Quantum Groups with Partial Commutation Relations

Roland Speicher

Saarland University
Saarbrücken, Germany

supported by ERC Advanced Grant
"Non-Commutative Distributions in Free Probability"

Section 1

Quantum Groups

European Research Council
Established by the European Commission

What Are Quantum Groups?

- are generalizations of groups G (actually, of $C(G)$)
- are supposed to describe non-classical symmetries
- are Hopf algebras, with some additional structure ...

European Research Council
Established by the European Commission

What Are Quantum Groups?

Deformation of classical symmetries: $\mathbf{G} \rightsquigarrow \mathbf{G_q}$

- quantum groups are often deformations G_q of classical groups, depending on some parameter q, such that for $q \to 1$, they go to the classical group $G = G_1$
- G_q and G_1 are incomparable, none is stronger than the other
  - G_1 is supposed to act on commuting variables
  - G_q is the right replacement to act on q-commuting variables

Strengthening of classical symmetries: $\mathbf{G} \rightsquigarrow \mathbf{G^+}$

- there are situations where a classical group G has a genuine non-commutative analogue G^+ (no interpolations)
- G^+ is "stronger" than G: $\qquad G \subset G^+$
  - G acts on commuting variables
  - G^+ is the right replacement for acting on maximally non-commuting variables

Orthogonal Hopf Algebras

We are interested in quantum versions of **real compact matrix groups**. Think of

- orthogonal matrices or permutation matrices

Such quantum versions are captured by the notion of **orthogonal Hopf algebra**.

Definition

An **orthogonal Hopf algebra** is a C^*-algebra A, given with a system of n^2 self-adjoint generators $u_{ij} \in A$ ($i, j = 1, \dots, n$), subject to the following conditions:

- The inverse of $u = (u_{ij})$ is the transpose matrix $u^t = (u_{ji})$.
- $\Delta(u_{ij}) = \Sigma_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.
- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.
- $S(u_{ij}) = u_{ji}$ defines a morphism $S : A \to A^{op}$.

Orthogonal Hopf Algebras

Definition

An **orthogonal Hopf algebra** is a C^*-algebra A, given with a system of n^2 self-adjoint generators $u_{ij} \in A$ $(i, j = 1, \dots, n)$, subject to the following conditions:

- The inverse of $u = (u_{ij})$ is the transpose matrix $u^t = (u_{ji})$.
- $\Delta(u_{ij}) = \Sigma_k \, u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.
- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.
- $S(u_{ij}) = u_{ji}$ defines a morphism $S : A \to A^{op}$.

- These are compact quantum groups in the sense of Woronowicz.
- In the spirit of non-commutative geometry, we are thinking of $\mathbf{A} = \mathbf{C}(\mathbf{G}^+)$ as the continuous functions, generated by the coordinate functions u_{ij}, on some (non-existing) quantum group G^+, replacing a classical group G.

Theorem

The dual $C(O_n)$ of the classical orthogonal group is the universal commutative unital C^*-algebra generated by commuting selfadjoint u_{ij} $(i,j=1,\dots,n)$ subject to the relation: $u=(u_{ij})_{i,j=1}^n$ is an orthogonal matrix; i.e., for all i,j we have

$$\sum_{k=1}^n u_{ik}u_{jk}=\delta_{ij} \qquad \text{and} \qquad \sum_{k=1}^n u_{ki}u_{kj}=\delta_{ij}$$

Definition (**Quantum Orthogonal Group O_n^+ (Wang 1995)**)

The quantum orthogonal group $A_o(n)=C(O_n^+)$ is the universal unital C^*-algebra generated by selfadjoint u_{ij} $(i,j=1,\dots,n)$ subject to the relation: $u=(u_{ij})_{i,j=1}^n$ is an orthogonal matrix; i.e., for all i,j we have

$$\sum_{k=1}^n u_{ik}u_{jk}=\delta_{ij} \qquad \text{and} \qquad \sum_{k=1}^n u_{ki}u_{kj}=\delta_{ij}$$

Theorem

The dual $C(S_n^+)$ of the classical permutation group is the universal commutative unital C^*-algebra generated by commuting u_{ij} $(i,j=1,\dots,n)$ subject to the relations

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i,j=1,\dots,n$
- each row and column of $u=(u_{ij})_{i,j=1}^n$ is a partition of unity:

$$\sum_{j=1}^n u_{ij} = 1 \qquad \text{and} \qquad \sum_{i=1}^n u_{ij} = 1$$

Definition (Quantum Permutation Group S_n^+ (Wang 1998))

The quantum permutation group $A_s(n) = C(S_n^+)$ is the universal unital C^*-algebra generated by u_{ij} $(i,j=1,\dots,n)$ subject to the relations

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i,j=1,\dots,n$
- $$\sum_{j=1}^n u_{ij} = 1 \qquad \text{and} \qquad \sum_{i=1}^n u_{ij} = 1$$

Actions of Quantum Groups

Definition

The action of a matrix quantum group $u = (u_{ij})_{i,j=1}^n$ on a vector of operators $x = (x_1, \dots, x_n)$ is given by

$$x \mapsto y = u \odot x, \qquad \text{i.e.}$$

$$(x_1, \dots, x_n) \mapsto (y_1, \dots, y_n) \qquad \text{where} \qquad y_i = \sum_{j=1}^n u_{ij} \otimes x_j$$

Invariance under such actions corresponds to interesting geometric or probabilistic properties

Example (classical examples)

- the classical sphere is invariant under the orthogonal group
- distribution of classical i.i.d. (independent, identically distributed) random variables is invariant under permutations of the variables

Actions of Quantum Groups

Definition

The action of a matrix quantum group $u = (u_{ij})_{i,j=1}^n$ on a vector of operators $x = (x_1, \dots, x_n)$ is given by

$$x \mapsto y = u \odot x, \qquad \text{i.e.}$$

$$(x_1, \dots, x_n) \mapsto (y_1, \dots, y_n) \qquad \text{where} \qquad y_i = \sum_{j=1}^n u_{ij} \otimes x_j$$

Invariance under such actions corresponds to interesting geometric or probabilistic properties

Example (non-commutative examples)

- the "free" sphere is invariant under the quantum orthogonal group
- non-commutative distribution of f.i.d. (free, identically distributed) random variables is invariant under quantum permutations

de Finetti Theorems

Theorem (de Finetti 1931, Hewitt+Savage 1955)

The following are equivalent for an infinite sequence of classical random variables:

- the sequence is exchangeable (i.e., invariant under each S_n)
- the sequence is independent and identically distributed with respect to the conditional expectation E onto the tail σ-algebra of the sequence

Theorem (Köstler+Speicher 2009)

The following are equivalent for an infinite sequence of non-commutative random variables:

- the sequence is quantum exchangeable (i.e., invariant under each S_n^+)
- the sequence is free and identically distributed with respect to the conditional expectation E onto the tail-algebra of the sequence

Section 2

Mixtures of Classical and Free Independence

European Research Council
Established by the European Commission

History

- Młotkowski (2004): Λ-freeness
- Wysoczanksi + Speicher (2016): ε-independence
- Speicher + Weber (2016): new corresponding quantum groups

European Research Council
Established by the European Commission

Non-commutative probability spaces given by groups

$$G \quad \rightsquigarrow \quad (\mathbb{C}G, \tau) \quad \text{or} \quad (L(G), \tau)$$

nc (W^*-) probability space

where

$$\tau(\sum_g \alpha_g g) = \alpha_1 \quad \text{or} \quad \tau(x) = \langle x\delta_1, \delta_1 \rangle.$$

freeness/independence $\hat{=}$ basic group constructions: free/direct product $\star_{i\in I} G_i, \quad \times_{i\in I} G_i$

$\uparrow$

only "universal" unital constructions which are uniform in $i \in I$

More General, Non-Homogeneous, Group Constructions

direct product = free product / commutation relations between all G_i

Generalize this to : partial commutation relations between groups

Fix symmetric matrix $(\varepsilon_{ij})_{i,j\in I} = (\Lambda_{ij})_{i,j\in I}$ with $0/1$ entries, with the idea that for $i \neq j$

$$\varepsilon_{ij} = \varepsilon_{ji} = 1 \quad \hat{=} \quad [G_i, G_j] = 0$$

$$\varepsilon_{ij} = \varepsilon_{ji} = 0 \quad \hat{=} \quad \text{no relation}$$

[ε_{ii} is unspecified or sometimes it is convenient to put $\varepsilon_{ii} = 0$.]

European Research Council
Established by the European Commission

Definition (Green 1990)

Let G_i $(i \in I)$ be groups. The **graph product** $\star_\varepsilon G_i$ (corresponding to graph with adjacency matrix ε) is

$$\star_{i\in I} G_i \Big/ \; G_i,\, G_j \; (i \neq j) \text{ commute whenever } \varepsilon_{ij} = 1$$

Example

If $G_i \mathbb{Z}$, then $\star_\varepsilon G_i$ has various names

- right angled Artin group
- free partially commutative group
- trace group
- etc

also: Cartier-Foata monoid, trace monoid, ...

European Research Council
Established by the European Commission

Definition (Młotkowski 2004)

Subalgebras $(\mathcal{A}_i)_{i\in I}$ are **ε-independent** in $(\mathcal{A},\varphi)$ if

- $[\mathcal{A}_i,\mathcal{A}_j]=0$ for all $i\neq j$ with $\varepsilon_{ij}=1$
- Consider $a_j\in\mathcal{A}_{i(j)}$ $(i=1,\dots,n)$ with
  - $\varphi(a_j)=0 \quad \forall j$
  - $i(j)\neq i(j+1)$ modulo commutation relations

  Then $\varphi(a_1\cdots a_n)=0$

"modulo commutation relations" means: if there are $k<l$ with $i(k)=i(l)$,

$$\cdots\ a_k\ \cdots \qquad \cdots\ a_l\ \cdots$$

then there exists p with $k<p<l$ such that $i(k)\neq i(p)\neq i(l)$ and $\varepsilon_{i(k)i(p)}=0$

Definition (Młotkowski 2004)

Subalgebras $(\mathcal{A}_i)_{i\in I}$ are **ε-independent** in $(\mathcal{A},\varphi)$ if

- $[\mathcal{A}_i,\mathcal{A}_j]=0$ for all $i\neq j$ with $\varepsilon_{ij}=1$
- Consider $a_j\in\mathcal{A}_{i(j)}$ $(i=1,\dots,n)$ with
  - $\varphi(a_j)=0 \quad \forall j$
  - $i(j)\neq i(j+1)$ modulo commutation relations

  Then $\varphi(a_1\cdots a_n)=0$

Example of mixed moments (can be calculated via centering)

For $x\in\mathcal{A}_i$, $y\in\mathcal{A}_j$, $i\neq j$ we have

$$\varphi(xyx)=\varphi(x^2)\varphi(y) \qquad \text{always}$$

$$\varphi(xyxy)=\begin{cases}\varphi(x^2)\varphi(y^2), & \varepsilon_{ij}=1\\ \varphi(x^2)\varphi(y)^2+\varphi(x)^2\varphi(y^2)-\varphi(x)^2\varphi(y)^2, & \varepsilon_{ij}=0\end{cases}$$

Definition (Młotkowski 2004)

Subalgebras $(\mathcal{A}_i)_{i\in I}$ are **ε-independent** in $(\mathcal{A},\varphi)$ if

- $[\mathcal{A}_i,\mathcal{A}_j]=0$ for all $i\neq j$ with $\varepsilon_{ij}=1$
- Consider $a_j\in\mathcal{A}_{i(j)}$ $(i=1,\dots,n)$ with
  - $\varphi(a_j)=0 \quad \forall j$
  - $i(j)\neq i(j+1)$ modulo commutation relations

  Then $\varphi(a_1\cdots a_n)=0$

Theorem (Młotkowski)

There is a ε-product construction, which has all nice properties.

Proposition (Speicher+Wysoczanski)

The subalgebras $\mathbb{C}G_i$ are ε-independent in the graph product group algebra $\mathbb{C}(\star_\varepsilon G_i)$ with respect to the canonical trace τ.

European Research Council
Established by the European Commission

Theorem (Speicher+Wysoczanski)

Let $(\mathcal{A}_i)_{i\in I}$ be ε-independent. Let $a_j \in \mathcal{A}_{i(j)}$. Then

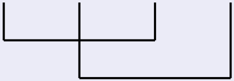
$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC^\varepsilon[i(1),\dots,i(n)]} k_\pi(a_1,\dots,a_n),$$

where

- k_π is the product of corresponding free cumulants, according to block structure of π
- with $i = (i(1),\dots,i(n))$ we have

$$NC^\varepsilon[i] := \{\pi \in \mathcal{P}(n) \mid \pi \leq \ker i, \text{and } \pi \text{ is } (\varepsilon, i)\text{-noncrossing}\}$$

$$\cdots p_1 \cdots q_1 \cdots p_2 \cdots q_2 \cdots$$



then $\varepsilon_{i(p_1)i(q_1)} = 1$

Canonical Question

Question

- What are quantum symmetries behind ε-independence?
- More generally: What are quantum symmetries going with partial commutation relations?

European Research Council
Established by the European Commission

Section 3

ε-Spheres and Its Symmetries

European Research Council
Established by the European Commission

Quantum Spheres (Real Versions)

Quantum spheres are given by deformations of the classical sphere

Definition (classical sphere)

$$C(S_{\mathbb{R}}^{n-1}) = C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_{i=1}^{n} x_i^2 = 1, \; x_i x_j = x_j x_i \; \forall i, j\}$$

- by deformations of the commutation relations as well as the sphere relation by some parameters (Podlés 1987, Connes+Dubois-Violette 2002, Connes+Landi 2001, etc)

Quantum Spheres (Real Versions)

Quantum spheres are given by deformations of the classical sphere

Definition (classical sphere)

$$C(S^{n-1}_{\mathbb{R}}) = C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_{i=1}^n x_i^2 = 1, \ x_i x_j = x_j x_i \ \forall i, j\}$$

- by dropping the commutation relation or replacing it by some other "rigid" relation, but keeping the sphere relation (Banica+Goswami 2010, Banica 2015)

Definition (free sphere, Banica+Goswami 2010)

$$C(S^{n-1}_{\mathbb{R},+}) := C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_{i=1}^n x_i^2 = 1\}$$

European Research Council
Established by the European Commission

The ε-Sphere (and Its Symmetries)

Definition (Speicher+Weber 2016)

For given $\varepsilon = (\varepsilon_{ij})_{i,j=1}^n$, the ε-sphere is defined by

$$C(S_{\mathbb{R},\varepsilon}^{n-1}) := C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_{i=1}^n x_i^2 = 1, \; x_i x_j = x_j x_i \text{ if } \varepsilon_{ij} = 1\}$$

What are the symmetries?

We have

$$C(S_{\mathbb{R}}^{n-1}) \leftarrow C(S_{\mathbb{R},\varepsilon}^{n-1}) \leftarrow C(S_{\mathbb{R},+}^{n-1})$$

The ε-Sphere (and Its Symmetries)

Definition (Speicher+Weber 2016)

For given $\varepsilon = (\varepsilon_{ij})_{i,j=1}^n$, the ε-sphere is defined by

$$C(S^{n-1}_{\mathbb{R},\varepsilon}) := C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_{i=1}^n x_i^2 = 1, \; x_i x_j = x_j x_i \text{ if } \varepsilon_{ij} = 1\}$$

What are the symmetries?

We have

$$S^{n-1}_{\mathbb{R}} \subset S^{n-1}_{\mathbb{R},\varepsilon} \subset S^{n-1}_{\mathbb{R},+}$$

and

- $S^{n-1}_{\mathbb{R}}$ is invariant under the orthogonal group O_n
- $S^{n-1}_{\mathbb{R},+}$ is invariant under the quantum orthogonal group O_n^+
- What is the invariance quantum group for the ε-sphere $S^{n-1}_{\mathbb{R},\varepsilon}$?

Symmetries of ε-Sphere: ε-Orthogonal Group

Theorem (Speicher+Weber)

The maximal quantum group acting on the ε-sphere $S^{n-1}_{\mathbb{R},\varepsilon}$ is given by the ε-orthogonal quantum group O_n^ε, which is defined by

$$C(O_n^\varepsilon) = C^*(u_{ij}(i,j=1,\dots,n) \mid u_{ij}^* = u_{ij}, u = (u_{ij}) \text{ is orthogonal}, R^\varepsilon\}$$

Definition (relations R^ε)

$$u_{ik}u_{jl} = \begin{cases} u_{jl}u_{ik}, & e_{ij}=1, e_{kl}=1 \\ u_{jk}u_{il}, & e_{ij}=1, e_{kl}=0 \\ u_{il}u_{jk}, & e_{ij}=0, e_{kl}=1 \end{cases}$$

European Research Council
Established by the European Commission

O_n^ε acts on $S^{n-1}_{\mathbb{R},\varepsilon}$ means that

- if $(x_1, \dots, x_n) \in C(S^{n-1}_{\mathbb{R},\varepsilon})$
- and $y = u \odot x$, i.e.,

$$y_i = \sum_k u_{ik} \otimes x_k$$

- then $(y_1, \dots, y_n) \in C(S^{n-1}_{\mathbb{R},\varepsilon})$

Note that in general

$$O_n \not\subset O_n^\varepsilon \subset O_n^+$$

European Research Council
Established by the European Commission

Section 4

ε-Independence and Its Symmetries

European Research Council
Established by the European Commission

The ε-Permutation Group

Definition (Speicher+Weber)

We set

$$S_n^\varepsilon := S_n^+ / R^\varepsilon, \qquad \text{i.e.}$$

$$C(S_n^\varepsilon) = C^*(u_{ij}(i,j=1,\dots,n) \mid u_{ij}^* = u_{ij} = u_{ij}^2, R^\varepsilon, \sum_k u_{ik} = 1 = \sum_k u_{kj} \ \forall i,j\}$$

European Research Council
Established by the European Commission

Definition (Speicher+Weber)

$$C(S_n^\varepsilon) = C^*(u_{ij}(i,j=1,\dots,n) \mid u_{ij}^* = u_{ij} = u_{ij}^2, R^\varepsilon, \sum_k u_{ik} = 1 = \sum_k u_{kj}\ \forall i,j\}$$

Theorem (Speicher+Weber)

1) This is the same as replacing R^ε by

$$u_{ik}u_{jl} = \begin{cases} u_{jl}u_{ik}, & e_{ij}=1, e_{kl}=1 \\ 0, & e_{ij}=1, e_{kl}=0 \\ 0, & e_{ij}=0, e_{kl}=1 \end{cases}$$

2) Hence, $C(S_n^\varepsilon)$ is the quantum automorphism group of Bichon (2003) of the graph with adjacency matrix ε.

Invariance of ε-Independence Under ε-Permutations

Theorem (Speicher+Weber)

Let $x_1, \dots, x_n$ be ε-independent and identically distributed. Then their distribution is invariant under S_n^ε, i.e., with $(u_{ij})_{i,j=1}^n \in S_n^\varepsilon$:

$$\varphi(x_{j(1)} \cdots x_{j(k)}) = \sum_{i(1),\dots,i(k)=1}^{n} \varphi(x_{i(1)} \cdots x_{i(k)}) u_{i(1)j(1)} \cdots u_{i(k)j(k)}$$

European Research Council
Established by the European Commission

Proof.

Recall

$$\varphi(x_{i(1)} \cdots x_{i(k)}) = \sum_{\pi \in NC^\varepsilon[i]} k_\pi(x_{i(1)}, \ldots, x_{i(k)}) = \sum_{\pi \in \mathcal{P}(k)} \delta(\pi \in NC^\varepsilon[i]) \cdot k_\pi$$

Hence

$$\sum_{i(1),\ldots,i(k)=1}^{n} \varphi(x_{i(1)} \cdots x_{i(k)}) u_{i(1)j(1)} \cdots u_{i(k)j(k)}$$

$$= \sum_{\pi \in \mathcal{P}(k)} k_\pi \underbrace{\left(\sum_{i(1),\ldots,i(k)} \delta(\pi \in NC^\varepsilon[i]) \cdot u_{i(1)j(1)} \cdots u_{i(k)j(k)} \right)}_{=\delta(\pi \in NC^\varepsilon[j])}$$

$$= \varphi(x_{j(1)} \cdots x_{j(k)})$$

□

Summary

- O_n^ε = quantum symmetry of the ε-sphere $S^{n-1}_{\mathbb{R},\varepsilon}$
- S_n^ε = quantum symmetry of ε-independence

- $C(S^{n-1}_{\mathbb{R},\varepsilon}) := C^*(x_1,\dots,x_n \mid x_i = x_i^*, \sum_i x_i^2 = 1, x_i x_j = x_j x_i, \text{ if } \varepsilon_{ij} = 1)$
- $C(O_n^\varepsilon) := C^*(u_{ij}, i,j = 1,\dots,n \mid u_{ij} = u_{ij}^*, u \text{ is orthogonal}, R^\varepsilon)$
- $C(S_n^\varepsilon) := C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1\ \forall i,j \text{ and } \mathring{R}^\varepsilon)$

$$u_{ik}u_{jl} = \begin{cases} R^\varepsilon & \mathring{R}^\varepsilon & \\ u_{jl}u_{ik} & u_{jl}u_{ik} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ u_{jk}u_{il} & 0 & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\ u_{il}u_{jk} & 0 & \text{if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \end{cases}$$

Speicher,Weber: Quantum groups with partial commutation relations, 1603.09192

Speicher,Wysoczanski: Mixtures of classical and free independence, 1603.08758

European Research Council
Established by the European Commission