Free Probability and Non-Commutative Symmetries

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1 Introduction

Here we will address the relevance of quantum symmetries in non-commutative and free probability theory. In particular, the quantum permutation and the quantum orthogonal group, which we already encountered in Chapters **??** and **??** as important examples of compact quantum groups, will feature prominently and give rise to the notion of "easy quantum groups", which present general classes of quantum symmetries.

First we will give a concise introduction to free probability theory, with an emphasis on its combinatorial side. Then we will present the main link between free probability theory and quantum groups: the free de Finetti theorem. It shows that freeness arises quite canonically via non-commutative symmetries. Finally we will examine the representation theory of (quantum) orthogonal and (quantum) permutation groups in terms of their intertwiner spaces and use this as motivation for the defintion of easy quantum groups. We survey the classification of easy quantum groups and show how the Weingarten formula for integration with respect to the Haar functional on those quantum groups can be used to derive various asymptotic properties.

2 Free Probability Theory

Free probability was introduced in the mid 1980's by Voiculescu [21] as a tool to attack the notorious free group isomorphism problem for von Neumann algebras. Since then there have emerged connections to quite different subjects in mathematics and also physics and engineering. Here we will give a concise introduction to

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free probability which puts its emphasis on the combinatorial side of free probability; this combinatorics will be instrumental for the relation to quantum symmetries. For general literature on free probability we refer to [25, 11, 24], for the combinatorial approach to [15].

2.1 The origin of freeness: free group factors

The moment point of view on von Neumann algebras

Free (and more generally, non-commutative) probability theory investigates operators on Hilbert spaces by looking at moments of those operators. Many methods and concepts for understanding those moments are inspired by analogues from classical probability theory.

Here is a bit of non-commutative language: A *non-commutative probability space* $(\mathcal{A},\varphi)$ consists of a unital algebra $\mathcal{A}$ and a unital linear functional $\varphi:\mathcal{A}\to\mathbb{C}$, $\varphi(1)=1$. Consider *(non-commutative) random variables* $a_1,\dots,a_n\in\mathcal{A}$. Expressions of the form $\varphi(a_{i(1)}\cdots a_{i(k)})$ for $k\in\mathbb{N}$, $1\le i(1),\dots,i(k)\le n$ are called *moments* of $a_1,\dots,a_n$.

Remark 1. 1) It is an easy but quite fundamental observation that moments of generators with respect to a faithful normal state determine a von Neumann algebra. Namely, let $\mathcal{A}$, $\mathcal{B}$ be two von Neumann algebras sucht that $\mathcal{A}=\mathrm{vN}(a_1,\dots,a_n)$, $\mathcal{B}=\mathrm{vN}(b_1,\dots,b_n)$, with selfadjoint generators a_i and b_i. Furthermore, let $\varphi:\mathcal{A}\to\mathbb{C}$ and $\psi:\mathcal{B}\to\mathbb{C}$ be faithful and normal states, and assume that for all $k\in\mathbb{N}$ and $1\le i(1),\dots,i(k)\le n$ we have $\varphi(a_{i(1)}\cdots a_{i(k)})=\psi(b_{i(1)}\cdots b_{i(k)})$. Then $\mathcal{A}$ is isomorph to $\mathcal{B}$ via the mapping $a_i\mapsto b_i$ $(i=1,\dots,n)$.

2) As a consequence of this we have the motto: moments can be useful. All questions on operators, which depend only on the generated operator algebra – like: spectrum, polar decomposition, existence of hyperinvariant subspaces, inequalities for L^p-norms – can in principle be answered by the knowledge of the moments of the operators with respect to a faithful normal state.

3) This insight is in general not very helpful, since moments are usually quite complicated. However, in many special (and interesting) situations moments have a special structure; this is the realm of free probability theory.

A main difference between measure theory and classical probability theory is given by the notion of *independence*. Similarly, a difference between von Neumann algebra theory and free probability theory is given by the notion of *freeness* or *free independence*. Freeness describes the special structure of moments arising from group von Neumann algebras $L(G)$, if G is the free product of subgroups

The structure of a group von Neumann algebra $L(G)$

Let G be a discrete group. The corresponding group von Neumann algebra is the closure of the left regular representation (where the group algebra acts on the group algebra by left multiplication) in the strong operator topology,

$$L(G) := \overline{\mathbb{C}G}^{\text{STOP}}.$$

If G is i.c.c. (all non-trivial conjugacy classes are infinite), then $L(G)$ is a II_1-factor.

In particular, the neutral element e of G induces a trace τ on $L(G)$, which is faithful and normal, via $\tau(a) := \langle ae, e\rangle$.

If G is amenable then $L(G)$ is the hyperfinite II_1-factor.

If $G = \mathbb{F}_n$ is the free group on n generators, then $L(\mathbb{F}_n)$ is, as already shown by Murray and von Neumann, not hyperfinite. It is Voiculescu's philosophy that those *free group factors* $L(\mathbb{F}_n)$ are the next interesting class of von Neumann algebras after the hyperfinite one.

Free probability theory was created in order to understand $L(\mathbb{F}_n)$ and similar von Neumann algebras; in particular, to attack the most famous, and still open, problem in this context, the *isomorphism problem of the free group factors:* Is it true or false that $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ for $n \neq m$ ($n, m \geq 2$).

The notion of freeness

It makes sense to say that a group is on an algebraic level the free product of subgroups, $G = G_1 * G_2$, just by requiring that there are no non-trivial relations between elements from G_1 and G_2. This can be extended to the group algebras, $\mathbb{C}G = \mathbb{C}G_1 * \mathbb{C}G_2$. But how about if we go over to the weak closure and the corresponding von Neumann algebras. Since elements in the von Neumann algebras are given by infinite sums, it is not apriori clear what we actually mean with $L(G) = L(G_1) * L(G_2)$. In order to make sense out of this we should rewrite the algebraic condition "absence of relations" in a form which can be extended to the von Neumann algebra closure.

That G_1, G_2 are free in G (as subgroups) means: If we consider $g_i \in G_{j(i)}$, such that $g_i \neq e$ for all i and such that $j(1) \neq j(2) \neq \cdots \neq j(k)$, then this implies that $g_1 \cdots g_k \neq e$. This can be reformulated with the help of the trace τ. Recall that τ is on G only different from zero if applied to e, thus $g \neq e$ can be rewritten as $\tau(g) = 0$, and the above characterization reads then as: If we consider $g_i \in G_{j(i)}$, such that $\tau(g_i) = 0$ for all i and such that $j(1) \neq \cdots \neq j(k)$, then this implies that $\tau(g_1 \cdots g_k) = 0$.

This characterisation goes now over not only to finite but as well to infinite sums in the von Neumann algebra; note that τ is normal.

2.2 Freeness

The considerations from the previous section motivated Voiculescu [21] to make the following definition.

Definition 1 (Voiculescu 1983). Let $(\mathcal{A}, \varphi)$ be *non-commutative probability space*, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional, $\varphi(1) = 1$.

Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are *free* or *freely independent*, if $\varphi(a_1 \cdots a_n) = 0$ whenever we have: $a_i \in \mathcal{A}_{j(i)}$, where $j(i) \in I$ for all i; $j(1) \neq j(2) \neq \cdots \neq j(n)$; $\varphi(a_i) = 0$ for all i.

Random variables $x_1, \dots, x_n \in \mathcal{A}$ are free, if their generated unital subalgebras $\mathcal{A}_i := \text{algebra}(1, x_i)$ are so.

Freeness between a and b is, by definition, an infinite set of equations relating various moments in a and b:

$$\varphi\Big(p_1(a)q_1(b)p_2(a)q_2(b)\cdots\Big) = 0.$$

A basic observation is that freeness between a and b is actually a *rule for calculating mixed moments* in a and b from the moments of a and the moments of b:

$$\varphi\Big(a^{n_1}b^{m_1}a^{n_2}b^{m_2}\cdots\Big) = \text{polynomial}\big(\varphi(a^i), \varphi(b^j)\big).$$

Example 1. By the definition of freeness we have

$$\varphi\Big(\big(a^n - \varphi(a^n)1\big)\big(b^m - \varphi(b^m)1\big)\Big) = 0,$$

thus

$$\varphi(a^n b^m) - \varphi(a^n \cdot 1)\varphi(b^m) - \varphi(a^n)\varphi(1 \cdot b^m) + \varphi(a^n)\varphi(b^m)\varphi(1 \cdot 1) = 0,$$

and hence

$$\varphi(a^n b^m) = \varphi(a^n) \cdot \varphi(b^m)$$

In the same way any mixed moment can be reduced to moments of a and moments of b. So we see that freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables. This is the reason that freeness is also called *free independence*.

One should, however, note that free independence is a different rule from classical independence; free independence occurs typically for *non-commuting random variables*, like operators on Hilbert spaces or (random) matrices

Example 2. As before, we have by definition

$$\varphi\Big(\big(a - \varphi(a)1\big) \cdot \big(b - \varphi(b)1\big) \cdot \big(a - \varphi(a)1\big) \cdot \big(b - \varphi(b)1\big)\Big) = 0,$$

which results (after some cancellations) in the formula

$$\varphi(abab) = \varphi(aa) \cdot \varphi(b) \cdot \varphi(b) + \varphi(a) \cdot \varphi(a) \cdot \varphi(bb) - \varphi(a) \cdot \varphi(b) \cdot \varphi(a) \cdot \varphi(b).$$

This latter result is very different from the factorisation $\varphi(abab) = \varphi(aa)\varphi(bb)$ for classically independent random variables.

Whereas freeness was introduced in the context of free group von Neumann algebras, Voiculescu discovered later [22] that also random matrices become asymptotically free (if their size tends to infinity) with respect to the trace. This unexpected relation between operator algebras and random matrices had a big effect on the development of the theory and is at the basis of many spectacular results. We will here not say more on the random matrix side of free probabilty; for more information on this one might consult [20].

2.3 The emergence of the combinatorics of freeness

We will motivate the combinatorial structure of freeness by the free central limit theorem; and also contrast this to the classical central limit theorem.

Consider $a_1, a_2, \cdots \in (\mathcal{A}, \varphi)$ which are identically distributed, centered and normalized ($\varphi(a_i) = 0$ and $\varphi(a_i^2) = 1$) and either classically independent or freely independent. A central limit theorem asks the question: What can we say about

$$S_n := \frac{a_1 + \cdots + a_n}{\sqrt{n}} \quad \overset{n\to\infty}{\longrightarrow} \quad ???$$

We say that S_n converges (in distribution) to s if

$$\lim_{n\to\infty} \varphi(S_n^m) = \varphi(s^m) \qquad \forall m \in \mathbb{N}.$$

We have

$$\varphi(S_n^m) = \frac{1}{n^{m/2}} \varphi\big[(a_1 + \cdots a_n)^m\big] = \frac{1}{n^{m/2}} \sum_{i(1),\dots,i(m)=1}^{n} \varphi[a_{i(1)} \cdots a_{i(m)}].$$

Now note that $\varphi[a_{i(1)} \cdots a_{i(m)}] = \varphi[a_{j(1)} \cdots a_{j(m)}]$ whenever $\ker i = \ker j$, where $\ker i$ denotes the maximal partition of $\{1, \dots, m\}$ such that i is constant on the blocks. (For a precise definition of "partition" and "block" see Definition 2 in the next section.) We denote the common value of those moments by $\kappa_{\ker i}$.

For example, for $i = (1,3,1,5,3)$ and $j = (3,4,3,6,4)$ we have

$$\varphi[a_1 a_3 a_1 a_5 a_3] = \varphi[a_3 a_4 a_3 a_6 a_4],$$

because independence/freeness allows to express (with the same polynomial)

$$\varphi[a_1a_3a_1a_5a_3] = \text{polynomial}\big(\varphi(a_1), \varphi(a_1^2), \varphi(a_3), \varphi(a_3^2), \varphi(a_5)\big)$$
$$\varphi[a_3a_4a_3a_6a_4] = \text{polynomial}\big(\varphi(a_3), \varphi(a_3^2), \varphi(a_4), \varphi(a_4^2), \varphi(a_6)\big)$$

and, by the identical distribution, we have $\varphi(a_1) = \varphi(a_3)$, $\varphi(a_1^2) = \varphi(a_3^2)$, $\varphi(a_3) = \varphi(a_4)$, $\varphi(a_3^2) = \varphi(a_4^2)$, $\varphi(a_5) = \varphi(a_6)$. We put in this case $\kappa_\pi := \varphi(a_1a_3a_1a_5a_3)$, where $\pi := \ker i = \ker j = \{\{1,3\},\{2,5\},\{4\}\}$. $\pi \in \mathcal{P}(5)$ is here a partition of $\{1,2,3,4,5\}$.

In our general calculation we can now continue as follows, where $\mathcal{P}(m)$ denotes the partitions of the set $\{1,\dots,m\}$.

$$\varphi(S_n^m) = \frac{1}{n^{m/2}} \sum_{i(1),\dots,i(m)=1}^{n} \varphi[a_{i(1)}\cdots a_{i(m)}] = \frac{1}{n^{m/2}} \sum_{\pi\in\mathcal{P}(m)} \kappa_\pi \cdot \#\{i : \ker i = \pi\}.$$

Note that

$$\#\{i : \ker i = \pi\} = n(n-1)\cdots(n-\#\pi-1) \sim n^{\#\pi}$$

for large n, and so we get

$$\varphi(S_n^m) \sim \sum_{\pi\in\mathcal{P}(m)} \kappa_\pi \cdot n^{\#\pi - m/2}.$$

Now consider a $\pi \in \mathcal{P}(m)$ with singleton, $\pi = \{\dots,\{k\},\dots\}$. Note that both for classical as well as for free independence we have the factorization rule $\varphi(abc) = \varphi(ac)\varphi(b)$ if b is independent/free from $\{a,b\}$. Thus we have for such a π:

$$\kappa_\pi = \varphi(a_{i(1)}\cdots a_{i(k)}\cdots a_{i(m)}) = \varphi(a_{i(1)}\cdots a_{i(k-1)}a_{i(k+1)}\cdots a_{i(m)}) \cdot \varphi(a_{i(k)}) = 0.$$

Thus: $\kappa_\pi = 0$ if π has singleton; i.e., in order to have $\kappa_\pi \neq 0$ we need that $\pi = \{V_1,\dots,V_r\}$ with $\#V_j \geq 2$ for all j, which implies $r = \#\pi \leq \frac{m}{2}$.

So in

$$\varphi(S_n^m) \sim \sum_{\pi\in\mathcal{P}(m)} \kappa_\pi \cdot n^{\#\pi - m/2}$$

only those π survive for $n \to \infty$ with

- π has no singleton, i.e., no block of size 1,
- π has exactly $m/2$ blocks.

Such π are exactly those, where each block has size 2, i.e.,

$$\pi \in \mathcal{P}_2(m) := \{\pi \in \mathcal{P}(m) \mid \pi \text{ is pairing}\}.$$

Thus we finally have:

$$\lim_{n\to\infty} \varphi(S_n^m) = \sum_{\pi\in\mathcal{P}_2(m)} \kappa_\pi.$$

This means in particular that odd moments are zero (because there are no pairings of an odd number of elements), thus the limit distribution is symmetric.

The main question is now: What are the even moments? This depends on the κ_π's. The actual value of those is now different for the classical and the free case!

Classical CLT

If the a_i commute and are independent, then $\kappa_\pi = \varphi(a_{i(1)} \cdots a_{i(2k)}) = 1$ for all $\pi \in \mathcal{P}_2(2k)$; recall our normalization $\varphi(a_i^2) = 1$. Thus

$$\lim_{n\to\infty} \varphi(S_n^m) = \#\mathcal{P}_2(m) = \begin{cases} 0, & m \text{ odd} \\ (m-1)(m-3)\cdots 5\cdot 3\cdot 1, & m \text{ even} \end{cases}.$$

Those limit moments are the moments of a Gaussian distribution of variance 1.

Free CLT

If the a_i are free, then, for $\pi \in \mathcal{P}_2(2k)$, we have

$$\kappa_\pi = \begin{cases} 0, & \pi \text{ is crossing} \\ 1, & \pi \text{ is non-crossing} \end{cases}.$$

as made plausible by the following two examples:

$$\begin{aligned} \kappa_{\{1,6\},\{2,5\},\{3,4\}} = \varphi(a_1a_2a_3a_3a_2a_1) &= \varphi(a_3a_3)\cdot\varphi(a_1a_2a_2a_1) \\ &= \varphi(a_3a_3)\cdot\varphi(a_2a_2)\cdot\varphi(a_1a_1) = 1, \end{aligned}$$

but

$$\kappa_{\{1,5\},\{2,3\},\{4,6\}\}} = \varphi(a_1a_2a_2a_3a_1a_3) = \varphi(a_2a_2)\cdot\underbrace{\varphi(a_1a_3a_1a_3)}_{0}.$$

The vanishing of the last term is by the definition of freeness.

We put now (for formal definition, see Definition 2)

$$NC_2(m) := \{\pi \in \mathcal{P}_2(m) \mid \pi \text{ is non-crossing}\}.$$

Then we have

$$\lim_{n\to\infty} \varphi(S_n^m) = \#NC_2(m) = \begin{cases} 0, & m \text{ odd} \\ c_k = \frac{1}{k+1}\binom{2k}{k}, & m = 2k \text{ even} \end{cases}.$$

Those limit moments are the moments of a semicircular distribution of unit variance,

$$\lim_{n\to\infty} \varphi(S_n^m) = \frac{1}{2\pi}\int_{-2}^{2} t^m\sqrt{4-t^2}\,dt.$$

The even moments $c_k := \#NC_2(2k)$ satisfy the recursion

$$c_k = \sum_{\pi \in NC_2(2k)} 1 = \sum_{i=1}^{k} \sum_{\pi = \{1,2i\} \cup \pi_1 \cup \pi_2} 1 = \sum_{i=1}^{k} c_{i-1} c_{k-i}.$$

This recursion, together with $c_0 = 1, c_1 = 1$, determines the sequence of *Catalan numbers*: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$.

2.4 Free cumulants

For a better understanding of the freeness rule it is advantageous to consider "free cumulants". Those are given as monomials in moments, the precise nature of this connection is given by summing over "non-crossing partitions". It will turn out that freeness is much easier to describe on the level of free cumulants, by the "vanishing of mixed cumulants". Free cumulants were introduced by Speicher [18], and used quite extensively in work of Nica and Speicher, see [15].

Definition 2. 1) A *partition* of $\{1, \ldots, n\}$ is a decomposition $\pi = \{V_1, \ldots, V_r\}$ with $V_i \neq \emptyset$, $V_i \cap V_j = \emptyset$ for $i \neq j$, and $\bigcup_i V_i = \{1, \ldots, n\}$. The V_i are the *blocks* of π. By $\mathcal{P}(n)$ we denote the set of all partitions of $\{1, \ldots, n\}$.

2) π is *non-crossing* if we do not have $p_1 < q_1 < p_2 < q_2$ such that p_1, p_2 are in same block, q_1, q_2 are in same block, but those two blocks are different. By $NC(n)$ we denote the non-crossing partitions of $\{1, \ldots, n\}$.

Definition 3 (Speicher 1994). For a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ we define *cumulant functionals* $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ (for all $n \geq 1$) as multi-linear functionals by the *moment-cumulant relations*

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi[a_1, \ldots, a_n],$$

where κ_π denotes a product of κ_i according to the block structure of π.

Note: classical cumulants are defined by a similar formula, where only $NC(n)$ is replaced by $\mathcal{P}(n)$.

Example 3. 1) For $n = 1$, we have one element in $NC(1)$, whichs gives $\varphi(a_1) = \kappa_1(a_1)$

2) For $n = 2$, we have two elements in $NC(2)$, which gives

$$\varphi(a_1 a_2) = \sqcup + \mid \; \mid = \kappa_2(a_1, a_2) + \kappa_1(a_1)\kappa_1(a_2)$$

and thus

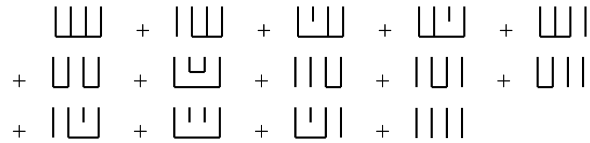
$$\kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2).$$

3) For $n = 3$, we have 5 elements in $NC(3)$, which gives

$\varphi(a_1a_2a_3) =$ 

$$= \kappa_3(a_1,a_2,a_3) + \kappa_1(a_1)\kappa_2(a_2,a_3) + \kappa_2(a_1,a_2)\kappa_1(a_3) + \kappa_2(a_1,a_3)\kappa_1(a_2) + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3).$$

4) For $n = 4$, there are 14 elements in $NC(4)$ (one of the 15 partitions is crossing), so we get

$\varphi(a_1a_2a_3a_4) =$ 

$$\begin{aligned} = \; & \kappa_4(a_1,a_2,a_3,a_4) + \kappa_1(a_1)\kappa_3(a_2,a_3,a_4) \\ & + \kappa_1(a_2)\kappa_3(a_1,a_3,a_4) + \kappa_1(a_3)\kappa_3(a_1,a_2,a_4) \\ & + \kappa_3(a_1,a_2,a_3)\kappa_1(a_4) + \kappa_2(a_1,a_2)\kappa_2(a_3,a_4) \\ & + \kappa_2(a_1,a_4)\kappa_2(a_2,a_3) + \kappa_1(a_1)\kappa_1(a_2)\kappa_2(a_3,a_4) \\ & + \kappa_1(a_1)\kappa_2(a_2,a_3)\kappa_1(a_4) + \kappa_2(a_1,a_2)\kappa_1(a_3)\kappa_1(a_4) \\ & + \kappa_1(a_1)\kappa_2(a_2,a_4)\kappa_1(a_3) + \kappa_2(a_1,a_4)\kappa_1(a_2)\kappa_1(a_3) \\ & + \kappa_2(a_1,a_3)\kappa_1(a_2)\kappa_1(a_4) + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3)\kappa_1(a_4). \end{aligned}$$

The main point of making such a definiton is that free cumulants can be used to describe freeness very effectively, namely it corresponds to the *vanishing of mixed cumulants*.

Theorem 1 (Speicher 1994). *The fact that $x_1,\dots,x_m$ are free is equivalent to the fact that $\kappa_n(x_{i(1)},\dots,x_{i(n)}) = 0$ whenever: $n \geq 2$ and there are p,r such that $i(p) \neq i(r)$.*

2.5 Operator-valued extension of free probability

Voiculescu defined from the very beginning a more general version of free probability theory, where the expectation onto scalars is replaced by more general expectations onto subalgebras, see [21, 23]. This corresponds to taking conditional expectations onto sub-σ-algebras in the classical setting. The combinatorial theory of this operator-valued version of free probability was developed in [19].

Definition 4 (Voiculescu 1983). 1) Let $\mathcal{B} \subset \mathcal{A}$. A linear map $E : \mathcal{A} \to \mathcal{B}$ is a *conditional expectation* if $E[b] = b$ for all $b \in \mathcal{B}$ and

$$E[b_1ab_2] = b_1E[a]b_2 \qquad \forall a \in \mathcal{A}, \quad \forall b_1,b_2 \in \mathcal{B}.$$

2) An *operator-valued probability space* consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$. The *operator-valued distribution* of a random variable $a \in \mathcal{A}$

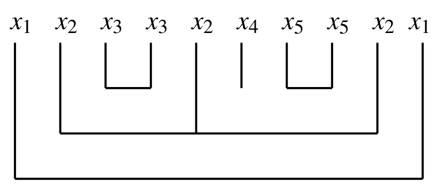
is then given by all operator-valued moments

$$E[ab_1ab_2\cdots b_{n-1}a] \in \mathcal{B} \qquad (n \in \mathbb{N}, b_1, \ldots, b_{n-1} \in \mathcal{B}).$$

3) Random variables $x_i \in \mathcal{A}$ $(i \in I)$ are *free with respect to E* (or *free with amalgamation over $\mathcal{B}$)* if $E[a_1\cdots a_n] = 0$ whenever $a_i \in \mathcal{B}\langle x_{j(i)}\rangle$ are polynomials in some $x_{j(i)}$ with coefficients from $\mathcal{B}$ and $E[a_i] = 0$ for all i and $j(1) \neq j(2) \neq \cdots \neq j(n)$.

Note that in an operator-valued setting the "scalars" from $\mathcal{B}$ and our random variable x do not commute in general! This has the consequence that operator-valued freeness works mostly like ordinary freeness, but one has to take care of the order of the variables; in all expressions they have to appear in their original order!

Example 4. 1) Assume that x_1, x_2, x_3, x_4, x_5 are free. Then one has, as in the scalar-valued case, a factorizaton of "non-crossing" moments, but this cannot be separated into a product, but one has to respect the nested structure of the non-crossing partitions. So for a moment like



we have the factorization:

$$E[x_1x_2x_3x_3x_2x_4x_5x_5x_2x_1] = E\Big[x_1 \cdot E\big[x_2 \cdot E[x_3x_3] \cdot x_2 \cdot E[x_4] \cdot E[x_5x_5] \cdot x_2\big] \cdot x_1\Big].$$

2) For "crossing" moments one also has analogous formulas as in the scalar-valued case, modulo respecting the order of the variables. For example, the formula

$$\varphi(x_1x_2x_1x_2) = \varphi(x_1x_1)\varphi(x_2)\varphi(x_2) + \varphi(x_1)\varphi(x_1)\varphi(x_2x_2) - \varphi(x_1)\varphi(x_2)\varphi(x_1)\varphi(x_2)$$

has now to be written as

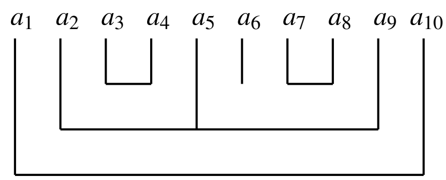
$$E[x_1x_2x_1x_2] = E\big[x_1E[x_2]x_1\big] \cdot E[x_2] + E[x_1] \cdot E\big[x_2E[x_1]x_2\big] - E[x_1]E[x_2]E[x_1]E[x_2].$$

Definition 5 (Speicher 1998). Consider an operator-valued probability space $E : \mathcal{A} \to \mathcal{B}$. We define the *(operator-valued) free cumulants* $\kappa_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}$ by

$$E[a_1\cdots a_n] = \sum_{\pi \in NC(n)} \kappa_\pi^{\mathcal{B}}[a_1, \ldots, a_n].$$

The arguments of $\kappa_\pi^{\mathcal{B}}$ are distributed according to the blocks of π, but now the cumulants are nested inside each other according to the nesting of the blocks of π.

Example 5. For the partition $\pi = \big\{\{1,10\},\{2,5,9\},\{3,4\},\{6\},\{7,8\}\big\} \in NC(10)$



we have

$$\kappa_\pi^{\mathcal{B}}[a_1,\dots,a_{10}] = \kappa_2^{\mathcal{B}}\Big(a_1\cdot\kappa_3^{\mathcal{B}}\big(a_2\cdot\kappa_2^{\mathcal{B}}(a_3,a_4),a_5\cdot\kappa_1^{\mathcal{B}}(a_6)\cdot\kappa_2^{\mathcal{B}}(a_7,a_8),a_9\big),a_{10}\Big).$$

One should note that elements from $\mathcal{B}$ can be moved over the commas, e.g., for $b\in\mathcal{B}$ one has $\kappa_2^{\mathcal{B}}(a_1 b,a_2) = \kappa_2^{\mathcal{B}}(a_1,ba_2)$.

Again one has the characterization that freeness is equivalent to the vanishing of mixed cumulants.

3 Non-Commutative de Finetti Theorem, Quantum Permutation Group and Non-Crossing Partitions

Now we will switch to the side of quantum symmetries. In classical probability theory one has a huge body of results characterizing distributional symmetries, see [12]. The most fundamental of those is the de Finetti theorem, which says that invariance of the joint distribution of a sequence of random variables is equivalent to the conditional independence of those variables. We will aim at getting statements of this kind in a non-commutative setting.

3.1 The classical de Finetti theorem

Let us first recall the classical de Finetti theorem. Consider a probability space $(\Omega,\mathfrak{A},P)$. We denote the expectation by φ,

$$\varphi(Y) = \int_\Omega Y(\omega)dP(\omega).$$

We say that random variables $X_1,X_2,\dots$ are *exchangeable* if their joint distribution is invariant under finite permutations, i.e. if

$$\varphi(X_{i(1)}\cdots X_{i(n)}) = \varphi(X_{\pi(i(1))}\cdots X_{\pi(i(n))})$$

for all $n\in\mathbb{N}$, all $i(1),\dots,i(n)\in\mathbb{N}$, and all permutations π.

For example, for an exchangeable sequence we have $\varphi(X_1^n) = \varphi(X_7^n)$ for all $n \in \mathbb{N}$ (i.e., the variables are in particular identically distributed), or $\varphi(X_1^3 X_3^7 X_4) = \varphi(X_8^3 X_2^7 X_9)$.

Note that the X_i might all contain some common component; e.g., if all X_i are the same, then clearly the sequence $X, X, X, X, X \ldots$ is exchangeable.

The classical theorem of de Finetti says that an infinite sequence of exchangeable random variables is independent modulo its common part.

We formalize the common part via the *tail σ-algebra* of the sequence $X_1, X_2, \ldots$

$$\mathfrak{A}_{\text{tail}} := \bigcap_{i \in \mathbb{N}} \sigma(X_k \mid k \geq i).$$

We denote by E the conditional expectation onto this tail σ-algebra

$$E : L^\infty(\Omega, \mathfrak{A}, P) \to L^\infty(\Omega, \mathfrak{A}_{\text{tail}}, P).$$

Now we can formulate the classical theorem of de Finetti.

Theorem 2 (de Finetti 1931; Hewitt, Savage 1955). *The following are equivalent for an infinite sequence of random variables:*

- *the sequence is exchangeable*
- *the sequence is independent and identically distributed with respect to the conditional expectation E onto the tail σ-algebra of the sequence*

3.2 Symmetries for non-commutative random variables

Now we want to investigate analogues of the classical de Finetti theorem in a non-commutative context. So we replace random variables by operators on Hilbert spaces, and the expectation by a state on the algebra generated by those operators. Since now analysis will be important, we will consider in the following always a W^*-probability space.

Definition 6. A *W^*-probability space* $(\mathcal{A}, \varphi)$ is a non-commutative probability space with:

- $\mathcal{A}$ is von Neumann algebra, i.e., a weakly closed subalgebra of bounded operators on Hilbert space,
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a faithful state on $\mathcal{A}$, i.e., $\varphi(aa^*) \geq 0$ for all $a \in \mathcal{A}$ and $\varphi(aa^*) = 0$ if and only if $a = 0$.

Clearly, we can also extend the notion of exchangeability to such a non-commutative setting.

Definition 7. Consider non-commutative random variables $x_1, x_2, \cdots \in \mathcal{A}$. They are *exchangeable* if

$$\varphi(x_{i(1)} \cdots x_{i(n)}) = \varphi(x_{\pi(i(1))} \cdots x_{\pi(i(n))})$$

for all $n \in \mathbb{N}$, all $i(1), \ldots, i(n) \in \mathbb{N}$, and all permutations π.

So we are now facing the question: Does exchangeability imply anything like independence in this general non-commutative setting? The answer is: only partially. Exchangeability gives, by work of Koestler [13], some weak form of independence (special factorization properties), but does not fully determine all mixed moments; there are just too many possibilities out there in the non-commutative world, and exchangeability is too weak a condition!

However, one should realize that invariance under permutations is in a sense also a commutative concept – and should be replaced by a non-commutative counterpart in the non-commutative world! It should not come as a surprise here that we should replace the permutation group by its quantum group analogue.

We have already seen this object in Chapter **??**. Let us recall the basic facts.

The permutation group S_k can be identified with $k \times k$ permutation matrices, and by dualizing we get the functions on S_k as

$$C(S_k) = \left\{ f : S_k \to \mathbb{C}; g \mapsto \left((u_{ij}(g) \right)_{i,j=1}^{k} \right\}.$$

(Here, $u_{ij} : S_k \to \mathbb{C}$ is the coordinate function mapping a matrix g to its (i, j)-entry.) This $C(S_k)$ can also be described as the universal commutative C^*-algebra generated by u_{ij} $(i, j = 1, \ldots, k)$, subject to the relations

$$u_{ij}^* = u_{ij} = u_{ij}^2 \quad \forall i, j \qquad \text{and} \qquad \sum_j u_{ij} = 1 = \sum_j u_{ji} \quad \forall i.$$

The group structure of S_k is, in this dual picture, captured by the Hopf algebra structure, which can be described on the algebra $\mathrm{alg}(u_{ij} \mid i, j = 1, \ldots, k)$ (which is dense in $C(S_k)$) by coproduct Δ, co-unit ε, and antipode S:

$$\Delta u_{ij} = \sum_k u_{ik} \otimes u_{kj}, \qquad \varepsilon(u_{ij}) = \delta_{ij}, \qquad S(u_{ij}) = u_{ji}.$$

The non-commutative analogue of this was introduced by S. Wang [27].

Definition 8 (Wang 1998). The *quantum permutation group* is given by the universal unital C^*-algebra $A_s(k)$ generated by u_{ij} $(i, j = 1, \ldots, k)$ subject to the relations:

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i, j = 1, \ldots, k$;
- each row and column of $u = (u_{ij})_{i,j=1}^k$ is a partition of unity:

$$\sum_{j=1}^{k} u_{ij} = 1 \quad \forall i \qquad \text{and} \qquad \sum_{i=1}^{k} u_{ij} = 1 \quad \forall j.$$

(Note: elements within a row or within a column are orthogonal.)

$A_s(k)$ is a compact quantum group in the sense of Woronowicz [29], with the same formulas for coproduct, co-unit, and antipode as above. We will also write:

$A_s(k) = C(S_k^+)$, and think of $A_s(k)$ as functions on the (non-existing) "quantum space" S_k^+.

Informally, we also think of a quantum permutation as any matrix $u = (u_{ij})$ of operators on a Hilbert space satisfying the relations from Definition 8. If

$$u_1 = (u_{ij}^{(1)})_{i,j=1}^k \qquad \text{and} \qquad u_2 = (u_{ij}^{(2)})_{i,j=1}^k$$

are quantum permutations, then so is

$$u_1 \odot u_2 := \Big(\sum_k u_{ik}^{(1)} \otimes u_{kj}^{(2)}\Big)_{i,j=1}^k.$$

Example 6. Examples of $u = (u_{ij})_{i,j=1}^k$ satisfying the quantum permutation relations are:

- permutation matrices
- the basic non-commutative example is of the form (for $k = 4$):

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & 1 \end{pmatrix}$$

for (in general, non-commuting) projections p and q

Note: $S_2^+ = S_2$, $S_3^+ = S_3$, but $S_k^+ \neq S_k$ for $k \geq 4$.

Definition 9 (Köstler, Speicher 2009). A sequence $x_1, \ldots, x_k$ in $(\mathcal{A}, \varphi)$ is *quantum exchangeable* if its distribution does not change under the action of quantum permutations S_k^+, i.e., if we have the following. Let the quantum permutation $u = (u_{ij}) \in C(S_k^+)$ act on $(x_1, \ldots, x_k)$ by

$$y_i := \sum_{j=1}^k u_{ij} \otimes x_j \in C(S_k^+) \otimes \mathcal{A} \qquad (i = 1, \ldots, k).$$

Then, for each k, $(x_1, \ldots, x_k) \in (\mathcal{A}, \varphi)$ has the same distribution as $(y_1, \ldots, y_k) \in (C(S_k^+) \otimes \mathcal{A}, \mathrm{id} \otimes \varphi)$. This means that we have

$$\varphi(x_{i(1)} \cdots x_{i(n)}) \cdot 1_{C(S_k^+)} = \mathrm{id} \otimes \varphi(y_{i(1)} \cdots y_{i(n)})$$

for all $n \geq 1$ and $1 \leq i(1), \ldots, i(n) \leq k$.

Note that this condition means concretely that

$$\varphi(x_{i(1)} \cdots x_{i(n)}) \cdot 1 = \sum_{j(1),\ldots,j(n)=1}^k u_{i(1)j(1)} \cdots u_{i(n)j(n)} \varphi(x_{j(1)} \cdots x_{j(n)})$$

for all $u = (u_{ij})_{i,j=1}^k$ which satisfy the defining relations for $A_s(k)$.

Since in particular permutation matrices satisfy the defining relations for the quantum permutation group it follows that quantum exchangeability implies exchangeability. The latter is a stronger form of invariance than the former one. One should, however, note that commuting variables are usually not quantum exchangeable.

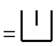
3.3 A non-commutative de Finetti theorem

The first hint of a de Finetti type theorem in this context is now given by the following fact which shows that freeness goes together nicely with the quantum permutation group.

Proposition 1. *Consider $x_1, \dots, x_k \in (\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_1, \dots, x_k$ are quantum exchangeable.*

Proof. We have to show the equality of the moments of the x_i's and of the y_i's. By the moment-cumulant formula, this is the same as showing for all $n \in \mathbb{N}$, all $1 \le i(1), \dots, i(n) \le k$ and all $\pi \in NC(n)$ that

$$\mathrm{id} \otimes \kappa_\pi(y_{i(1)}, \dots, y_{i(n)}) = \kappa_\pi(x_{i(1)}, \dots, x_{i(n)})$$

We will give the idea of the proof of this by considering $n = 3$ and $\pi =$  . In this case we have for the left-hand side of the above equation

$$\begin{aligned}
LHS &= \sum_{j(1),j(2),j(3)} u_{i(1)j(1)} u_{i(2)j(2)} u_{i(3)j(3)} \cdot \underbrace{\kappa_\pi(x_{j(1)}, x_{j(2)}, x_{j(3)})}_{\kappa_2(x_{j(1)}, x_{j(3)}) \cdot \kappa_1(x_{j(2)})} \\
&= \sum_{j(1),j(2),j(3)} u_{i(1)j(1)} \underbrace{u_{i(2)j(2)}}_{\sum_{j(2)} \to 1} u_{i(3)j(3)} \cdot \kappa_2(x_{j(1)}, x_{j(3)}) \cdot \underbrace{\kappa_1(x_{j(2)})}_{\kappa_1(x)} \\
&= \sum_{j(1),j(3)} u_{i(1)j(1)} u_{i(3)j(3)} \cdot \underbrace{\kappa_2(x_{j(1)}, x_{j(3)})}_{\delta_{j(1)j(3)} \cdot \kappa_2(x,x)} \cdot \kappa_1(x) \\
&= \sum_{j(1)=j(3)} u_{i(1)j(1)} u_{i(3)j(3)} \cdot \kappa_2(x,x) \cdot \kappa_1(x) \\
&= \sum_{j(1)} \underbrace{u_{i(1)j(1)} u_{i(3)j(1)}}_{\delta_{i(1)i(3)} u_{i(1)j(1)}} \cdot \kappa_2(x,x) \cdot \kappa_1(x) \\
&= \delta_{i(1)i(3)} \cdot \kappa_2(x,x) \cdot \kappa_1(x) \\
&= \kappa_2(x_{i(1)}, x_{i(3)}) \cdot \kappa_1(x_{i(2)}) \\
&= \kappa_\pi(x_{i(1)}, x_{i(2)}, x_{i(3)}) \\
&= RHS \qquad \square
\end{aligned}$$

Now we want to address the general question: What does quantum exchangeability for an infinite sequence $x_1, x_2, \dots$ imply?

As before, constant sequences are trivially quantum exchangeable, thus we have to take out the common part of all the x_i. In our non-commutative W^*-probability space setting this common part is now given by the *tail algebra* of the sequence:

$$\mathcal{A}_{\text{tail}} := \bigcap_{i \in \mathbb{N}} \mathrm{vN}(x_k \mid k \geq i).$$

One can then show that there exists a conditional expectation from all variables onto the tail algebra, $E : \mathrm{vN}(x_i \mid i \in \mathbb{N}) \to \mathcal{A}_{\text{tail}}$. Then one has the following non-commutative de Finetti theorem [14].

Theorem 3 (Köstler, Speicher 2009). *The following are equivalent for an infinite sequence of non-commutative random variables in a W^*-probability space:*

- *the sequence is quantum exchangeable;*
- *the sequence is free and identically distributed with respect to the conditional expectation E onto the tail-algebra of the sequence.*

Proof. We want to give an idea of the proof of the main direction, namely that the quantum symmetry implies freeness. Actually, first we want to address non-crossing expressions like $E[x_9 x_7 x_2 x_7 x_9]$. To determine those we only need exchangeability; namely we have

$$\begin{aligned} E[x_9 x_7 x_2 x_7 x_9] &= \frac{1}{N}\big(E[x_9 x_7 x_{10} x_7 x_9] + E[x_9 x_7 x_{11} x_7 x_9] + \cdots + E[x_9 x_7 x_{9+N} x_7 x_9]\big) \\ &= E\left[x_9 x_7 \cdot \frac{1}{N}\sum_{i=1}^{N} x_{9+i} \cdot x_7 x_9\right]. \end{aligned}$$

However, by the mean ergodic theorem, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_{9+i} = E[x_9] = E[x_2].$$

Thus $E\big[x_9 x_7 x_2 x_7 x_9\big] = E\big[x_9 x_7 E[x_2] x_7 x_9\big]$. We do now the same trick for $x_7 E[x_2] x_7$.

$$E\big[x_9 x_7 E[x_2] x_7 x_9\big] = \lim_{N\to\infty} E\left[x_9 \Big(\frac{1}{N}\sum_{i=1}^{N} x_{13+i} E[x_2] x_{13+i}\Big) x_9\right] = E\big[x_9 E[x_7 E[x_2] x_7] x_9\big].$$

So we finally get

$$E[x_9 x_7 x_2 x_7 x_9] = E\big[x_9 E[x_7 E[x_2] x_7] x_9\big].$$

In the same way (by always working on interval blocks) one gets factorizations for all non-crossing terms in an iterative way. Thus exchangeability implies factorizations for all non-crossing moments. This was shown by Köstler [13].

For commuting variables this factorizaton of non-crossing moments determines everything. However, for non-commuting variables there are many more expressions which cannot be treated like this. The basic example for such a situation is $E[x_1x_2x_1x_2]$. Exchangeability cannot make a statement for such mixed moments. To determine those we need quantum exchangeability!

So let us consider $E[x_1x_2x_1x_2]$ and assume, for convenience, that $E[x_1] = E[x_2] = 0$. By quantum exchangeability we have

$$\begin{aligned} E[x_1x_2x_1x_2] &= \sum_{j(1),\dots,j(4)=1}^{k} u_{1j(1)}u_{2j(2)}u_{1j(3)}u_{2j(4)}E[x_{j(1)}x_{j(2)}x_{j(3)}x_{j(4)}] \\ &= \sum_{j(1)\neq j(2)\neq j(3)\neq j(4)} u_{1j(1)}u_{2j(2)}u_{1j(3)}u_{2j(4)}E[x_{j(1)}x_{j(2)}x_{j(3)}x_{j(4)}] \\ &= \sum_{j(1)=j(3)\neq j(2)=j(4)} u_{1j(1)}u_{2j(2)}u_{1j(3)}u_{2j(4)}E[x_1x_2x_1x_2]. \end{aligned}$$

For the restriction of the summation in the second line we have used that u_{ik} and u_{jk} are orthogonal for $i \neq j$; and in the last step we have used the fact that we already know that non-crossing moments factorize, hence the other possible cases like $E[x_1x_2x_3x_4]$ or $E[x_1x_2x_3x_2]$ do all vanish.

It is now a relatively easy exercise to show that

$$\sum_{j(1)=j(3)\neq j(2)=j(4)} u_{1j(1)}u_{2j(2)}u_{1j(3)}u_{2j(4)}$$

is not equal to 1 for a general quantum permutation matrix (u_{ij}) and hence we must have that $E[x_1x_2x_1x_2] = 0$.

Thus we have shown: if $E[x_1] = 0 = E[x_2]$, then $E[x_1x_2x_1x_2] = 0$. In general, one shows in the same way that $E\big[p_1(x_{i(1)})p_2(x_{i(2)})\cdots p_n(x_{i(n)})\big] = 0$ whenever: $n \in \mathbb{N}$ and $p_1, \dots, p_n \in \mathcal{A}_{\text{tail}}\langle X\rangle$ are polynomials in one variable; $i(1) \neq i(2) \neq i(3) \neq \cdots \neq i(n)$; and $E[p_j(x_{i(j)})] = 0$ for all $j = 1, \dots, n$. But this is exactly the definition of freeness with respect to E. □

4 Quantum Symmetries in Non-Commutative Probability: Easy Quantum Groups

4.1 Motivation and definition of easy quantum groups

Now we want to have a more general look on possible quantum symmetries in non-commutative probability theory. Let us recall that quantum groups are generalizations of groups G (actually, of $C(G)$), which are supposed to describe non-classical symmetries.

Very often, quantum groups are deformations G_q of classical groups, depending on some parameter q, such that for $q \to 1$, they go to the classical group $G = G_1$. In such cases, G_q and G_1 are incomparable, none is stronger than the other; and, whereas G_1 is supposed to act on commuting variables, G_q is the right replacement to act on some special non-commuting (like q-commuting) variables; i.e., we are replacing the commutativity condition by some other non-commutative relation.

However, the quantum permutation group is not of this type. It is, quite to the contrary, a quantum group which strengthens a classical symmetry in a non-commutative context. More generally, there are situations where a classical group G has a genuine non-commutative analogue G^+, which is "stronger" than G: $G \subset G^+$. Whereas G acts on commuting variables, G^+ is now the right replacement for acting on maximally non-commuting variables; we do not replace commutativity by some other relation, but the commutativity condition is just dropped.

We will now be interested in quantum versions of *real compact matrix groups*. For the latter one should think of orthogonal matrices or permutation matrices. Such quantum versions are captured by the notion of orthogonal Hopf algebras.

Definition 10. An *orthogonal Hopf algebra* is a C^*-algebra A, given with a system of n^2 self-adjoint generators $u_{ij} \in A$ $(i, j = 1, \ldots, n)$, subject to the following conditions:

- The inverse of $u = (u_{ij})$ is the transpose matrix $u^t = (u_{ji})$.
- $\Delta(u_{ij}) = \Sigma_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.
- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.
- $S(u_{ij}) = u_{ji}$ defines a morphism $S : A \to A^{op}$.

These are compact quantum groups in the sense of Woronowicz [29].

In the spirit of non-commutative geometry, we are thinking of $A = C(G^+)$ as the continuous functions, generated by the coordinate functions u_{ij}, on some (non-existing) quantum group G^+, replacing a classical group G.

Besides the quantum permutation group we know at least one other such quantum symmetry, see [26].

Definition 11 (Wang 1995). The *quantum orthogonal group* is given by the universal unital C^*-algebra $A_o(n) = C(O_n^+)$, generated by selfadjoint u_{ij} $(i, j = 1, \ldots, n)$ subject to the relation that the matrix $u = (u_{ij})_{i,j=1}^n$ is an orthogonal matrix. Explicitly, this means: for all $i, j = 1, \ldots, n$ we have

$$\sum_{k=1}^n u_{ik} u_{jk} = \delta_{ij} \qquad \text{and} \qquad \sum_{k=1}^n u_{ki} u_{kj} = \delta_{ij}.$$

Since also the classical permutation and orthogonal group fit into this frame we have now four orthogonal Hopf algebras sitting inside each other like this:

$$\begin{array}{ccc} S_n^+ & \subset & O_n^+ \\ \cup & & \cup \\ S_n & \subset & O_n \end{array}$$

We want to ask the following question: are there more non-commutative versions G_n^+ of classical groups G_n. Or more general; are there more nice non-commutative quantum groups G_n^*, stronger than S_n?

$$
\begin{array}{ccccc}
S_n^+ & \subset & G_n^+ & \subset & O_n^+ \\
\cup & & \cup & & \cup \\
S_n & \subset & G_n & \subset & O_n
\end{array}
\qquad\qquad
\begin{array}{ccccc}
S_n^+ & & \subset & & O_n^+ \\
 & & & \diagup & \\
\cup & & G_n^* & & \cup \\
 & \diagup & & & \\
S_n & & \subset & & O_n
\end{array}
$$

In the above, with having (non-classical) quantum groups sitting inside each other, $S_n \subset G_n^* \subset O_n^+$, we mean of course that we have homomorphisms between the corresponding orthogonal quantum groups: $C(S_n) \leftarrow C(G_n^*) \leftarrow C(O_n^+)$.

Of course, it is not apriori clear how we should describe and understand intermediate quantum groups. The guiding principle of "liberation" – write down the defining relations in the classical case and drop the commutativity requirement – is somehow clear in the case of orthogonal and permutation group; in more general cases, however, it is usually not so clear what the canonical form of such equations is. A better way for dealing with such intermediate quantum groups is to look on their representations. And a good way to deal with those is by describing them by spaces of intertwiners.

Definition 12. Associated to an orthogonal Hopf algebra $(A = C(G_n^*), (u_{ij})_{i,j=1}^n)$ are the *spaces of intertwiners*:

$$\mathbf{I}_{G_n^*}(k,l) = \{T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l} \mid Tu^{\otimes k} = u^{\otimes l}T\},$$

where $u^{\otimes k}$ is the $n^k \times n^k$ matrix $(u_{i_1 j_1} \dots u_{i_k j_k})_{i_1 \dots i_k, j_1 \dots j_k}$; i.e., if we consider $u \in M_n(A)$ as a mapping $u : \mathbb{C}^n \to \mathbb{C}^n \otimes A$, then $u^{\otimes k}$ is a mapping

$$u^{\otimes k} : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k} \otimes A.$$

Note: if $T \in \mathbf{I}_{G_n^*}(0,l)$, then $\eta := T1 \in (\mathbb{C}^n)^{\otimes l}$ is a fixed vector unter $u^{\otimes l}$. Namely, $Tu^{\otimes 0} = u^{\otimes l}T$ implies that

$$\eta = Tu^{\otimes 0}1 = u^{\otimes l}T1 = u^{\otimes l}\eta.$$

Furthermore, we always have that $\xi := \sum_i e_i \otimes e_i \in \mathbf{I}_{G_n^*}(0,2)$. To check this we have to see that $\left(u^{\otimes 2}\xi\right)_{i_1,i_2} = \xi_{i_1,i_2}$. That this is indeed the case follows like this:

$$\begin{aligned}
\left(u^{\otimes 2}\sum_i e_i \otimes e_i\right)_{i_1,i_2} &= \sum_i \sum_{j_1,j_2} u_{i_1 j_1} u_{i_2 j_2} (e_i \otimes e_i)_{j_1,j_2} \\
&= \sum_i \sum_{j_1,j_2} u_{i_1 j_1} u_{i_2 j_2} \delta_{i j_1} \delta_{i j_2} = \sum_i u_{i_1 i} u_{i_2 i} = \delta_{i_1 i_2} = \left(\sum_i e_i \otimes e_i\right)_{i_1,i_2}
\end{aligned}$$

It is easy to check that the intertwiner space of our quantum groups has the following properties.

Proposition 2. *The space of intertwiners* $\mathbf{I}_{G_n^*}$ *of an orthogonal Hopf algebra is a* tensor category with duals, *i.e., it is a collection of vector spaces* $\mathbf{I}_{G_n^*}(k,l)$ *with the following properties:*

- $T, T' \in \mathbf{I}_{G_n^*}$ *implies* $T \otimes T' \in \mathbf{I}_{G_n^*}$.
- *If* $T, T' \in \mathbf{I}_{G_n^*}$ *are composable, then* $TT' \in \mathbf{I}_{G_n^*}$.
- $T \in \mathbf{I}_{G_n^*}$ *implies* $T^* \in \mathbf{I}_{G_n^*}$.
- $id(x) = x$ *is in* $\mathbf{I}_{G_n^*}(1,1)$.
- $\xi = \sum e_i \otimes e_i$ *is in* $\mathbf{I}_{G_n^*}(0,2)$.

It follows from Woronowicz's fundamental Tannaka-Krein theory for compact quantum groups [30] that the space of intertwiners contains all relevant information about the quantum groups.

Theorem 4 (Woronowicz 1988). *The compact quantum group* G_n^* *can actually be rediscovered from its space of intertwiners. Thus there is a one-to-one correspondence between:*

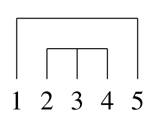
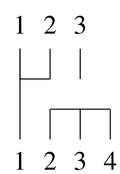
- *orthogonal Hopf algebras* $C(O_n^+) \to C(G_n^*) \to C(S_n)$
- *tensor categories with duals* $\mathbf{I}_{O_n^+} \subset \mathbf{I}_{G_n^*} \subset \mathbf{I}_{S_n}$.

So the question is now whether we have some concrete description of the relevant spaces of intertwiners. Since all of them have to sit inside the space of intertwiners for the classical permutation group, we will first take a look on those. It turns out that they can actually be described in combinatorial terms via partitions.

We denote by $P(k,l)$ the set of partitions of the set with repetitions $\{1,\dots,k,1,\dots,l\}$. Such a partition will be pictured as

$$p = \begin{array}{c} 1\dots k \\ \mathcal{P} \\ 1\dots l \end{array}$$

where $\mathcal{P}$ is a diagram joining the elements in the same block of the partition. Here are two examples of such partitions.

$p =$  $\in P(0,5)$ $\qquad$ $q =$  $\in P(3,4)$

Associated to any partition $p \in P(k,l)$ is a linear map $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ given by

$$T_p(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \cdots j_l} \delta_p(i,j)\, e_{j_1} \otimes \dots \otimes e_{j_l},$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$, and where

$$\delta_p(i,j) = \begin{cases} 1, & \text{if all indices which are connected by } p \text{ are the same} \\ 0, & \text{otherwise.} \end{cases}$$

Example 7.

$$T_{|\,|}(e_a \otimes e_b) = e_a \otimes e_b, \qquad T_{\{|\!-\!|\}}(e_a \otimes e_b) = \delta_{ab}\, e_a \otimes e_a,$$

$$T_{\left\{\substack{\sqcup \\ |\,|}\right\}}(e_a \otimes e_b) = \delta_{ab} \sum_{cd} e_c \otimes e_d, \qquad T_{\left\{\substack{|\,\rfloor \\ \lceil\,}\right\}}(e_a \otimes e_b) = e_b.$$

One can now check that all those T_p are intertwiners for the permutation groups S_n. Namely, take $u \hat{=} \pi$ permutation matrix, i.e., $ue_i = e_{\pi^{-1}(i)}$. Then

$$\begin{aligned} T_p u^{\otimes k} e_{i_1} \otimes \cdots \otimes e_{i_k} &= T_p e_{\pi^{-1}(i_1)} \otimes \cdots \otimes e_{\pi^{-1}(i_k)} \\ &= \sum_j \delta_p(\pi^{-1}(i_1), \ldots, \pi^{-1}(i_k), j_1, \ldots, j_l) e_{j_1} \otimes \cdots \otimes e_{j_l} \end{aligned}$$

and

$$\begin{aligned} u^{\otimes l} T_p e_{i_1} \otimes \cdots \otimes e_{i_k} &= u^{\otimes l} \sum_r \delta_p(i_1, \ldots, i_k, r_1, \ldots, r_l) e_{r_1} \otimes \cdots \otimes e_{r_l} \\ &= \sum_r \delta_p(i_1, \ldots, i_k, r_1, \ldots, r_l) e_{\pi^{-1}(r_1)} \otimes \cdots \otimes e_{\pi^{-1}(r_l)} \\ &= \sum_j \delta_p(i_1, \ldots, i_k, \pi(j_1), \ldots, \pi(j_l)) e_{j_1} \otimes \cdots \otimes e_{j_l}. \end{aligned}$$

But then the two calculations give the same, because we have

$$\delta_p(\pi^{-1}(i_1), \ldots, \pi^{-1}(i_k), j_1, \ldots, j_l) = \delta_p(i_1, \ldots, i_k, \pi(j_1), \ldots, \pi(j_l)).$$

Actually, the T_p form a basis and we have

$$\mathbf{I}_{S_n}(k,l) = \operatorname{span}(T_p | p \in P(k,l)).$$

Also for the other three basic orthogonal Hopf algebras their intertwiner space is spanned by the T_p; in those cases we are, however, not running over all p, but only over some subsets in $P(k,l)$.

Let $NC(k,l) \subset P(k,l)$ be the subset of noncrossing partitions and denote by P_2 the subset of all pairings and by NC_2 the subset of non-crossing pairings. Then we have

$$\begin{array}{ccc} \operatorname{span}(T_p | p \in NC(k,l)) = \mathbf{I}_{S_n^+}(k,l) & \supset & \mathbf{I}_{O_n^+}(k,l) = \operatorname{span}(T_p | p \in NC_2(k,l)) \\ \cap & & \cap \\ \operatorname{span}(T_p | p \in P(k,l)) = \mathbf{I}_{S_n}(k,l) & \supset & \mathbf{I}_{O_n}(k,l) = \operatorname{span}(T_p | p \in P_2(k,l)) \end{array}$$

These observations led to the following definition, see [5].

Definition 13 (Banica, Speicher 2009). A quantum group $S_n \subset G_n^* \subset O_n^+$ is called *easy* if its associated tensor category is of the form

$$\mathbf{I}_{S_n} = \operatorname{span}(T_p \mid p \in P) \quad \subset \quad \mathbf{I}_{G_n^*} = \operatorname{span}(T_p \mid p \in P_{G^*}) \quad \subset \quad \mathbf{I}_{O_n^+} = \operatorname{span}(T_p \mid p \in NC_2)$$

for a certain collection of subsets $P_{G^*} \subset P$.

There are now several questions arising canonically in this context. We are interested in

- classification of easy (and more general) quantum groups,
- understanding of meaning/implications of symmetry under such quantum groups; in particular, under quantum permutations S_n^+, or quantum rotations O_n^+,
- treating series of such quantum groups (like S_n^+ or O_n^+) as fundamental examples of non-commuting random matrices.

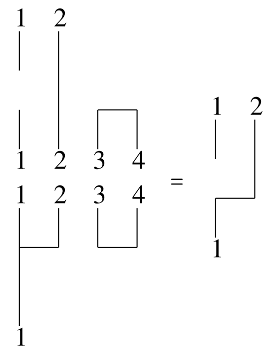
4.2 Classification results for easy quantum groups

The subsets $P_{G^*} \subset P$ of partitions, which appear in the definition of easy quantum groups, cannot be arbitrary, but they must have quite some structure. Namely they are "category of partitions" in the following sense.

Definition 14 (Banica, Speicher 2009). A *category of partitions* P_{G^*} is a subset of P which satisfies:

- P_{G^*} is stable by tensor product.
- P_{G^*} is stable by composition.
- P_{G^*} is stable by involution.
- P_{G^*} contains the "unit" partition $|$.
- P_{G^*} contains the "duality" partition $\sqcap$.

Example 8. The only operation which is not selfexplanatory is the composition. Two partitions are composed by identifying the lower line of the first with the upper line of the second (which have to agree, in order to be defined); possibly appearing loops will be removed. Here is an example of a composition $P(2,4) \times P(4,1) \to P(2,1)$:



The classification of the free and the classical easy quantum groups is relatively straightforward and was achieved in [5, 28].

Theorem 5 (Banica, Speicher 2009; Weber 2011). *1) There are*

- *7 categories of noncrossing partitions and*
- *6 categories of partitions containing the basic crossing.*

In the non-crossing case the seven categories are given by

$$\begin{array}{ccccccc}
\left\{\begin{matrix}\text{singletons}\\ \text{pairings}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons}\\ \text{pairings}\\ (\textit{even part})\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons}\\ \text{pairings}\\ (\textit{resp. parity})\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all}\\ \text{pairings}\end{matrix}\right\} \\
\cap & & \cap & & & & \cap \\
\left\{\begin{matrix}\text{all}\\ \text{partitions}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all partitions}\\ \text{(even part)}\end{matrix}\right\} & \supset & & & \left\{\begin{matrix}\text{blocks of}\\ \text{even size}\end{matrix}\right\}
\end{array},$$

in the classical case the middle upper two collapse to one.

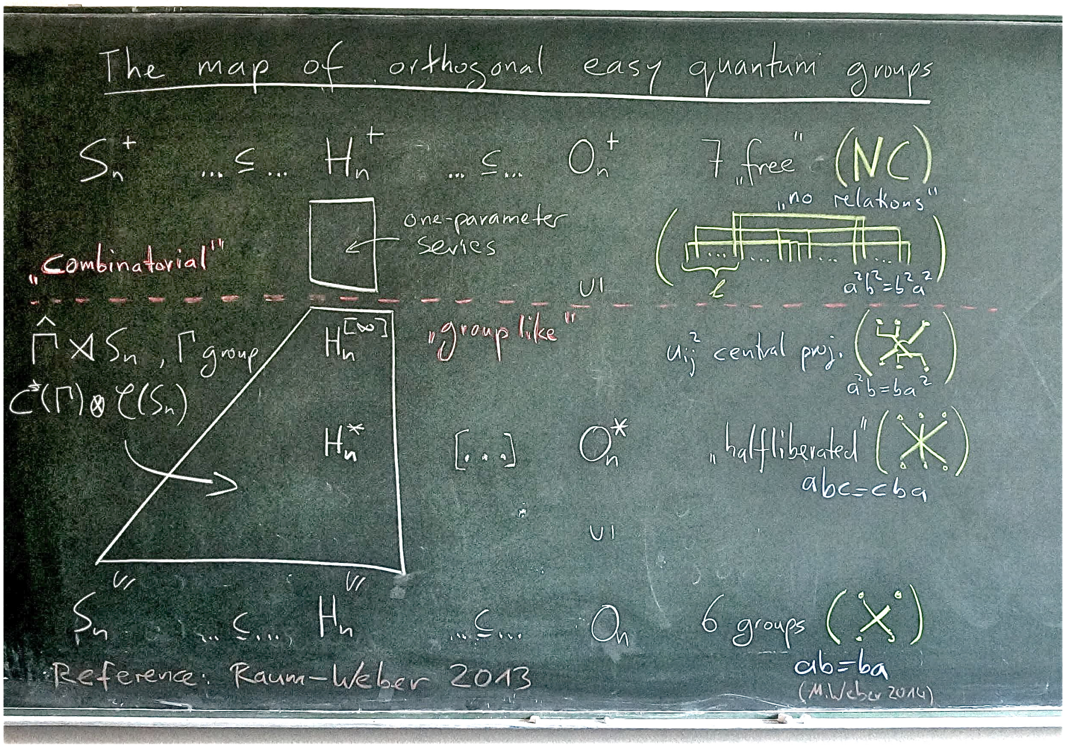
2) Thus there are seven free easy quantum groups $S_n^+ \subset G_n^+ \subset O_n^+$:

$$\begin{array}{ccccccc}
B_n^+ & \subset & {B'}_n^+ & \subset & B_n^{\#+} & \subset & O_n^+ \\
\cup & & \cup & & & & \cup \\
S_n^+ & \subset & {S'}_n^+ & & \subset & & H_n^+
\end{array}$$

and six classical easy groups $S_n \subset G_n \subset O_n$; in the classical case B_n' and $B_n^\sharp$ collapse to one group. The six classical easy groups are:

- *O_n and S_n,*
- *$H_n = \mathbb{Z}_2 \wr S_n$: the hyperoctahedral group, consisting of monomial matrices with ± 1 nonzero entries,*
- *$B_n \simeq O_{n-1}$: the bistochastic group, consisting of orthogonal matrices having sum 1 in each row and each column,*
- *$S_n' = \mathbb{Z}_2 \times S_n$: permutation matrices multiplied by ± 1,*
- *$B_n' = \mathbb{Z}_2 \times B_n$: bistochastic matrices multiplied by ± 1.*

The general case of easy quantum groups, which are not necessarily free nor classical, but just sitting between S_n and O_n^+, is much harder and it took a while before a complete classification was achieved. After some first results by Banica, Curran and Speicher [2], this picture was completed in a series of impressive papers by Raum and Weber [16, 17]. We present here only the final picture.



4.3 Non-commutative random matrices

Weingarten formula

The philosophy behind easy quantum groups is that they are defined in combinatorial terms and thus their description and properties should also rely essentially on these combinatorial data. One instance where this philosophy could be implemented successfully is [10], where the representation theory and fusion rules of easy quantum groups were described in terms of the underlying category of partitions. Other instances are de Finetti theorems and more stochastic properties of the "random matrices" u. Those rely on the Haar state. There exists, as for any compact quantum group, a unique Haar state on the easy quantum groups, thus one can integrate/average over those quantum groups. In accordance with the philosophy above there exists for easy quantum groups a nice and "concrete" combinatorial formula for the calculation of this Haar state.

Theorem 6 (Weingarten formula for an easy quantum group). *Denote by* $D = (D(k))_{k\in\mathbb{N}}$ *the category of partitions for the easy quantum group* G_n^**; where* $D(k) := D(0,k)$*. Then*

$$\int_{G_n^*} u_{i_1 j_1} \cdots u_{i_k j_k} du = \sum_{\substack{p,q\in D(k)\\ p\le \ker i\\ q\le \ker j}} W_n(p,q),$$

where $W_{k,n} = (W_n(p,q))_{p,q\in D(k)} = G_{k,n}^{-1}$ *is the inverse of the Gram matrix*

$$G_{k,n} = (G_n(p,q))_{p,q\in D(k)} \qquad \text{where} \qquad G_n(p,q) = n^{|p\vee q|}.$$

Note: $p\vee q$ *is here always the supremum in the lattice of all partitions; i.e.,* $p\vee q$ *is not necessarily in D.*

This theorem is from [5]. For earlier special cases see [1, 6]. Compare also Chapter **??**.

Example 9. We want to integrate $u_{21}u_{23}$. Then $i = (2,2)$, $j = (1,3)$, hence

$$\ker i = \sqcup\,, \qquad \ker j = |\,|$$

and thus

$$\int_{G_n} u_{21}u_{23} du = W(\sqcup, |\,|) + W(|\,|, |\,|)$$

Similarly,

$$\int_{G_n} u_{23}u_{23} du = W(\sqcup, \sqcup) + W(\sqcup, |\,|) + W(|\,|, \sqcup) + W(|\,|, |\,|)$$

All probabilistic properties are now encoded in this Weingarten function. For finite n this is a quite complicated object, which is not easy to handle. Much more can be said asymptotically if n goes to infinity. In particular, we have the asymptotics

$$W_n(p,q) = O(n^{|p\vee q|-|p|-|q|}).$$

Based on this one can derive de Finetti theorems for various easy quantum groups, see [7, 8, 4].

Distribution of traces of powers

Equipped with the Haar state one can now treat our matrix $u = (u_{ij})$ as a non-commutative analogue of a random matrix and address questions – in analogy to classical work of Diaconis and Shahshahani [9] – about the distribution (with respect to the Haar state) of traces of u. The following basic result was derived in [3].

Theorem 7 (Banica, Curran, Speicher 2011). *Let G be an easy quantum group. Consider $s \in \mathbb{N}$, $k_1, \dots, k_s \in \mathbb{N}$, $k := \sum_{i=1}^{s} k_i$, and denote*

$$\gamma := (1,2,\dots,k_1)(k_1+1,k_1+2,\dots,k_1+k_2)\cdots(\cdots,k) \in S_k$$

Then we have, for any n such that G_{kn} is invertible:

$$\int_{G_n} \mathrm{Tr}(u^{k_1})\dots\mathrm{Tr}(u^{k_s})\,du = \#\{p \in D(k) \mid p = \gamma(p)\} + O(1/n).$$

If G is a classical easy group, then this formula is exact, without any lower order corrections in n.

Proof. By using the Weingarten formula we can calculate the expectation as follows.

$$\begin{aligned}
I &:= \int_G \mathrm{Tr}(u^{k_1})\dots\mathrm{Tr}(u^{k_s})\,du \\
&= \sum_{i_1\dots i_k} \int_G (u_{i_1 i_2}\dots u_{i_{k_1} i_1})\dots(u_{i_{k-k_s+1} i_{k-k_s+2}}\dots u_{i_k i_{k-k_s+1}}) \\
&= \sum_{i_1\dots i_k} \int_G u_{i_1 i_{\gamma(1)}}\dots u_{i_k i_{\gamma(k)}} \\
&= \sum_{i_1\dots i_k=1}^{n} \sum_{\substack{p,q\in D_k \\ p\le\ker\mathbf{i},\, q\le\ker\mathbf{i}\circ\gamma}} W_{kn}(p,q) \\
&= \sum_{i_1\dots i_k=1}^{n} \sum_{\substack{p,q\in D_k \\ p\le\ker\mathbf{i},\, \gamma(q)\le\ker\mathbf{i}}} W_{kn}(p,q) \\
&= \sum_{p,q\in D_k} \sum_{\substack{i_1\dots i_k=1 \\ p\le\ker\mathbf{i},\, \gamma(q)\le\ker\mathbf{i}}}^{n} W_{kn}(p,q) \\
&= \sum_{p,q\in D_k} n^{|p\vee\gamma(q)|} W_{kn}(p,q) \\
&= \sum_{p,q\in D_k} n^{|p\vee\gamma(q)|} n^{|p\vee q|-|p|-|q|}(1+O(1/n)).
\end{aligned}$$

The leading order of $n^{|p\vee\gamma(q)|+|p\vee q|-|p|-|q|}$ is n^0, which is achieved if and only $p = q = \gamma(q)$.

In the classical case, instead of using the approximation for $W_{nk}(p,q)$, we can write $n^{|p\vee\gamma(q)|}$ as $G_{nk}(\gamma(q),p)$. (Note that this only makes sense if we know that

$\gamma(q)$ is also an element in D_k; and this is only the case for the classical partition lattices.) Then one can continue as follows:

$$I = \sum_{p,q\in D_k} G_{nk}(\gamma(q),p)W_{kn}(p,q) = \sum_{q\in D_k} \delta(\gamma(q),q) = \#\{q \in D_k | q = \gamma(p)\}.$$

□

This description can be used to calculate the distribution of $u_r := \lim_{n\to\infty} \mathrm{Tr}(u^r)$. We list here some prominent cases from [3].

Variable	O_n	O_n^+
u_1	real Gaussian	semicircular
u_2	real Gaussian	semicircular
u_r $(r \geq 3)$	real Gaussian	circular

Variable	S_n	S_n^+
u_1	Poisson	free Poisson
$u_2 - u_1$	Poisson	semicircular
$u_r - u_1$ $(r \geq 3)$	sum of Poissons	circular

One should note that in the non-commutative situation traces of powers are not selfadjoint in general. Whereas $\mathrm{Tr}(u)$ and $\mathrm{Tr}(u^2)$ are selfadjoint, this is not true for $\mathrm{Tr}(u^3)$ in the general non-commutative situation! We have $u_1 = \sum u_{ii} = u_1^*$ and $u_2 = \sum u_{ij}u_{ji} = \sum u_{ji}u_{ij} = u_2^*$, but $u_3 = \sum u_{ij}u_{jl}u_{li} \neq \sum u_{li}u_{jl}u_{ij} = u_3^*$.

Eigenvalues: The final frontier

One should also note that in the classical case, knowledge about traces of powers of the matrices is the same as knowledge about the eigenvalues of the matrices. This raises our final question:

What actually are eigenvalues of a non-commutative matrix?

Unfortunately, at the moment we have nothing to say about this, and we have to remain with Wittgenstein's dictum:

"Whereof one cannot speak, thereof one must be silent".

References

1. T. Banica and B. Collins: Integration over quantum permutation groups. *J. Funct. Anal.* 242 (2007), 641–657.
2. T. Banica, S. Curran, and R. Speicher: Classification results for easy quantum groups. *Pac. J. Math.* 247 (2010), 1–26.

3. T. Banica, S. Curran, and R. Speicher: Stochastic aspects of easy quantum groups. *Prob. Th. Related Fields* 149 (2011), 435–462.
4. T. Banica, S. Curran, and R. Speicher: De Finetti theorems for easy quantum groups. *Ann. Probab.* 40 (2012), 401–435.
5. T. Banica and R. Speicher: Liberation of orthogonal Lie groups. *Advances Math.* 222 (2009), 1461–1501.
6. B. Collins and P. Sniady: Integration with respect to the Haar measure on the unitary, orthogonal and symplectic group, *Comm. Math. Phys.* 264 (2006), 773-795.
7. S. Curran: Quantum exchangeable sequences of algebras. *Indiana Univ. Math. J.* 58 (2009), 1097-1126.
8. S. Curran: Quantum rotability. *Trans. Amer. Math. Soc.* 362 (2010), 4831–4851.
9. P. Diaconis and M. Shahshahani: On the eigenvalues of random matrices. *J. Appl. Probab.* (1994), 49–62.
10. A. Freslon and M. Weber: On the representation theory of partition (easy) quantum groups. *Journal für die reine und angewandte Mathematik (Crelles Journal)* (to appear), arXiv:1308.6390, 42 pages (2013).
11. F. Hiai and D. Petz: The semicircle law, free random variables and entropy. AMS, Providence, RI, 2000.
12. O. Kallenberg: Probabilistic symmetries and invariance principles. Probability and its Applications, Springer Verlag, New York 2005.
13. C. Köstler: A noncommutative extended de Finetti theorem. *J. Funct. Anal.* 258 (2010), 1073–1120.
14. C. Köstler and R. Speicher: A noncommutative de Finetti theorem: invariance under quantum permutations is equivalent to freeness with amalgamation. *Comm. Math. Phys.* 291 (2009), 473–490.
15. A. Nica and R. Speicher: Lectures on the combinatorics of free probability. *London Mathematical Society Lecture Note Series*, 335. Cambridge University Press, Cambridge, 2006. xvi+417 pp.
16. S. Raum and M. Weber: Easy quantum groups and quantum subgroups of a semi-direct product quantum group. arXiv:1311.7630, 26 pages (2013).
17. S. Raum and M. Weber: The full classification of orthogonal easy quantum groups. arXiv:1312.3857, 38 pages (2013).
18. R. Speicher: Multiplicative functions on the lattice of non-crossing partitions and free convolution. *Math. Ann.* 298 (1994), 611–628.
19. R. Speicher: Combinatorial theory of the free product with amalgamation and operator-valued free probability theory. Memoirs of the American Math. Society 132 (1998), x+88.
20. R. Speicher: Free probability and random matrices. In *The Oxford handbook of random matrix theory* (eds G. Akemann, J. Baik, and P. Di Francesco), Oxford University Press, 2011,
21. D. Voiculescu: Symmetries of some reduced free product C^*-algebras. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, Lecture Notes in Math., vol. 1132, Springer, Berlin, 1985, pp. 556–588.
22. D. Voiculescu: Limit laws for random matrices and free products. *Invent. Math.* 104 (1991), 201–220.
23. D. Voiculescu: Operations on certain non-commutative operator-valued random variables. Recent advances in operator algebras (Orlans, 1992). Astérisque No. 232 (1995).
24. D. Voiculescu: Lectures on free probability theory. In Lectures on probability theory and statistics, St. Flour 28 (1998), Springer Lecture Notes in Mathematics vol. 1738, 279–349.
25. D. Voiculescu, K. Dykema, and A. Nica: Free Random Variables. CRM Monograph Series, Vol. 1, American Math. Society, 1992.
26. S. Wang: Free products of compact quantum groups. *Comm. Math. Phys.* 195 (1995), 671–692.
27. S. Wang: Quantum symmetry groups of finite spaces. *Comm. Math. Phys.* 195 (1998), 195-211.

28. M. Weber: On the classification of easy quantum groups. *Advances Math.* 245 (2013), 500-533.
29. S. Woronowicz: Compact matrix pseudogroups. *Comm. Math. Phys.* 111 (1987), 613-665.
30. S. Woronowicz: Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups. *Invent. Math.* 93 (1988), 33–76.