the histogram for the 4000 eigenvalues of one realization of a 4000 × 4000 Gaussian random matrix and the semicircle distribution; as the agreement between histogram and semicirle suggests, Wigner's semicircle law does not only hold for the averaged eigenvalue distributions, but also almost surely for generic realizations

which is exactly the recursion for the *Catalan numbers*. Thus,

$$c_m = \frac{1}{m+1}\binom{2m}{m}.$$

One can show that these are exactly the moments for the semicircular law, i.e.

$$c_m = \frac{1}{2\pi}\int_{-2}^{2} t^m \sqrt{4-t^2}dt$$

and hence we get the theorem:

Theorem 2.9 (Wigner's semicircle law)**.** *The asymptotic eigenvalue distribution of a GUE A is given by the semicircle law, i.e.*

$$\lim_{N\to\infty} \mu_A = \mu_S \quad \textit{(weak convergence)},$$

where

$$d\mu_S(t) = \frac{1}{2\pi}\sqrt{4-t^2}dt \quad \textit{on } [-2,2].$$

3. The Free Central Limit Theorem

We will now switch to the free probability side and see that the semicircle distribution appears there also as one of the basic distributions. For this we will look on the free analogue of the central limit theorem. This free central limit theorem was one of the first theorems of Voiculuescu in free probability theory. It was also my entry point into the free world. From the work of my PhD supervisor, Wilhelm von Waldenfels, I was aware of combinatorial approaches to classical and bosonic/fermionic central limit theorems, and I tried to understand in this spirit Voiculescu's result. In the following I will present this combinatorial approach.

In order to illuminate the parallels (and also the differences) between classical and free, we will give a uniform treatment of both the classical and the free central limit theorem.

Consider a sequence $(a_i)_{i=1}^{\infty}$ of elements of a non-commutative probability space $(\mathcal{A}, \varphi)$ which are

- identically distributed,
- centered, i.e. $\varphi(a_1) = 0$,
- normalized, i.e. $\varphi(a_1^2) = 1$,
- either classically independent or freely independent.

Note that we require the existence of moments here.
What can we say about

$$S_N = \frac{a_1 + \cdots + a_N}{\sqrt{N}},$$

when $N \to \infty$?

Definition 3.1. We say that $S_N \in (\mathcal{A}_N, \varphi_N)$ $(N \in \mathbb{N})$ *converges in distribution* to $x \in (\mathcal{A}, \varphi)$, and denote this by $S_N \overset{dist.}{\to} x$, if

$$\lim_{N\to\infty} \varphi_N(S_N^m) = \varphi(x^m)$$

for all $m \in \mathbb{N}$.

Let us see whether we can control the moments of S_N when N goes to ∞. We have

$$\varphi(S_N^m) = \frac{1}{N^{\frac{m}{2}}} \varphi((a_1 + \cdots + a_N)^m) = \frac{1}{N^{\frac{m}{2}}} \sum_{i(1),\dots,i(m)=1}^{N} \varphi(a_{i(1)} \dots a_{i(m)}).$$

For $i = (i(1), \dots, i(m))$ we denote by $\ker(i)$ the maximal partition of $\{1, \dots, m\}$ such that i is constant on blocks. For example, $i = (1, 3, 1, 5, 3)$ and $j = (3, 4, 3, 6, 4)$ have the same kernel. Now we note that by the fact that all our variables are identically distributed and by basic properties of either independence or freeness we have

$$\varphi(a_{i(1)} \dots a_{i(m)}) = \varphi(a_{j(1)} \dots a_{j(m)}),$$

whenever $\ker i = \ker j$ and hence (by denoting by $\mathcal{P}(m)$ the set of all partitions of $\{1,\ldots,m\}$; for a formal definition see Def. 4.1)

$$\begin{aligned}\varphi(S_N^m) &= \frac{1}{N^{\frac{m}{2}}}\sum_{\pi\in\mathcal{P}(m)} \kappa_\pi \; \#\{i : \{1,\ldots,m\}\to\{1,\ldots,N\} \;:\; \ker i = \pi\}\\ &= \frac{1}{N^{\frac{m}{2}}}\sum_{\pi\in\mathcal{P}(m)} \kappa_\pi \; N(N-1)\cdot\cdots\cdot(N-(\#\pi-1))\\ &\sim \sum_{\pi\in\mathcal{P}(m)} \kappa_\pi \; N^{\#\pi-\frac{m}{2}}\end{aligned}$$

for large N, where $\kappa_\pi = \varphi(a_{i(1)}\ldots a_{i(m)})$ if $\ker i = \pi$.

Let π have a singleton, i.e., a block consisting of just one element (meaning that one of the appearing indices is different from all the others.) Then

$$\kappa_\pi = \varphi(a_{i(1)}\ldots a_{i(k)}\ldots a_{i(m)}) = \varphi(a_{i(1)}\ldots a_{i(k-1)}a_{i(k+1)}\ldots a_{i(m)})\varphi(a_{i(k)}) = 0,$$

where $i(k)$ is the index that differs from all the others. Here we used the free/classical independence of the sets $\{a_{i(1)},\ldots,a_{i(k-1)},a_{i(k+1)},\ldots a_{i(m)}\}$ and $\{a_{i(k)}\}$ and the fact that all our variables are centered. Hence $\kappa_\pi\neq 0$ implies that $\pi = \{V_1,\ldots,V_r\}$ with $|V_j|\geq 2$ for all $j=1,\ldots,r$ and thus $r=\#\pi\leq\frac{m}{2}$. Altogether we get

$$\lim_{N\to\infty}\varphi(S_N^m) = \sum_{\substack{\pi\in\mathcal{P}(m)\\ \pi \text{ has no singleton}\\ \#\pi=\frac{m}{2}}} \kappa_\pi = \sum_{\pi\in\mathcal{P}_2(m)}\kappa_\pi,$$

where $\mathcal{P}_2(m)$ denotes, as in the previous section, the set of pair-partitions. In particular it follows that

$$\lim_{N\to\infty}\varphi(S_N^m) = 0$$

for odd m.

Now we want to distinguish the classical and the free case:

1) If we consider the a_i's to be classical (commutative) independent random variables, then we have for even m

$$\kappa_\pi = \varphi(a_{i(1)}\ldots a_{i(m)}) = 1$$

for all $\pi\in\mathcal{P}_2(m)$. Thus we have

$$\lim_{N\to\infty}\varphi(S_N^m) = \#\mathcal{P}_2(m) = \begin{cases}0, & m \text{ odd}\\ (m-1)(m-3)\cdot\cdots\cdot 1, & m \text{ even}\end{cases},$$

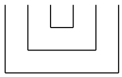
which are exactly the moments of the Gaussian distribution. This proves the classical central limit theorem, in the case where all moments exist.

2) If the a_i's are free, we get

$$\kappa_\pi = \begin{cases}0, & \pi \text{ is crossing}\\ 1, & \pi\in\mathcal{NC}_2(m).\end{cases}$$

For instance, if

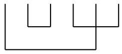
Note to self: Partitionen einfgen

$$\pi = \{(1,6),(2,5),(3,4)\} =$$ 

we obtain

$$\kappa_\pi = \varphi(a_1a_2a_3a_3a_2a_1) = \varphi(a_3a_3)\varphi(a_1a_2a_2a_1) = \varphi(a_3a_3)\varphi(a_2a_2)\varphi(a_1a_1) = 1,$$

and if

$$\pi = \{(1,5),(2,3),(4,6)\} =$$ 

we have

$$\kappa_\pi = \varphi(a_1a_2a_2a_3a_1a_3) = \varphi(a_2a_2)\varphi(a_1a_3a_1a_3) = 0,$$

by definition of freeness and the fact that all $\varphi(a_i) = 0$.
So we get in the limit of the free central limit theorem that the moments are counted by the number of non-crossing pair-partitions; hence the same moments as in the limit of Gaussian random matrices.

Theorem 3.2. *Assume $a_1, a_2, \dots \in (\mathcal{A}, \varphi)$ are free and identically distributed with $\varphi(a_1) = 0$ and $\varphi(a_1^2) = 1$. Then*

$$\frac{a_1 + \cdots + a_N}{\sqrt{N}} \overset{dist.}{\to} s,$$

where s is a semicircular element, i.e.

$$\varphi(s^m) = \frac{1}{2\pi}\int_{-2}^{2} t^m \sqrt{4-t^2}\, dt = \#\mathcal{NC}_2(m).$$

The free central limit theorem can be easily generalized to a multivariate version:

Theorem 3.3. *Let $\{a_1^{(i)} \mid i \in \mathcal{I}\}$, $\{a_2^{(i)} \mid i \in \mathcal{I}\}, \dots \subset (\mathcal{A}, \varphi)$ be a sequence of freely independent with identical distribution with $\varphi(a_r^{(i)}) = 0$ for all $r \in \mathbb{N}$, $i \in \mathcal{I}$. We denote the covariance by*

$$c_{ij} = \varphi(a_r^{(i)} a_r^{(j)}) \quad (i, j \in \mathcal{I}).$$

Then

$$\left(\frac{a_1^{(i)} + \cdots + a_N^{(i)}}{\sqrt{N}}\right)_{i\in\mathcal{I}} \overset{dist.}{\to} (s_i)_{i\in\mathcal{I}},$$

where $(s_i)_{i\in\mathcal{I}}$ is a semicircular family of covariance $(c_{ij})_{i,j\in\mathcal{I}}$, i.e.

$$\varphi(s_{i(1)} \dots s_{i(m)}) = \sum_{\pi\in\mathcal{NC}_2(m)} \prod_{(r,p)\in\pi} c_{i(r)i(p)}$$

for all $m \in \mathbb{N}$.

4. Non-crossing Partitions and Free Cumulants

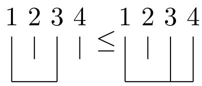
After having realized that the transition from the classical to the free central limit theorem consists, on a combinatorial level, in replacing all pair-partitions by non-crossing pair-partitions it was tempting to try to develop a general approach to free probability theory based on this observation. For this the combinatorial description of classical probability theory in terms of cumulants and the lattice of all partitions, as presented in the work of Gian-Carlo Rota and his coworkers, was instrumental. Motivated by this I developed the following general combinatorial approach to free probability theory, resting on the notion of free cumulants.

Definition 4.1. A *partition* of $\{1, \dots, n\}$ is a collection $\pi = \{V_1, \dots, V_r\}$ of subsets of $\{1, \dots, n\}$ with

- $V_i \neq \varnothing$ for all $i = 1, \dots r$,
- $V_i \cap V_j = \varnothing$ for $i \neq j$ and
- $\bigcup_{i=1}^{r} V_i = \{1, \dots, n\}$.

The V_i's are called the *blocks* of π.
The partition π is *non-crossing* if we do not have $p_1, p_2, q_1, q_2 \in \{1, \dots, n\}$ such that $p_1 < q_1 < p_2 < q_2$ and p_1, p_2 belong to the same block, q_1, q_2 belong to the same block, but those two blocks are different. We denote the set of all partitions of $\{1, \dots, n\}$ by $\mathcal{P}(n)$ and the subset of all non-crossing partitions by $\mathcal{NC}(n)$.

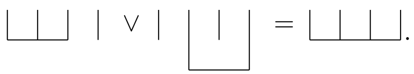
We can define a partial order on $\mathcal{NC}(n)$ by: $\pi_1 \leq \pi_2$ iff each block of π_1 is contained in a block of π_2. For instance, we have



This partial order induces a lattice structure on $\mathcal{NC}(n)$, i.e. for all $\pi, \sigma \in \mathcal{NC}(n)$ there is a minimal partition $\pi \vee \sigma$ that is larger than π and larger than σ (called the *join* of π and σ) and a maximal partition $\pi \wedge \sigma$ that is smaller than π and smaller than σ (called the *meet* of π and σ).

Example 4.2. We have

and



The lattice $\mathcal{NC}(n)$ has the maximal element 1_n consisting of one block of size n and the minimal element 0_n consisting of n blocks of size one.

Definition 4.3. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. The *free cumulants* $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ $(n \geq 1)$ are inductively defined by the moment-cumulant formulas

$$\varphi(a_1 \dots a_n) = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi(a_1, \dots, a_n),$$

where, for $\pi = \{V_1, \dots, V_r\}$,

$$\kappa_\pi(a_1, \dots, a_n) = \prod_{i=1}^{r} \kappa_{|V_i|}((a_j)_{j \in V_i}).$$

Example 4.4. For $n = 1$, we get

$$\kappa_1(a_1) = \varphi(a_1)$$

and for $n = 2$, the moment-cumulant formula yields

$$\varphi(a_1 a_2) = \kappa_2(a_1, a_2) + \kappa_1(a_1)\kappa_2(a_2).$$

Hence

$$\kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2).$$

For $n = 3$, we have five non-crossing partitions and hence κ_3 is determined by

$$\begin{aligned}\varphi(a_1 a_2 a_3) = \kappa_3(a_1, a_2, a_3) &+ \kappa_1(a_1)\kappa_2(a_2, a_3) \\ &+ \kappa_1(a_2)\kappa_2(a_1, a_3) + \kappa_1(a_3)\kappa_2(a_1, a_2) \\ &+ \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3).\end{aligned}$$

One can use an inductive argument to show that κ_n is a n-linear functional. Now we want to have a look at the behaviour of κ_n with respect to products of elements of $\mathcal{A}$. We consider the following example:

$$\begin{aligned}\kappa_2(a_1 a_2, a_3) &= \varphi((a_1 a_2) a_3) - \varphi(a_1 a_2)\varphi(a_3) \\ &= \varphi(a_1 a_2 a_3) - \varphi(a_1 a_2)\varphi(a_3) \\ &= \kappa_3(a_1, a_2, a_3) + \kappa_1(a_1)\kappa_2(a_2, a_3) + \kappa_1(a_2)\kappa_2(a_1, a_3).\end{aligned}$$

We note, that the cumulants appearing in the last equation are exactly the ones that correspond to partitions in $\mathcal{NC}(3)$ that connect the blocks $\{1, 2\}$ and $\{3\}$. This can be generalized to the following result.

Theorem 4.5. *Consider $a_1, \dots, a_n \in \mathcal{A}$ and multiply some of them together to $A_1, \dots, A_m$ $(m \leq n)$ such that $A_1 \cdots A_m = a_1 \cdots a_n$. Then*

$$\kappa_m(A_1, \dots, A_m) = \sum_{\substack{\pi \in \mathcal{NC}(n) \\ \pi \vee \sigma = 1_n}} \kappa_\pi(a_1, \dots, a_n),$$

where i, j belong to the same block of σ if and only if a_i and a_j are factors in the same A_k.

Remark 4.6. We note that the condition $\pi \vee \sigma = 1_n$ appearing in the sum in the last theorem means that one has to consider all partitions π that couple all blocks of σ.

Now we present the main result on free cumulants, which connects them to freeness.

Theorem 4.7. *Let* $(\mathcal{A}, \varphi)$ *be a non-commutative probability space. Unital subalgebras* $\mathcal{A}_1, \dots, \mathcal{A}_s \subset \mathcal{A}$ *are free if and only if all mixed cumulants vanish, i.e.*

$$\kappa_n(a_1, \dots, a_n) = 0$$

whenever $a_j \in \mathcal{A}_{i(j)}$ *for all* $j = 1, \dots, n$ *and there are* $k, l \in \{1, \dots, n\}$ *such that* $i(k) \neq i(l)$*.*

Proof. It is easy to show by induction that $\mathcal{A}_1, \dots, \mathcal{A}_s$ are free whenever mixed cumulants vanish. On the other hand, if $\mathcal{A}_1, \dots, \mathcal{A}_s$ are free, we note that another easy inductive argument shows that $\kappa_n(a_1, \dots, a_n) = 0$ whenever $\varphi(a_i) = 0$ for all $i = 1, \dots, n$ and $i(j) \neq i(j+1)$ for all $j = 1, \dots, n-1$. The difficult part of the proof is to weaken this condition to the one needed in the theorem. To do so, one shows first that $\kappa_n(a_1, \dots, a_n) = 0$ if $1 \in \{a_1, \dots, a_n\}$ and $n \geq 2$. But then we get that

$$\kappa_n(a_1, \dots, a_n) = \kappa_n(a_1 - \varphi(a_1)1, \dots, a_n - \varphi(a_n)1)$$

and hence we can get rid of the condition $\varphi(a_i) = 0$ for all $i = 1, \dots, n$. Therefore $\kappa_n(a_1, \dots, a_n) = 0$ whenever $i(j) \neq i(j+1)$ for all $j = 1, \dots, n-1$. Let there be $k, l \in \{1, \dots, n\}$ such that $i(k) \neq i(l)$. We multiply neighbors from the same algebra together to get elements $A_1, \dots, A_m$ such that $A_1 \cdots A_m = a_1 \cdots a_n$ and A_j, A_{j+1} are from different algebras for all $j = 1, \dots, m-1$. Hence

$$\kappa_m(A_1, \dots, A_m) = 0$$

but we also have

$$\begin{aligned}\kappa_m(A_1, \dots, A_m) &= \sum_{\substack{\pi \in \mathcal{NC}(n) \\ \pi \vee \sigma = 1_n}} \kappa_\pi(a_1, \dots, a_n) \\ &= \kappa_n(a_1, \dots, a_n) + \sum_{\substack{\pi \neq 1_n \\ \pi \vee \sigma = 1_n}} \kappa_\pi(a_1, \dots, a_n).\end{aligned}$$

Now we assume that we know the statement for κ_l, $l < n$. Then we have that $\kappa_\pi(a_1, \dots, a_n) \neq 0$ only for partitions π that couple elements a_i from the same algebra. But as σ also does so and $\pi \vee \sigma = 1_n$, all a_i must be from the same algebra. But this contradicts the condition that there are $k, l \in \{1, \dots, n\}$ such that $i(k) \neq i(l)$. □

Applying the product formula once more, one can also show:

Theorem 4.8. *Let* $(\mathcal{A}, \varphi)$ *be a non-commutative probability space. Elements* $b_1, \dots, b_s \in \mathcal{A}$ *are free if and only if for all* n

$$\kappa_n(b_{i(1)}, \dots, b_{i(n)}) = 0$$

whenever there are $k, l \in \{1, \dots, n\}$ *such that* $i(k) \neq i(l)$*.*

5. Sums and Products of Free Variables

5.1. **Sums.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $a, b \in \mathcal{A}$ be free. How can we describe the distribution of $a + b$ in terms of the distributions of a and b? Of course, one can calculate the moments of $a + b$ in terms of moments of a and b, but moments turn out not to be the adequate tool to deal with sums of free variables, as calculating the moments of $a + b$ gets increasingly complicated for higher powers of $a + b$. Using cumulants is more promising.
Let us denote $\kappa_n^a = \kappa_n(a, a, \ldots, a)$. Then we have

$$\begin{aligned}\kappa_n^{a+b} &= \kappa_n(a+b, a+b, \ldots, a+b)\\ &= \kappa_n(a, a, \ldots, a) + \kappa_n(b, b, \ldots, b) + \kappa_n(\text{mixed terms})\\ &= \kappa_n^a + \kappa_n^b,\end{aligned}$$

as mixed cumulants vanish by the results of last chapter. Hence cumulants are additive w.r.t. what we want to call *free convolution*. However, at the moment the relation between moments and cumulants is just given on a combinatorial level by summations over large sets of non-crossing partitions. In order to be really useful, we need analytic tools for a better understanding of this relation between moments and cumulants.

Theorem 5.1. *We denote the n-th moment $\varphi(a^n)$ of $a \in \mathcal{A}$ by m_n and we consider the formal power series*

$$M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n \quad (\textit{moment series})$$

and

$$C(z) = 1 + \sum_{n=1}^{\infty} \kappa_n^a z^n \quad (\textit{cumulant series}).$$

Then the moment-cumulant relation

$$m_n = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi^a$$

is equivalent to

$$M(z) = C(zM(z)).$$

Proof. To simplify notation, we will write κ_π instead of κ_π^a. The crucial observation is now that we can encode a non-crossing partition by its first block $V = (j_1 = 1, j_2, \ldots, j_s)$ and non-crossing partitions $\pi_1, \ldots, \pi_s$ of the points between consecutive points of V; i.e., π_r is a non-crossing partition of $i_r := j_{r+1} - j_r - 1$ points (where we put $j_{s+1} := n$). This leads to the following rewriting of our moment-cumulant formula.

$$m_n = \sum_{\pi \in \mathcal{NC}(n)} \kappa_\pi = \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\dots+i_s+s=n}} \sum_{\pi_1 \in \mathcal{NC}(i_1)} \cdots \sum_{\pi_s \in \mathcal{NC}(i_s)} \kappa_s \kappa_{\pi_1} \cdots \kappa_{\pi_s}$$
$$= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\dots+i_s+s=n}} \kappa_s m_{i_1} \cdots m_{i_s}.$$

Hence we have that

$$\begin{aligned} M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n &= 1 + \sum_{n=1}^{\infty} \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\dots+i_s+s=n}} \kappa_s z^s m_{i_1} z^{i_1} \cdots m_{i_s} z^{i_s} \\ &= 1 + \sum_{s=1}^{\infty} \sum_{i_1,\dots,i_s \geq 0} \kappa_s z^s m_{i_1} z^{i_1} \cdots m_{i_s} z^{i_s} \\ &= 1 + \sum_{s=1}^{\infty} \kappa_s z^s M(z)^s \\ &= C(zM(z)). \end{aligned}$$

□

Remark 5.2. Classical cumulants (c_n) are defined by the moment-cumulant formula

$$m_n = \sum_{\pi \in \mathcal{P}(n)} c_\pi.$$

In terms of

$$A(z) = 1 + \sum_{n=1}^{\infty} \frac{m_n}{n!} z^n \qquad \text{and} \qquad B(z) = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n,$$

this is equivalent to

$$B(z) = \log A(z).$$

Thus classical cumulants are essentially the coefficients of the logarithm of the Fourier transform (or characteristic function) of the considered random variable.

To be able to use analytic methods, it is useful to rewrite $M(z)$ and $C(z)$ in terms of the *Cauchy transform*

$$G(z) = \varphi\left(\frac{1}{z-a}\right) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{z^{n+1}} = \frac{M(\frac{1}{z})}{z}$$

and Voiculescu's *R-transform*

$$R(z) = \sum_{n=0}^{\infty} \kappa_{n+1} z^n = \frac{C(z) - 1}{z}.$$

Then, as
$$M(z) = \frac{G(\frac{1}{z})}{z} \qquad \text{and} \qquad C(z) = zR(z) + 1,$$
the relation $M(z) = C(zM(z))$ can be rewritten as
$$\frac{G(\frac{1}{z})}{z} = zM(z)R(zM(z)) + 1 = G(1/z)R(G(1/z)) + 1.$$
Replacing z by $1/z$ leads to
$$zG(z) = G(z)R(G(z)) + 1$$
and hence
$$R(G(z)) + \frac{1}{G(z)} = z,$$
i.e. $R(z) + 1/z$ and $G(z)$ are inverses under composition. Thus,
$$G(R(z) + 1/z) = z$$
also holds.
The advantage of $G(z)$ over $M(z)$ is that
$$G(z) = \varphi\left(\frac{1}{z-a}\right) = \int \frac{1}{z-t}\, d\mu_a(t)$$
defines an analytic function $G : \mathbb{C}^+ \to \mathbb{C}^-$ and we can recover $\mu = \mu_a$ from G by the *Stieltjes inversion formula*
$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} G(t + i\varepsilon).$$

Example 5.3. We consider the semicircular distribution μ_s which is characterized by the moments being given by the Catalan numbers (counting noncrossing pair-partitions) or equivalently
$$\kappa_n = \begin{cases} 0 & \text{if } n \neq 2 \\ 1 & \text{if } n = 2. \end{cases}$$
Thus, $R(z) = \kappa_2 z = z$ and
$$z = R(G(z)) + \frac{1}{G(z)} = G(z) + \frac{1}{G(z)}.$$
This implies $G(z)^2 + 1 = zG(z)$ and this is solved by
$$G(z) = \frac{z \pm \sqrt{z^2 - 4}}{2}.$$
As $G(z) \sim 1/z$ for $z \to \infty$, we have
$$G(z) = \frac{z - \sqrt{z^2 - 4}}{2}.$$
The Stieltjes inversion formula finally shows that
$$d\mu_s(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im}\left(\frac{t - \sqrt{t^2 - 4}}{2}\right) = \begin{cases} \frac{1}{2\pi}\sqrt{4 - t^2}, & \text{if } t \in [-2, 2] \\ 0, & \text{otherwise,} \end{cases}$$

which indeed describes the semicircle on the interval $[-2,2]$.

Altogether, we have found an analytic way to calculate the free convolution $\mu = \mu_1 \boxplus \mu_2$ of two distributions μ_1 and μ_2 by calculating their Cauchy and R-transforms G_1, G_2 and R_1, R_2 and adding R_1 and R_2 to get the R-transform $R = R_1 + R_2$ of μ. Hence we can recover the Cauchy transform G of μ from R and use the Stieltjes inversion formula to get μ.

Exercise 5.4. (1) Calculate $(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\boxplus n}$ for $n = 2, 3, \ldots$.
(2) Prove: If a, b are free, then

$$G_{a+b}(z) = G_a[z - R_b(G_{a+b}(z))].$$

5.2. **Products.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $a, b \in \mathcal{A}$ be free. As in the sum case, we want to find a way of recovering $\mu_{ab} = \mu_a \boxtimes \mu_b$ from μ_a and μ_b. We note that this is not an operation on real probability measures as ab is in general not be selfadjoint, even if a and b are. However, if $b \geq 0$, then $b^{1/2}ab^{1/2}$ is selfadjoint and has the the same moments as ab and thus, we also have

$$\mu_a \boxtimes \mu_b = \mu_{b^{1/2}ab^{1/2}}.$$

Calculating the moments directly, again, turns out to be rather complicated, so we need other methods to determine $\mu_a \boxtimes \mu_b$.

Let σ denote the pair partition $\{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\}$ for $n \in \mathbb{N}$. Then, for $\{a_1,\ldots,a_n\}$, $\{b_1,\ldots,b_n\}$ free, we have, by Theorem 4.5

$$\kappa_n(a_1b_1, a_2b_2, \ldots, a_nb_n) = \sum_{\substack{\pi \in \mathcal{NC}(2n) \\ \pi \vee \sigma = 1_{2n}}} \kappa_\pi(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$$

and we can decompose π into $\pi = \pi_1 \cup \pi_2$, where

$$\pi_1 \in \mathcal{NC}(\text{odd}) := \mathcal{NC}(1,3,5,\ldots,2n-1)$$

and

$$\pi_2 \in \mathcal{NC}(\text{even}) := \mathcal{NC}(2,4,6,\ldots,2n).$$

Hence we can write the expression above as

$$\sum_{\pi_1 \in \mathcal{NC}(\text{odd})} \kappa_{\pi_1}(a_1, a_2, \ldots, a_n) \sum_{\substack{\pi_2 \in \mathcal{NC}(\text{even}) \\ \pi_1 \cup \pi_2 \in \mathcal{NC}(2n) \\ (\pi_1 \cup \pi_2) \vee \sigma = 1_{2n}}} \kappa_{\pi_2}(b_1, b_2, \ldots, b_n).$$

Given $\pi_1 \in \mathcal{NC}(\text{odd})$, there exists a unique $\pi_2 \in \mathcal{NC}(\text{even})$ that fulfills the summing conditions above. This π_2 is called the *Kreweras complement* of π_1 and we write $\pi_2 = K(\pi_1)$.
The induced map $K : \mathcal{NC}(n) \to \mathcal{NC}(n)$ is an anti-isomorphism in the sense that if $\sigma \leq \pi$, then $K(\sigma) \geq K(\pi)$. Moreover, $K(\pi)$ is the maximal σ such that $\pi \in \mathcal{NC}(\text{odd})$, $\sigma \in \mathcal{NC}(\text{even})$ and $\pi \cup \sigma \in \mathcal{NC}(2n)$.

Theorem 5.5. *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space and let $\{a_1,\dots,a_n\}$, $\{b_1,\dots,b_n\}$ be free. Then we have*

$$\kappa_n(a_1b_1,a_2b_2,\dots,a_nb_n)=\sum_{\pi\in\mathcal{NC}(n)}\kappa_\pi(a_1,\dots,a_n)\cdot\kappa_{K(\pi)}(b_1,\dots,b_n)$$

and

$$\varphi(a_1b_1a_2b_2\cdots a_nb_n)=\sum_{\pi\in\mathcal{NC}(n)}\kappa_\pi(a_1,\dots,a_n)\cdot\varphi_{K(\pi)}(b_1,\dots,b_n).$$

Translating this into power series gives Voiculescu's description via the S-transform.

Theorem 5.6. *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space and let $a\in\mathcal{A}$. We denote the moment series of a by $M_a(z)$ and define the* S-transform *of a by*

$$S_a(z)=\frac{1+z}{z}M_a^{<-1>}(z).$$

Then, if $b,c\in\mathcal{A}$ are free, we have

$$S_{bc}(z)=S_b(z)S_c(z).$$

6. Asymptotic Freedom of Random Matrices

Finally, we want to come back to random matrices and find freeness itself (at least asymptotically) making its appearance there.

Definition 6.1. Two sequences of random matrices $(A_N)_{N\in\mathbb{N}}$ and $(B_N)_{N\in\mathbb{N}}$ are called *asymptotically free* if:

- $A_N,B_N\stackrel{dist.}{\to}a,b$, for some $a,b\in\mathcal{A}$, where $(\mathcal{A},\varphi)$ is some non-commutative probability space. Recall that convergence in distribution means

$$\lim_{N\to\infty}\varphi_N(p(A_N,B_N))=\varphi(p(a,b))$$

  for any polynomial $p\in\mathbb{C}\langle X_1,X_2\rangle$ in two non-commuting variables and that $\varphi_N=\mathbb{E}\otimes\mathrm{tr}$ denotes here the averaged trace on our $N\times N$-random matrices.
- a,b are free in $(\mathcal{A},\varphi)$.

As an appetizer let us first consider Voiculescu's generalization of Wigner's theorem to the case of several independent GUE random matrices $A^{(1)}=(a_{ij}^{(1)})_{i,j=1}^N,\dots,A^{(n)}=(a_{ij}^{(n)})_{i,j=1}^N$. Then

$$\{a_{ij}^{(p)}\mid i,j=1,\dots,N;\ p=1,\dots,n\}$$

is a Gaussian family with

$$\mathbb{E}[a_{ij}^{(p)}a_{kl}^{(r)}]=\frac{1}{N}\delta_{il}\delta_{jk}\delta_{pr}.$$

The same calculations as in Section 2 yield the mixed moments

$$\varphi(A^{(p(1))}\dots A^{(p(m))}) = \sum_{\pi\in\mathcal{P}_2^{(p)}(m)} N^{\#(\gamma\pi)-1-\frac{m}{2}},$$

where $\mathcal{P}_2^{(p)}(m)$ denotes the set of pair partititions π that respect the "coloring" of the matrices, i.e. those $\pi \in \mathcal{P}_2(m)$ for which $(r,s)\in\pi$ implies $p(r)=p(s)$. Thus

$$\lim_{N\to\infty}\varphi(A^{(p(1))}\dots A^{(p(m))}) = \#\mathcal{NC}_2^{(p)}(m).$$

These limiting mixed moments of $A^{(1)},\dots,A^{(n)}$ are exactly those of a *semicircular family* $s_1,\dots,s_n$ with diagonal covariance, i.e.

$$\varphi(s_{i(1)}\dots s_{i(m)}) = \sum_{\pi\in\mathcal{NC}_2(m)}\prod_{(r,p)\in\pi}\varphi(s_{i(r)}s_{i(p)})$$

and $\varphi(s_is_j)=\delta_{ij}$ for $i,j=1,\dots,n$. Note that we can also rewrite this in terms of cumulants as $\kappa_n(s_{i(1)},\dots,s_{i(n)})=0$ for $n\neq 2$ and

$$\kappa_2(s_i,s_j) = \begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}.$$

However, this shows that mixed cumulants in $s_1,\dots,s_n$ vanish and thus we have the following theorem.

Theorem 6.2. *Elements of a semicircular family with diagonal covariance are free. Therefore, independent GUE random matrices are asymptotically free.*

Now we want to go further and find asymptotic freeness not just for semicircular distributions. For this we consider a GUE of $N\times N$-random matrices $(A_N)_{N\in\mathbb{N}}$ and a sequence of deterministic matrices $(D_N)_{N\in\mathbb{N}}$ such that $\lim_{N\to\infty}\operatorname{tr}(D_N^m)$ exists for all $m\in\mathbb{N}$, i.e $D_N \overset{dist.}{\to} d$, where

$$\varphi(d^m) = \lim_{N\to\infty}\operatorname{tr}(D_N^m).$$

We know that $A_N \overset{dist.}{\to} s$ where s is semicircular and $D_N \overset{dist.}{\to} d$, but what about $A_N, D_N \overset{dist.}{\to}$? In other words, can we calculate mixed moments in A_N, D_N?

We recall that for $A_N=(a_{ij})_{i,j=1}^N$, we have the Wick formula

$$\mathbb{E}[a_{i(1)j(1)}\cdots a_{i(m)j(m)}] = \sum_{\pi\in\mathcal{P}_2(m)}\prod_{(r,s)\in\pi}\mathbb{E}[a_{i(r)j(r)}a_{i(s)j(s)}],$$

where

$$\mathbb{E}[a_{ij}a_{kl}] = \frac{1}{N}\delta_{il}\delta_{jk}.$$

Now, we consider a deterministic $N\times N$-matrix $D=(d_{ij})_{i,j=1}^N$ and write $D^{q(k)}=(d_{ij}^{(k)})_{i,j=1}^N$ for $q(k)\in\mathbb{N},\ k=1,\dots,m$. Then, we have for a general mixed moment

$$\begin{aligned}\varphi(AD^{q(1)}\dots AD^{q(m)})&=\frac{1}{N}\sum_{\substack{i(1),\dots,i(m)=1\\ j(1),\dots,j(m)=1}}^N \mathbb{E}[a_{i(1)j(1)}d^{(1)}_{j(1)i(2)}a_{i(2)j(2)}\cdots d^{(m)}_{j(m)i(1)}]\\
&=\frac{1}{N}\sum_{\substack{i(1),\dots,i(m)=1\\ j(1),\dots,j(m)=1}}^N \mathbb{E}[a_{i(1)j(1)}\cdots a_{i(m)j(m)}]d^{(1)}_{j(1)i(2)}\cdots d^{(m)}_{j(m)i(1)}\\
&=\frac{1}{N^{1+\frac{m}{2}}}\sum_{\pi\in\mathcal{P}_2(m)}\sum_{\substack{i(1),\dots,i(m)=1\\ j(1),\dots,j(m)=1}}^N\prod_{r=1}^m \delta_{i(r)j(\pi(r))}d^{(1)}_{j(1)i(2)}\cdots d^{(m)}_{j(m)i(1)}\\
&=\frac{1}{N^{1+\frac{m}{2}}}\sum_{\pi\in\mathcal{P}_2(m)}\sum_{j(1),\dots,j(m)=1}^N d^{(1)}_{j(1)j(\pi\gamma(1))}\cdots d^{(m)}_{j(m)j(\pi\gamma(m))},\end{aligned}$$

where we denote as before the permutation $(1,2,3,\dots,m-1,m)\in S_m$ by γ. If we denote also

$$\mathrm{tr}_\sigma(D^{q(1)},\dots,D^{q(m)})=\prod_{c=(d_1,\dots,d_l)\in\sigma}\mathrm{tr}(D^{q(d_1)}\cdots D^{q(d_l)})$$

(product of traces along the cycles of σ) for a permutation $\sigma\in S_m$, the above can be written as

$$\varphi(AD^{q(1)}\dots AD^{q(m)})=\sum_{\pi\in\mathcal{P}_2(m)}\mathrm{tr}_{\pi\gamma}(D^{q(1)},\dots,D^{q(m)})\cdot N^{\#(\pi\gamma)-1-\frac{m}{2}},$$

which converges for $N\to\infty$ to

$$\sum_{\pi\in\mathcal{NC}_2(m)}\mathrm{tr}_{\pi\gamma}(d^{q(1)},\dots,d^{q(m)}).$$

We recall that, if s,d are free, it holds that

$$\begin{aligned}\varphi(sd^{q(1)}\dots sd^{q(m)})&=\sum_{\pi\in\mathcal{NC}(m)}\kappa_\pi(s,\dots,s)\varphi_{K(\pi)}(d^{q(1)},\dots,d^{q(m)})\\
&=\sum_{\pi\in\mathcal{NC}_2(m)}\varphi_{K(\pi)}(d^{q(1)},\dots,d^{q(m)}),\end{aligned}$$

as

$$\kappa_\pi(s,\dots,s)=\begin{cases}1, & \text{if } \pi\in\mathcal{NC}_2(m)\\ 0, & \text{otherwise.}\end{cases}$$

Hence the asymptotic value of $\varphi(AD^{q(1)}\dots AD^{q(m)})$ is given by the moment $\varphi(sd^{q(1)}\dots sd^{q(m)})$, provided that $K(\pi)$ and $\pi\gamma$ coincide. This is indeed the case for general $\pi\in\mathcal{NC}_2(m)$. We will check this at the following example.

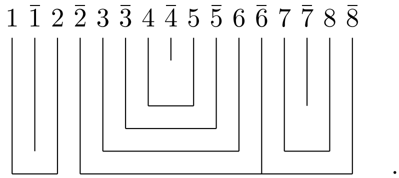
Example 6.3. Consider the non-crossing pairing

$$\pi = \{(1,2),(3,6),(4,5),(7,8)\}.$$

Then we have

$$\pi\gamma = (1)(2,6,8)(3,5)(7).$$

That this agrees with $K(\pi)$ can be seen from the graphical representation



Since s and d are free we get the asymptotic freeness of A_N and D_N. The above calculuations can be generalized to several independent GUE and deterministic matrices, resulting in the following theorem.

Theorem 6.4. *If $A_N^{(1)},\dots,A_N^{(p)}$ are p independent $N\times N$-random GUE matrices and $D_N^{(1)},\dots,D_N^{(q)}$ are q deterministic $N\times N$-matrices such that*

$$D_N^{(1)},\dots,D_N^{(q)} \overset{dist.}{\to} d_1,\dots,d_q$$

for some $d_1,\dots,d_q \in (\mathcal{A},\varphi)$, it holds that

$$A_N^{(1)},\dots,A_N^{(p)},D_N^{(1)},\dots,D_N^{(q)} \overset{dist.}{\to} s_1,\dots,s_p,d_1,\dots,d_q,$$

where $s_1,\dots,s_p$ are semicircular and $s_1,\dots,s_p,\{d_1,\dots,d_q\}$ are free.

Finally we want also mention a version of these asymptotic freeness results for Haar unitary random matrices: Let $\mathcal{U}(N)$ denote the set of unitary $N\times N$-matrices. As this is a compact group, we can equip $\mathcal{U}(N)$ with its Haar probability measure leading to the notion of Haar unitary random matrices.

Definition 6.5. (1) We equip the compact group $\mathcal{U}(N)$ with its Haar probability measure. Random matrices distributed according to this measure will be called *Haar unitary random matrices.*

(2) Let $(\mathcal{A},\varphi)$ be a non-commutative probability space. An element $u\in\mathcal{A}$ is called a *Haar unitary* if

- u is unitary,
- $\varphi(u^k)=\delta_{0k}$ for all $k\in\mathbb{Z}$.

Theorem 6.6. *Let $U_N^{(1)},\dots,U_N^{(p)}$ be p independent Haar unitary $N\times N$-random matrices and let $D_N^{(1)},\dots,D_N^{(q)}$ be q deterministic $N\times N$-matrices such that*

$$D_N^{(1)},\dots,D_N^{(q)} \overset{dist.}{\to} d_1,\dots,d_q$$

for some $d_1,\dots,d_q\in(\mathcal{A},\varphi)$. Then

$$U_N^{(1)},(U_N^{(1)})^*,\dots,U_N^{(p)},(U_N^{(p)})^*,D_N^{(1)},\dots,D_N^{(q)} \overset{dist.}{\to} u_1,u_1^*,\dots,u_p,u_p^*,d_1,\dots,d_q,$$

where $u_1, \dots, u_p$ are Haar unitaries and $\{u_1, u_1^*\}, \dots, \{u_p, u_p^*\}, \{d_1, \dots, d_q\}$ are free.

Remark 6.7. Note that, if u is a Haar unitary and $\{u, u^*\}$ is free from $\{a, b\}$, then a and ubu^* are free. Thus, if $(A_N)_{N\in\mathbb{N}}$ and $(B_N)_{N\in\mathbb{N}}$ are two sequences of deterministic matrices with limit distributions and $(U_N)_{N\in\mathbb{N}}$ is a sequence of Haar unitary random matrices, then $(A_N)_{N\in\mathbb{N}}$ and $(U_N B_N U_N^*)_{N\in\mathbb{N}}$ are asymptotically free.

7. Notes and Remarks

Random matrices have been studied in statistics and in physics since the influential papers of Wishart and Wigner, respectively. Random matrices appear nowadays in different fields of mathematics and physics (such as combinatorics, probability theory, statistics, operator theory, number theory, quantum chaos, quantum field theory etc.) or applied fields (such as electrical engineering). Some idea of the diversity of random matrix appearances can be gotten by looking on the collection of surveys in [1].

The genus expansion for Gaussian random matrices is a folklore result in physics; for a mathematical exposition see, for example, [11].

The notion of "asymptotic freeness" was introduced by Voiculescu in [9]. Our presentation of the asymptotic freeness results for Gaussian random matrices follows essentially the ideas of Voiculescu's original proofs in [9, 10]; however, our presentation is more streamlined by using the Wick formula and the genus expansion to make contact with the combinatorial description of freeness.

The combinatorial approach to free probability theory originated in my work [5] on free limit theorems. (At this time I was not aware of the work of Kreweras on non-crossing partitions and addressed the latter, not very imaginatively, as "admissible" partitions.) Inspired by the work of Rota [4] around the combinatorial structure of classical probability theory, featuring in particular multiplicative functions on the lattice of all partitions, I developped a few years later in [6] the full combinatorial description of freeness, resting on the notion of multiplicative functions on the lattice of non-crossing partitions and "free cumulants". Andu Nica showed a bit later in [2] how this combinatorial approach connects in general to Voiculescu's operator-theoretic approach in terms of creation and annihilation operators on the full Fock space. I teamed then up with Nica, pushing the combinatorial approach much further. Whereas in the beginning we were mainly driven by the desire to understand Voiculescu's work by giving new and "simpler" (at least for the combinatorially inclined) proofs of existing results of Voiculescu (like his R- and S-transform descriptions in [7, 8] for the free additive and multiplicative convolutions, respectively), later we could also initiate new directions in free probability. Prominent examples here are the determination of the distribution of the free commutator, the introduction of R-diagonal elements or the proof of the existence of the free convolution power semigroup $(\mu^{\boxplus t})_{t\geq 1}$. A good source for these developments and the combinatorial approach in general is our monograph [3].

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