

# Higher Order Freeness: A Survey

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Second order freeness and fluctuations of random matrices:

Mingo + Speicher:

I. Gaussian and Wishart matrices and cyclic Fock spaces

JFA 235 (2006), 226-270

Mingo + Sniady + Speicher:

II. Unitary random matrices

Adv. Math. 209 (2007), 212-240

Collins + Mingo + Sniady + Speicher:

III. Higher order freeness and free cumulants

Documenta Math. 12 (2007), 1-70

Kusallik + Mingo + Speicher: CRELLES 604 (2007), 1-46

Mingo + Speicher + Tan: arXiv:0708.0586 (to appear in TAMS)

## Warning

We deal only with **complex** random matrices.

Higher order freeness for the real case still has to be worked out!

We want to consider  $N \times N$  random matrices  $A_N$  in the limit  $N \rightarrow \infty$ .

Which kind of information about the random matrices do we want to keep in the limit  $N = \infty$ ?

Consider selfadjoint Gaussian  $N \times N$  random matrices  $X_N$ . One knows:

- empirical eigenvalue distribution of  $X_N$  converges almost surely to deterministic limit distribution  $\mu_X$
- one has a large deviation principle for convergence towards  $\mu_X$

## Wigner's semicircle law

$$N = 4000$$

Convergence of  $\mu_{X_N}$  towards  $\mu_X$  is governed by **large deviation principle**:

$$\text{Prob}(\mu_{X_N} \approx \nu) \sim e^{-N^2 I(\nu)},$$

where rate function  $\nu \mapsto I(\nu)$  is given as Legendre transform of

$$\mathbb{C}\langle X \rangle \ni p \mapsto \lim_{N \rightarrow \infty} \frac{1}{N^2} \log E \left\{ e^{-N^2 \text{tr}(p(X_N))} \right\}$$

Note:

$$\log E \left\{ e^{-N^2 \text{tr}(p(X_N))} \right\} = \sum_r \frac{(-1)^r}{r!} N^{2r} \cdot k_r \left( \text{tr}(p(X_N)), \dots, \text{tr}(p(X_N)) \right)$$

where

$k_r$  are classical cumulants

This motivates our **general assumption** on the considered random matrices  $A_N$ :

For all  $r \in \mathbb{N}$  and all  $k_1, \dots, k_r \in \mathbb{N}$  the following limits exists

$$\lim_{N \rightarrow \infty} N^{2r-2} \underbrace{k_r \left( \text{tr}(A_N^{k_1}), \dots, \text{tr}(A_N^{k_r}) \right)}_{\text{classical cumulants of traces of powers}} =: \alpha_{k_1, \dots, k_r}^A$$

The  $\alpha$ 's are the asymptotic **correlation moments** of our random matrix ensemble  $A_N$  and constitute its

**limiting distribution of all orders.**

Typical examples for random matrices where limiting distribution of all orders exists: Gaussian random matrices, Wishart random matrices, Haar unitary random matrices, and combinations of independent copies of such ensembles

Note: We are looking on random matrix ensembles whose eigenvalues have a correlation as for Gaussian random matrices:

$$\text{tr}(A_N^k) = \frac{\lambda_1^k + \dots + \lambda_N^k}{N}$$

Eigenvalues  $\lambda_1, \dots, \lambda_N$  of  $A_N$  are **not independent**, but feel some interaction



Contrast this with following situation:

$$D_N = \begin{pmatrix} \eta_1 & 0 & \cdots & 0 \\ 0 & \eta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_N \end{pmatrix},$$

where  $\eta_1, \eta_2, \dots$  are **independent** and identically distributed according to  $\eta$ .

Then

$$\text{tr}(D_N^k) = \frac{\eta_1^k + \cdots + \eta_N^k}{N} \rightarrow E[\eta^k]$$

with large deviation principle  $\sim e^{-\mathbf{N}H(\nu)}$ ; not  $\sim e^{-N^2 I(\nu)}$

In this case:

$$k_r(\text{tr}(D_N^{k_1}), \dots, \text{tr}(D_N^{k_r})) = N^{1-r} k_r(\eta^{k_1}, \dots, \eta^{k_r}),$$

and thus:  $D_N$  has no limiting distribution of all orders in our sense.

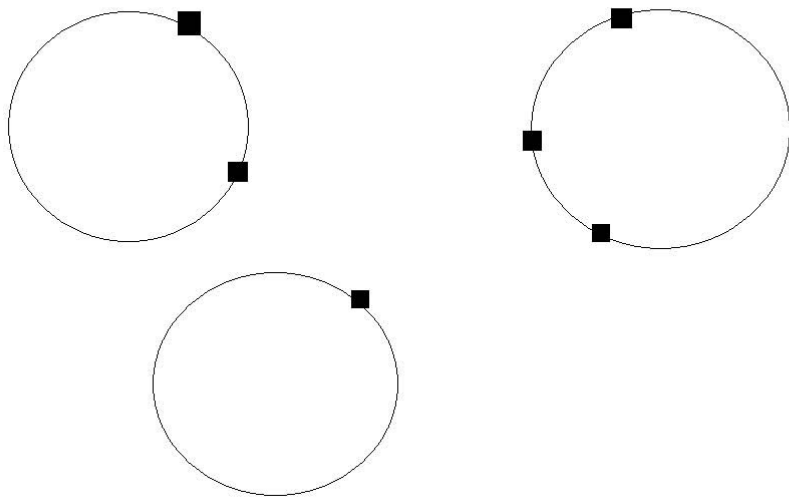
The Gaussian random matrices  $A_N$  and the above ensemble with a semicircle distribution for  $\eta$  have the same asymptotic eigenvalue distribution, but a quite different type of convergence towards the semicircle

Remarks:

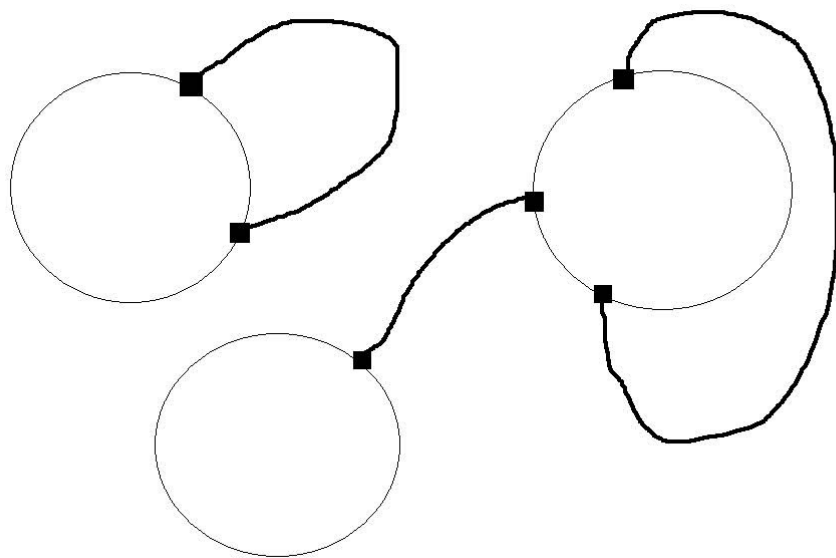
1) For Gaussian (and also for Wishart) random matrices there are nice combinatorial descriptions of the higher order limit distributions in terms of planar pictures

$\alpha_{k_1, \dots, k_r}^{Gaussian} = \# \text{NC-pairings of } r \text{ circles,}$   
with  $k_1$  points on first circle,  
 $k_2$  points on second circle, etc.  
such that all circles are connected by pairing

Consider  $\alpha_{2,3,1}$

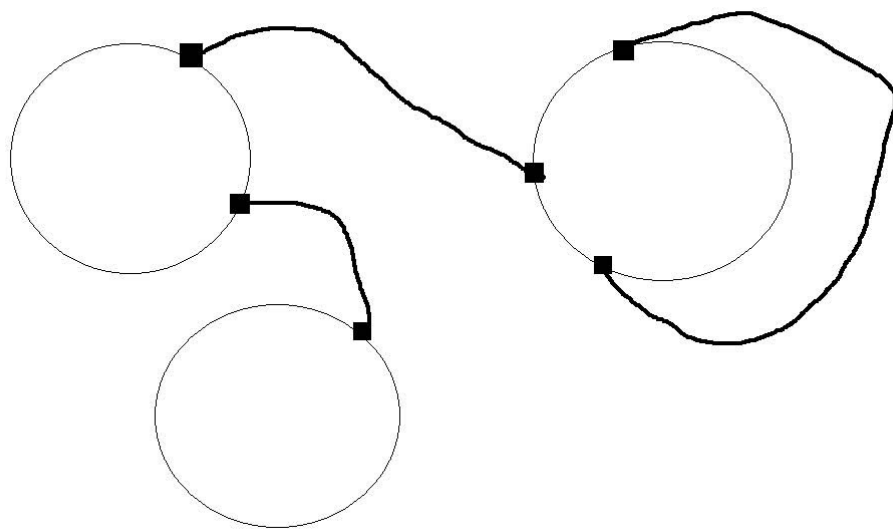


Consider  $\alpha_{2,3,1}$



does not count!

Consider  $\alpha_{2,3,1}$



counts!

2) Specialize general theory to second order: An  $N \times N$  random matrix ensemble  $(A_N)_{N \in \mathbb{N}}$  has a **second order limit distribution** if for all  $m, n \geq 1$  the limits

$$\alpha_n := \lim_{N \rightarrow \infty} E[\text{tr}(A_N^n)]$$

and

$$\alpha_{m,n} := \lim_{N \rightarrow \infty} \text{cov}(\text{Tr}(A_N^m), \text{Tr}(A_N^n))$$

exist and if all higher classical cumulants of  $\text{Tr}(A_N^m)$  go to zero.

This means that the family

$$\left( \text{Tr}(A_N^m) - E[\text{Tr}(A_N^m)] \right)_{m \in \mathbb{N}}$$

converges to a Gaussian family.

Example: Gaussian random matrix  $A$  ( $N = 40$ , trials=50.000)

$$\cdot \quad \text{Var}(\text{Tr}(A)) = 1 \quad \text{Var}(\text{Tr}(A^2)) = 2 \quad \text{Var}(\text{Tr}(A^4)) = 36$$



Now consider **two random matrix ensembles**  $A_N, B_N$

Relevant quantities are all joint correlation moments

$$\lim_{N \rightarrow \infty} N^{2r-2} k_r \left( \text{tr}(p_1(A_N, B_N)), \dots, \text{tr}(p_r(A_N, B_N)) \right)$$

for all  $r \in \mathbb{N}$  and all polynomials  $p_1, \dots, p_r$

**asymptotic joint distribution of all orders of  $A_N, B_N$**

**Theorem:** If  $A_N$  and  $B_N$  are in **generic position**, i.e.,

- $A_N$  and  $B_N$  are independent
- at least one of them is unitarily invariant

and if  $A_N$  as well as  $B_N$  have asymptotic distributions of all orders then also the asymptotic joint distribution of all orders of  $A_N, B_N$  exists and it is, furthermore, determined uniquely and in a universal way by the joint distribution of  $A$  and the joint distribution of  $B$ .

This universal calculation rule is the essence of

**freeness (of all orders)**

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \text{cov}(\text{Tr}(A_N B_N), \text{Tr}(A_N B_N)) \\
&= \lim_{N \rightarrow \infty} \left\{ E[\text{tr}(A_N A_N)] \cdot E[\text{tr}(B_N B_N)] \right. \\
&\quad - E[\text{tr}(A_N A_N)] \cdot E[\text{tr}(B_N)] \cdot E[\text{tr}(B_N)] \\
&\quad - E[\text{tr}(A_N)] \cdot E[\text{tr}(A_N)] \cdot E[\text{tr}(B_N B_N)] \\
&\quad + E[\text{tr}(A_N)] \cdot E[\text{tr}(A_N)] \cdot E[\text{tr}(B_N)] \cdot E[\text{tr}(B_N)] \\
&\quad + \text{cov}(\text{tr}(A_N), \text{tr}(A_N)) \cdot E[\text{tr}(B_N)] \cdot E[\text{tr}(B_N)] \\
&\quad \left. + E[\text{tr}(A_N)] \cdot E[\text{tr}(A_N)] \cdot \text{cov}(\text{tr}(B_N), \text{tr}(B_N)) \right\}
\end{aligned}$$

In order to understand this universal calculation rule use the

**idea of cumulants!**

Write our correlation moments

$$k_r(\text{tr}(A^{k_1}), \dots, \text{tr}(A^{k_r}))$$

in terms of cumulants of entries of our matrix,

$$k_r(a_{i(1)j(1)}, \dots, a_{i(r)j(r)}).$$

Asymptotically, the later will give the cumulants in our theory.

To make this connection explicit, consider

**unitarily invariant**  $A_N = (a_{ij})$ ,

i.e., the joint distribution of the entries of  $A_N$  is the same as the joint distribution of  $UA_NU^*$ , for any unitary  $N \times N$  matrix  $U$ .

Then, the only contributing cumulants of the entries are those with cycle structure in their indices!

We have a **Wick type formula**:

$$k_r(a_{i(1)j(1)}, \dots, a_{i(r)j(r)}) = \sum_{\pi \in S_n} \delta_{i,j \circ \pi} \kappa(\pi)$$

Examples:

$$k_1(a_{79}) = ?$$

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Note:  $\kappa$  depends actually on  $N$

$$k_1(a_{77}) = \kappa^{(N)}((1))$$

$$k_3(a_{79}, a_{95}, a_{57}) = \kappa^{(N)}((1, 2, 3))$$

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$$\pi \in S_r : \quad \kappa^{(N)}(\pi) \sim N^{-r+2-\#\pi}$$

$$\kappa(\pi) := \lim_{N \rightarrow \infty} N^{r-2+\#\pi} \kappa^{(N)}(\pi)$$

Consider

$$\alpha_{2,1} = \lim_{N \rightarrow \infty} N^2 k_2(\text{tr}(A^2), \text{tr}(A))$$

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$$\begin{aligned}\alpha_{2,1} &= \lim_{N \rightarrow \infty} N^2 k_2(\text{tr}(A^2), \text{tr}(A)) \\ &= \lim_{N \rightarrow \infty} N^2 \frac{1}{N^2} \sum_{i,j,k} k_2(a_{ij}a_{ji}, a_{kk})\end{aligned}$$

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 \alpha_{2,1} &= \lim_{N \rightarrow \infty} N^2 k_2(\text{tr}(A^2), \text{tr}(A)) \\
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 &\quad + k_2(a_{\mathbf{ij}}, a_{\mathbf{kk}})k_1(a_{\mathbf{ji}}) \\
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 &\quad + \kappa((1, 3))\kappa((2)) + \kappa((2)(3))\kappa((1)) + \kappa((2, 3))\kappa((1))
 \end{aligned}$$

Thus

$$\begin{aligned}\alpha_{2,1} = & \kappa((1, 2)(3)) \\ & + \kappa((1, 2, 3)) \\ & + \kappa((1, 3, 2)) \\ & + \kappa((1)(3))\kappa((2)) \\ & + \kappa((1, 3))\kappa((2)) \\ & + \kappa((2)(3))\kappa((1)) \\ & + \kappa((2, 3))\kappa((1))\end{aligned}$$

Thus

$$\begin{aligned}\alpha_{2,1} = & \kappa((1, 2)(3)) && \kappa_{2,1} \\ & + \kappa((1, 2, 3)) && \kappa_3 \\ & + \kappa((1, 3, 2)) && \kappa_3 \\ & + \kappa((1)(3))\kappa((2)) && \kappa_{1,2}\kappa_1 \\ & + \kappa((1, 3))\kappa((2)) && \kappa_2\kappa_1 \\ & + \kappa((2)(3))\kappa((1)) && \kappa_{1,1}\kappa_1 \\ & + \kappa((2, 3))\kappa((1)) && \kappa_2\kappa_1\end{aligned}$$

Thus

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 \alpha_{2,1} = & \kappa((1, 2)(3)) && \kappa_{2,1} && \kappa(\{1, 2, 3\}, (1, 2)(3)) \\
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 & + \kappa((2, 3))\kappa((1)) && \kappa_2\kappa_1 && \kappa(\{\mathbf{1}\}\{\mathbf{2}, \mathbf{3}\}, (1)(2, 3))
 \end{aligned}$$

general combinatorial object

**partitioned permutation**  $(\mathcal{V}, \pi) \in \mathcal{P}S_n$

$$\pi \in S_n, \quad \mathcal{V} \in \mathcal{P}_n, \quad \text{with} \quad \mathcal{V} \geq \pi$$

Index both correlation moments  $\varphi(\mathcal{V}, \pi)$  and cumulants  $\kappa(\mathcal{V}, \pi)$  with  $(\mathcal{V}, \pi)$ :

product of moments/cumulants according to blocks of  $\mathcal{V}$ , distribution into slots for arguments according to cycles of  $\pi$ :

Let  $C_1, \dots, C_9 \in \{A, B\}$ , with  $C_k = (c_{ij}^{(k)})_{i,j=1}^N$ .

$$\varphi\left(\{1, 3, 4, 6, 7\}\{2, 5, 8\}\{9\}, (1, 3, 4)(2, 8)(5)(6)(7)(9)\right)[C_1, \dots, C_9]$$

$$\kappa\left(\{1, 3, 4, 6, 7\}\{2, 5, 8\}\{9\}, (1, 3, 4)(2, 8)(5)(6)(7)(9)\right)[C_1, \dots, C_9]$$

Let  $C_1, \dots, C_9 \in \{A, B\}$ , with  $C_k = (c_{ij}^{(k)})_{i,j=1}^N$ .

$$\varphi(\{1, 3, 4, 6, 7\}\{2, 5, 8\}\{9\}, (1, 3, 4)(2, 8)(5)(6)(7)(9))[C_1, \dots, C_9]$$

$$= \lim_{N \rightarrow \infty} N^6 \cdot k_3(\text{tr}(C_1 C_3 C_4), \text{tr}(C_6), \text{tr}(C_7)) \dots$$

$$\kappa(\{1, 3, 4, 6, 7\}\{2, 5, 8\}\{9\}, (1, 3, 4)(2, 8)(5)(6)(7)(9))[C_1, \dots, C_9]$$

$$= \lim_{N \rightarrow \infty} N^9 \cdot k_5(c_{12}^{(1)}, c_{23}^{(3)}, c_{31}^{(4)}, c_{44}^{(6)}, c_{55}^{(7)}) \dots$$

Let  $C_1, \dots, C_9 \in \{A, B\}$ , with  $C_k = (c_{ij}^{(k)})_{i,j=1}^N$ .

$$\begin{aligned} & \varphi(\{1, 3, 4, 6, 7\}\{2, 5, 8\}\{9\}, (1, 3, 4)(2, 8)(5)(6)(7)(9))[C_1, \dots, C_9] \\ &= \lim_{N \rightarrow \infty} N^6 \cdot k_3(\text{tr}(C_1 C_3 C_4), \text{tr}(C_6), \text{tr}(C_7)) \cdot k_2(\text{tr}(C_2 C_8), \text{tr}(C_5)) \cdots \end{aligned}$$

$$\begin{aligned} & \kappa(\{1, 3, 4, 6, 7\}\{2, 5, 8\}\{9\}, (1, 3, 4)(2, 8)(5)(6)(7)(9))[C_1, \dots, C_9] \\ &= \lim_{N \rightarrow \infty} N^9 \cdot k_5(c_{12}^{(1)}, c_{23}^{(3)}, c_{31}^{(4)}, c_{44}^{(6)}, c_{55}^{(7)}) \cdot k_3(c_{12}^{(2)}, c_{21}^{(8)}, c_{33}^{(5)}) \cdots \end{aligned}$$

Let  $C_1, \dots, C_9 \in \{A, B\}$ , with  $C_k = (c_{ij}^{(k)})_{i,j=1}^N$ .

$$\begin{aligned} & \varphi(\{1, 3, 4, 6, 7\}\{2, 5, 8\}\{9\}, (1, 3, 4)(2, 8)(5)(6)(7)(9))[C_1, \dots, C_9] \\ &= \lim_{N \rightarrow \infty} N^6 \cdot k_3(\text{tr}(C_1 C_3 C_4), \text{tr}(C_6), \text{tr}(C_7)) \cdot k_2(\text{tr}(C_2 C_8), \text{tr}(C_5)) \cdot k_1(\text{tr}(C_9)) \end{aligned}$$

$$\begin{aligned} & \kappa(\{1, 3, 4, 6, 7\}\{2, 5, 8\}\{9\}, (1, 3, 4)(2, 8)(5)(6)(7)(9))[C_1, \dots, C_9] \\ &= \lim_{N \rightarrow \infty} N^9 \cdot k_5(c_{12}^{(1)}, c_{23}^{(3)}, c_{31}^{(4)}, c_{44}^{(6)}, c_{55}^{(7)}) \cdot k_3(c_{12}^{(2)}, c_{21}^{(8)}, c_{33}^{(5)}) \cdot k_1(c_{33}^{(9)}) \end{aligned}$$

Define **length function**

$$|(\mathcal{V}, \pi)| := n - (2\#\mathcal{V} - \#\pi)$$

We have triangle inequality  $((\mathcal{V}, \pi), (\mathcal{W}, \sigma) \in \mathcal{P}S_n)$

$$|(\mathcal{V} \vee \mathcal{W}, \pi\sigma)| \leq |(\mathcal{V}, \pi)| + |(\mathcal{W}, \sigma)|$$

Define product

$$(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = \begin{cases} (\mathcal{V} \vee \mathcal{W}, \pi\sigma), & |(\mathcal{V} \vee \mathcal{W}, \pi\sigma)| = |(\mathcal{V}, \pi)| + |(\mathcal{W}, \sigma)| \\ 0, & \text{otherwise} \end{cases}$$

Asymptotically, for  $N \rightarrow \infty$ , only the **geodesic terms** corresponding to equality in the triangle inequality contribute.

In particular, the relation between correlation moments and cumulants is given by the

**moment-cumulant formula for all orders**

$$\varphi(\mathcal{U}, \gamma)[C_1, \dots, C_n] = \sum_{\substack{(\mathcal{V}, \pi) \in \mathcal{PS}_n \\ (\mathcal{V}, \pi) \cdot (0, \gamma \pi^{-1}) = (\mathcal{U}, \gamma)}} \kappa(\mathcal{V}, \pi)[C_1, \dots, C_n]$$



If  $A_N$  and  $B_N$  are in generic position (i.e., asymptotically free of all orders), then we have for their asymptotic distribution

- **the vanishing of mixed cumulants**

$\kappa(1_n, \pi)[C_1, \dots, C_n] = 0$ , whenever  $C_1, \dots, C_n$  contain  $A$  as well as  $B$

- **convolution formula for cumulants of products**

$$\kappa(\mathcal{U}, \gamma)[AB, AB, \dots, AB]$$

$$= \sum_{(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = (\mathcal{U}, \gamma)} \kappa(\mathcal{V}, \pi)[A, A, \dots, A] \cdot \kappa(\mathcal{W}, \sigma)[B, B, \dots, B]$$

## Restrict now to special situation

Consider only first and second order, and restrict to problem of the sum of  $A$  and  $B$

If  $A$  and  $B$  are free, then the second order distribution (covariances) of  $A+B$  depends only on the expectations and covariances of  $A$  and of  $B$ .

Example: We have

$$\alpha_{1,2}^{A+B} = \alpha_{1,2}^A + \alpha_{1,2}^B + 2\alpha_1^A \cdot \alpha_{1,1}^B + 2\alpha_1^B \cdot \alpha_{1,1}^A,$$

i.e.,

$$\begin{aligned} \text{cov}\left(\text{Tr}(A + B), \text{Tr}((A + B)^2)\right) \\ = \text{cov}\left(\text{Tr}(A), \text{Tr}(A^2)\right) + \text{cov}\left(\text{Tr}(B), \text{Tr}(B^2)\right) \\ + 2E[\text{tr}(A)] \cdot \text{cov}\left(\text{Tr}(B), \text{Tr}(B)\right) \\ + 2E[\text{tr}(B)] \cdot \text{cov}\left(\text{Tr}(A), \text{Tr}(A)\right) \end{aligned}$$

**Moment-cumulant formulas** for first and second order say

$$\alpha_1 = \kappa_1$$

$$\alpha_2 = \kappa_2 + \kappa_1\kappa_1$$

$$\alpha_3 = \kappa_3 + \kappa_1\kappa_2 + \kappa_2\kappa_1 + \kappa_2\kappa_1 + \kappa_1\kappa_1\kappa_1$$

$$\alpha_4 = \kappa_4 + 4\kappa_1\kappa_3 + 2\kappa_2^2 + 6\kappa_1^2\kappa_2 + \kappa_1^4$$

⋮

$$\alpha_{1,1} = \kappa_{1,1} + \kappa_2$$

$$\alpha_{1,2} = \kappa_{1,2} + 2\kappa_1\kappa_1 + 2\kappa_3 + 2\kappa_1\kappa_2$$

$$\alpha_{2,2} = \kappa_{2,2} + 4\kappa_1\kappa_{1,2} + 4\kappa_1^2\kappa_{1,1} + 4\kappa_4 \\ + 8\kappa_1\kappa_3 + 2\kappa_2^2 + 4\kappa_1^2\kappa_2$$

⋮

Vanishing of mixed cumulants gives

**additivity of free cumulants** for free  $A, B$

$$\kappa_m^{A+B} = \kappa_m^A + \kappa_m^B \quad \forall m$$

and

$$\kappa_{m,n}^{A+B} = \kappa_{m,n}^A + \kappa_{m,n}^B \quad \forall m, n$$

Combinatorial relation between moments and cumulants can be rewritten in terms of generating power series

Recall: first order case (Voiculescu)

$$G(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_n}{x^{n+1}} \quad \text{Cauchy transform}$$

and

$$\mathcal{R}(x) = \sum_{n=1}^{\infty} \kappa_n x^{n-1} \quad \mathcal{R}\text{-transform}$$

are related by the relation

$$\frac{1}{G(x)} + \mathcal{R}(G(x)) = x.$$

## Second order $R$ -transform formula

$$G(x, y) := \sum_{m, n \geq 1} \alpha_{m, n} \frac{1}{x^{m+1}} \frac{1}{y^{n+1}}$$

and

$$\mathcal{R}(x, y) = \sum_{m, n \geq 1} \kappa_{m, n} x^{m-1} y^{n-1}$$

are related by the equation

$$G(x, y) = G'(x) \cdot G'(y) \cdot \mathcal{R}(G(x), G(y)) + \frac{\partial^2}{\partial x \partial y} \left[ \log \left( \frac{G(x) - G(y)}{x - y} \right) \right]$$

If second order free cumulants are zero, then formula reduces to

$$G(x, y) = \frac{\partial^2}{\partial x \partial y} \left[ \log \left( \frac{G(x) - G(y)}{x - y} \right) \right],$$

i.e. the fluctuations in such a case are determined by the eigenvalue distribution.

This is the formula of **Bai and Silverstein (2004)** for the fluctuations of general Wishart matrices.



$$G(x, y) = \frac{\partial^2}{\partial x \partial y} \left[ \log \left( \frac{G(x) - G(y)}{x - y} \right) \right],$$

Second order free cumulants are zero for example for

- Gaussian random matrices
- Wishart matrices
- independent sums of Gaussian and Wishart

## How do Wishart matrices fit in this theory?

Consider

$$A_N = X_N T_N X_N^*$$

where

- $X_N$  are  $N \times N$  non-selfadjoint Gaussian random matrices
- $T_N$  are random matrix ensemble such that second order limit distribution exists
- $X_N$  and  $T_N$  are independent (for example,  $T_N$  are deterministic)

Then, in first order,

$$A_N = X_N T_N X_N^*$$

converges to

$$A = CTC^*$$

where

- $C$  is circular
- $T$  has the limit distribution of the  $T_N$
- $C$  and  $T$  are  $*$ -free

And

$$A = CTC^*$$

is a

**free compound Poisson element,**

determined by the fact that

$$\kappa_n^A = \alpha_n^T \quad \text{for all } n$$

In terms of transforms this gives the fixed point equation of Marchenko-Pastur for the Cauchy transform of  $A$  in terms of the Cauchy transform of  $T$ .

In second order, the situation is exactly the same: The limit

$$A = CTC^*$$

of

$$A_N = X_N T_N X_N^*$$

is a

**free compound Poisson element of second order,**

determined by the fact that

$$\kappa_n^A = \alpha_n^T \quad \text{for all } n$$

and

$$\kappa_{m,n}^A = \alpha_{m,n}^T \quad \text{for all } m, n$$

$$\kappa_n^A = \alpha_n^T, \quad \kappa_{m,n}^A = \alpha_{m,n}^T \quad \text{for all } m, n$$

In terms of transforms this gives:

$$G^A(x, y) = \frac{G'(x) \cdot G'(y)}{G(x)^2 G(y)^2} \cdot G^T\left(\frac{1}{G(x)}, \frac{1}{G(y)}\right) + \frac{\partial^2}{\partial x \partial y} \left[ \log \left( \frac{G(x) - G(y)}{x - y} \right) \right]$$

If  $T_N$  are deterministic (i.e.,  $\kappa_{m,n}^A = \alpha_{m,n}^T = 0$ ), then this reduces to the formula of Bai-Silverstein