Free Probability Theory

and

Random Matrices

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What is Operator-Valued Free Probability and Why Should Engineers Care About it?

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Many approximations for calculating eigenvalue distribution of matrices consist in replacing

independent Gaussian random variables	$\longrightarrow$	free (semi)circular variables

Reasons for doing so:

- in limit $N \to \infty$, we have this transition asymptotically
- but even for finite N, this approximation is usually quite close to original problem
- this is an approximation which is usually exactly calculable for each N

Example: selfadjoint Gaussian $N \times N$ random matrix

$$X_N = (x_{ij})_{i,j=1}^N$$

with

- x_{ij} $(1 \leq i \leq j \leq)$ are independent complex $(i \neq j)$ or real $(i = j)$ Gaussian random variables

- $x_{ij} = \overline{x_{ji}}$

- $\varphi(x_{ij}) = 0, \qquad \varphi(x_{ij}\overline{x_{ij}}) = 1/N$

replacing

independent Gaussian variables

by

free (semi)circular variables

gives

selfadjoint noncommutative $N \times N$ random matrix

$$S_N = (c_{ij})_{i,j=1}^N$$

with

- c_{ij} $(1 \leq i \leq j \leq)$ are free circular $(i \neq j)$ or semicircular $(i = j)$ random variables

- $c_{ij} = c_{ji}^*$

- $\varphi(c_{ij}) = 0, \qquad \varphi(c_{ij}c_{ij}^*) = 1/N$

$$X_N = (x_{ij})_{i,j=1}^N \qquad \longrightarrow \qquad S_N = (c_{ij})_{i,j=1}^N$$

- X_N has a complicated averaged eigenvalue distribution (i.e., with respect to $\text{tr} \otimes \varphi$)

- S_N has a very simple distribution with respect to $\text{tr} \otimes \varphi$: for each N, S_N is a semicircular variable

$$X_N = (x_{ij})_{i,j=1}^N \qquad \longrightarrow \qquad S_N = (c_{ij})_{i,j=1}^N$$

Morale: S_N is an approximation for X_N with

- the approximation gets better for large N
- distr(S_N) can be calculated exactly

Why is S_N better treatable than X_N?

- taking matrices does not fit well with independence and Gaussianity

- freeness and (semi)circular variables, on the other hand, go very nicely with matrices freeness and (semi)circular variables go very nicely with matrices

caveat: this is true for free (semi)circulars which are centered and all have same variance freeness and (semi)circular variables go very nicely with matrices

caveat: this is true for free (semi)circulars which are centered and all have same variance

however: we might be interested in more general situations:

- x_{ij} might have different variances for different ij
- x_{ij} might not be centered ($\rightarrow$ Ricean model)
- there might even be correlations between different x_{ij} and x_{kl}

Example: non-zero mean, independent Gaussian variables with constant variance

$$Y_N = A_N + X_N$$

$\uparrow$	$\uparrow$
deterministic matrix of means	independent centered Gaussians constant variance

We replace this by . . .

$$A_N \quad + \quad S_N$$

A_N		S_N
$\uparrow$		$\uparrow$
same deterministic matrix as before		free centered (semi)circulars of constant variance

We have:

- A_N is free from S_N , thus
- $\text{distr}(A_N + S_N) = \text{distr}(A_N) \boxplus \text{distr}(S_N)$

Example: non-zero mean, independent Gaussian variables with varying variance

$$Y_N \quad = \quad A_N \quad + \quad X_N$$

$\uparrow$	$\uparrow$
deterministic matrix of means	independent centered Gaussians $\varphi(x_{ij}\overline{x_{ij}}) = \sigma_{ij}/N$

We replace this by . . .

$$A_N \qquad + \qquad S_N$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

same deterministic matrix as before	free centered (semi)circulars with $\varphi(c_{ij}c_{ij}^*) = \sigma_{ij}/N$

We replace this by . . .

$$A_N \qquad + \qquad S_N$$

$$\uparrow \qquad\qquad \uparrow$$

same deterministic matrix as before	free centered (semi)circulars with $\varphi(c_{ij}c_{ij}^*) = \sigma_{ij}/N$

Now we have a **problem**

- S_N is **not semicircular** in general
- A_N and S_N are **not free** in general

So what do we gain by replacing

independent Gaussians

by

free (semi)circulars

in such a case?

$$X = (x_{ij})_{i,j=1}^{N} \qquad Y = (y_{kl})_{k,l=1}^{N}$$

with

$\{x_{ij}\}$ and $\{y_{kl}\}$ independent

$$\Downarrow$$

X and Y are not independent

$$X = (x_{ij})_{i,j=1}^{N} \qquad Y = (y_{kl})_{k,l=1}^{N}$$

with

$\{x_{ij}\}$ and $\{y_{kl}\}$ independent

$$\Downarrow$$

X and Y are not independent

actually: relation between X and Y is untreatable

$$X = (x_{ij})_{i,j=1}^{N} \qquad Y = (y_{kl})_{k,l=1}^{N}$$

with

$\{x_{ij}\}$ and $\{y_{kl}\}$ free

$$\Downarrow$$

X and Y are not free

$$X = (x_{ij})_{i,j=1}^{N} \qquad Y = (y_{kl})_{k,l=1}^{N}$$

with

$\{x_{ij}\}$ and $\{y_{kl}\}$ free

$$\Downarrow$$

X and Y are not free

however: relation between X and Y is more complicated, but still treatable in terms of

operator-valued freeness

Let $(\mathcal{C}, \varphi)$ be non-commutative probability space.

Consider $N \times N$ matrices over $\mathcal{C}$:

$$M_N(\mathcal{C}) := \{(a_{ij})_{i,j=1}^N \mid a_{ij} \in \mathcal{C}\}$$

$$M_N(\mathcal{C}) = M_N(\mathbb{C}) \otimes \mathcal{C}$$

is a non-commutative probability space with respect to

$$\mathrm{tr}_N \otimes \varphi,$$

but there is also an intermediate level

Instead of

$$
\begin{array}{c}
M_N(\mathcal{C}) \\
\downarrow \mathsf{tr} \otimes \varphi \\
\mathbb{C}
\end{array}
$$

consider ...

$$
\begin{array}{ccc}
M_N(\mathcal{C}) = M_N(\mathbb{C}) \otimes \mathcal{C} & & \\
\downarrow \mathsf{id} \otimes \varphi & & \Big| \\
M_N(\mathbb{C}) & & \Big| \ \mathsf{tr} \otimes \varphi \\
\downarrow \mathsf{tr} & & \downarrow \\
\mathbb{C} & &
\end{array}
$$

$$
\begin{array}{ccl}
M_N(\mathcal{C}) = M_N(\mathbb{C}) \otimes \mathcal{C} =: \mathcal{A} & & \\
\downarrow \mathrm{id} \otimes \varphi =: E & & \Big| \\
M_N(\mathbb{C}) =: \mathcal{B} & & \Big| \ \mathrm{tr} \otimes \varphi \\
\downarrow \mathrm{tr} & & \downarrow \\
\mathbb{C} & &
\end{array}
$$

Let $\mathcal{B} \subset \mathcal{A}$. A linear map

$$E : \mathcal{A} \to \mathcal{B}$$

is a **conditional expectation** if

$$E[b] = b \qquad \forall b \in \mathcal{B}$$

and

$$E[b_1 a b_2] = b_1 E[a] b_2 \qquad \forall a \in \mathcal{A}, \quad \forall b_1, b_2 \in \mathcal{B}$$

An **operator-valued probability space** consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$

Consider an operator-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$. The **operator-valued distribution** of $a \in \mathcal{A}$ is given by all operator-valued moments

$$E[ab_1ab_2 \cdots b_{n-1}a] \in \mathcal{B} \qquad (n \in \mathbb{N}, b_1, \dots, b_{n-1} \in \mathcal{B})$$

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Random variables $x, y \in \mathcal{A}$ are **free with respect to E** (or **free with amalgamation over $\mathcal{B}$**) if

$$E[p_1(x)q_1(y)p_2(x)q_2(y) \cdots] = 0$$

whenever p_i, q_j are polynomials with coefficients from $\mathcal{B}$ and

$$E[p_i(x)] = 0 \quad \forall i \qquad \text{and} \qquad E[q_j(y)] = 0 \quad \forall j.$$

Note: polynomials in x with coefficients from $\mathcal{B}$ are of the form

- x^2
- $b_0 x^2$

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- x^2

- $b_0 x^2$

- $b_1 x b_2 x b_3$

b's and x do not commute in general!

Note: polynomials in x with coefficients from $\mathcal{B}$ are of the form

- x^2
- b_0x^2
- $b_1xb_2xb_3$
- $b_1xb_2xb_3 + b_4xb_5xb_6 + \cdots$
- etc.

b's and x do not commute in general!

Operator-valued freeness works mostly like ordinary freeness, one only has to take care of the order of the variables; in all expressions they have to appear in their original order!

Example: If x and $\{y_1, y_2\}$ are free, then one has

$$E[y_1 x y_2] = E\Big[y_1 E[x] y_2\Big];$$

and more general

$$E[y_1 b_1 x b_2 y_2] = E\Big[y_1 b_1 E[x] b_2 y_2\Big].$$

Consider $E : \mathcal{A} \to \mathcal{B}$.

Define **free cumulants**

$$\kappa_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}$$

by

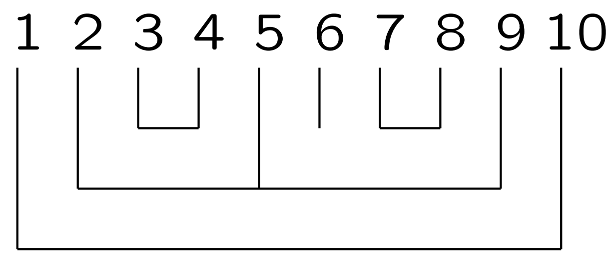
$$E[a_1 \cdots a_n] = \sum_{\pi \in NC(n)} \kappa_\pi^{\mathcal{B}}[a_1, \dots, a_n]$$

- arguments of $\kappa_\pi^{\mathcal{B}}$ are distributed according to blocks of π

- but now: cumulants are nested inside each other according to nesting of blocks of π

Example:

$$\pi = \Big\{\{1,10\},\{2,5,9\},\{3,4\},\{6\},\{7,8\}\Big\} \in NC(10),$$



$$\kappa_\pi^{\mathcal{B}}[a_1,\dots,a_{10}]$$

$$= \kappa_2^{\mathcal{B}}\Big(a_1 \cdot \kappa_3^{\mathcal{B}}\big(a_2 \cdot \kappa_2^{\mathcal{B}}(a_3,a_4), a_5 \cdot \kappa_1^{\mathcal{B}}(a_6) \cdot \kappa_2^{\mathcal{B}}(a_7,a_8), a_9\big), a_{10}\Big)$$

For $a \in \mathcal{A}$ define its **operator-valued Cauchy transform**

$$G_a(b) := E[\frac{1}{b-a}] = \sum_{n \geq 0} E[b^{-1}(ab^{-1})^n]$$

and **operator-valued R-transform**

$$\begin{aligned} R_a(b) &:= \sum_{n \geq 0} \kappa_{n+1}^{\mathcal{B}}(ab, ab, \ldots, ab, a) \\ &= \kappa_1^{\mathcal{B}}(a) + \kappa_2^{\mathcal{B}}(ab, a) + \kappa_3^{\mathcal{B}}(ab, ab, a) + \cdots \end{aligned}$$

Then

$$bG(b) = 1 + R(G(b)) \cdot G(b) \qquad \text{or} \qquad G(b) = \frac{1}{b - R(G(b))}$$

If x and y are free over $\mathcal{B}$, then

- mixed $\mathcal{B}$-valued cumulants in x and y vanish

- $R_{x+y}(b) = R_x(b) + R_y(b)$

- $G_{x+y}(b) = G_x\Big[b - R_y\Big(G_{x+y}(b)\Big)\Big]$ subordination

If s is a semicircle over $\mathcal{B}$ then

$$R_s(b) = \eta(b)$$

where $\eta : \mathcal{B} \to \mathcal{B}$ is a linear map given by

$$\eta(b) = E[sbs].$$

Back to random matrices

What can we say about

$$A_N \quad + \quad S_N$$

$$\uparrow \qquad\qquad \uparrow$$

deterministic matrix of means	free centered (semi)circulars with $\varphi(c_{ij}c_{ij}^*) = \sigma_{ij}/N$

Consider

$$T_N := A_N + S_N$$

We want Cauchy transform

$$g_T(z) = \mathrm{tr} \otimes \varphi[\frac{1}{z - T}]$$

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$$\begin{array}{ccc} M_N(\mathcal{C}) = M_N(\mathbb{C}) \otimes \mathcal{C} & & \\ \downarrow \mathrm{id} \otimes \varphi & & | \\ & & | \\ M_N(\mathbb{C}) & & | \ \mathrm{tr} \otimes \varphi \\ & & | \\ \downarrow \mathrm{tr} & & \downarrow \\ \mathbb{C} & & \end{array}$$

Consider

$$T := A + S$$

We want Cauchy transform

$$g_T(z) = \mathrm{tr} \otimes \varphi[\frac{1}{z-T}]$$

$$\begin{aligned} g_T(z) &= \mathrm{tr} \otimes \varphi[\frac{1}{z-T}] \\ &= \mathrm{tr}\Big[\underbrace{\mathrm{id} \otimes \varphi(\frac{1}{z-T})}_{G_T(z)}\Big] = \mathrm{tr}[G_T(z)] \end{aligned}$$

We have nice behavior as $M_N(\mathbb{C})$-operator-valued objects

- A_N, S_N are free over $M_N(\mathbb{C})$, i.e.,

$$G_T(z) = G_A\Big[z - R_S\Big(G_T(z)\Big)\Big]$$

- S_N is semicircular over $M_N(\mathbb{C})$, i.e.,

$$R_S(B) = \eta(B)$$

  with $\eta : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ linear, given by $\eta(B) = E[SBS]$.

Thus the distribution of $T_N = A_N + S_N$ is determined via its Cauchy-transform g_T according to

$$g_T(z) = \mathrm{tr}\Big[G_T(z)\Big]$$

and

$$G_T(z) = G_A\left[z - \eta\Big(G_T(z)\Big)\right] = E\left[\frac{1}{z - \eta\Big(G_T(z)\Big) - A}\right]$$

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Note: by [Helton, Far, Speicher IMRN 2007], there exists exactly one solution of above fixed point equation with the right positivity property!

Note: the more "symmetries" we have in entries of S_N, the "better" is the freeness between A_N and S_N!

For considered situation where different c_{ij} are free, we have for

$$\eta : M_N(\mathbb{C}) \to M_N(\mathbb{C})$$

that actually

$$[\eta(B)]_{ij} = E[SBS]_{ij} = \sum_{k,l} \varphi(c_{ik} b_{kl} c_{lj}) = \sum_{k,l} \underbrace{\varphi(c_{ik} c_{jl}^*)}_{\delta_{kl}\delta_{ij}\sigma_{ik}} b_{kl} = \delta_{ij} \sum_k \sigma_{ik} b_{kk}$$

thus η is effectively a mapping on diagonal matrices

Consider in addition to

$$E_M : M_N(\mathcal{C}) \to M_N(\mathbb{C}); \begin{pmatrix} a_{11} & \dots & a_{1N} \\ \vdots & \ddots & \dots \\ a_{N1} & \dots & a_{NN} \end{pmatrix} \mapsto \begin{pmatrix} \varphi(a_{11}) & \dots & \varphi(a_{1N}) \\ \vdots & \ddots & \vdots \\ \varphi(a_{N1}) & \dots & \varphi(a_{NN}) \end{pmatrix}$$

also

$$D_N(\mathbb{C}) := \{\text{diagonal matrices}\} \subset M_N(\mathbb{C})$$

and corresponding conditional expectation

$$E_D : M_N(\mathcal{C}) \to D_N(\mathbb{C}); \begin{pmatrix} a_{11} & \dots & a_{1N} \\ \vdots & \ddots & \dots \\ a_{N1} & \dots & a_{NN} \end{pmatrix} \mapsto \begin{pmatrix} \varphi(a_{11}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \varphi(a_{NN}) \end{pmatrix}$$

Then we have in our situation

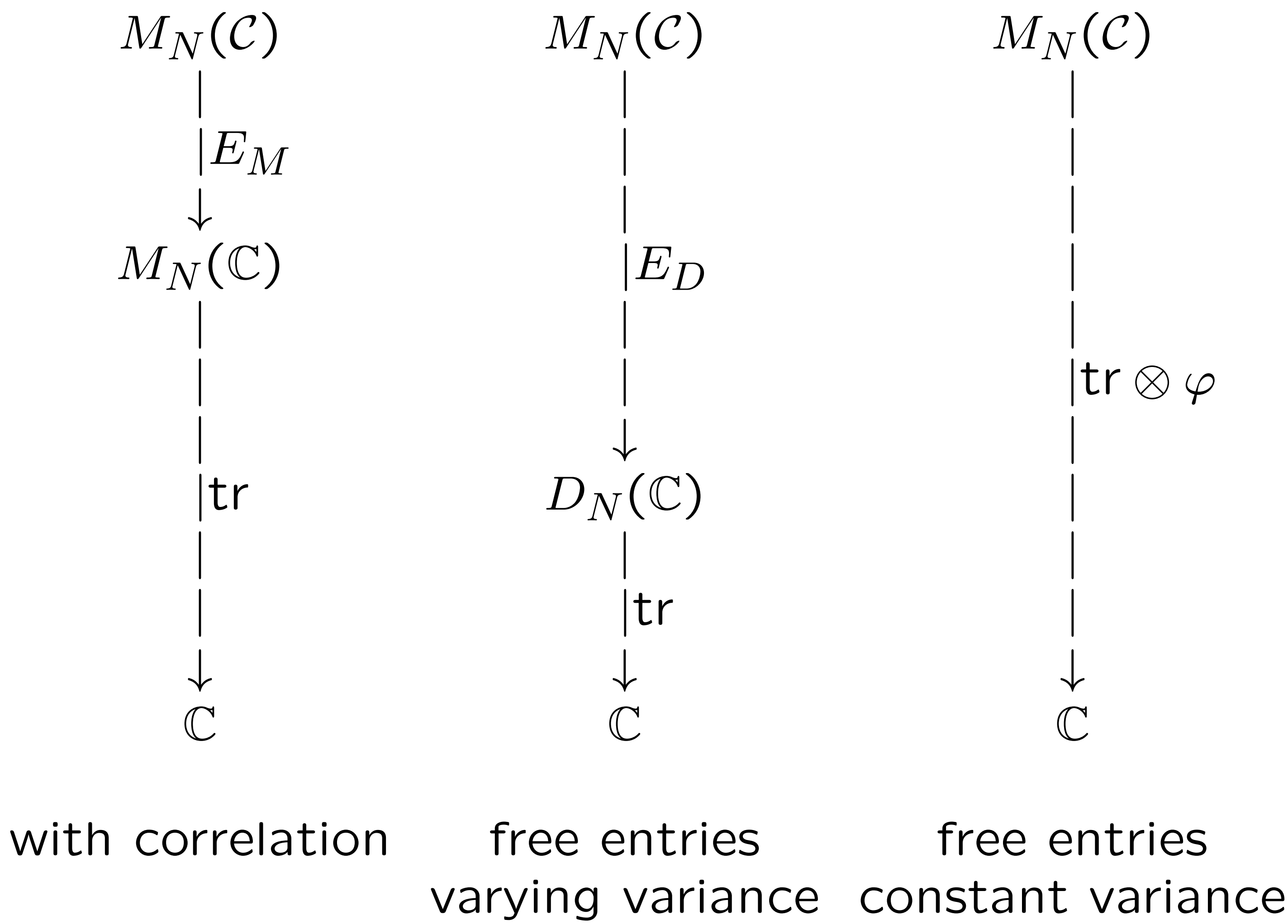
$$A_N \qquad + \qquad S_N$$

$$\uparrow \qquad\qquad \uparrow$$

deterministic matrix of means	free centered (semi)circulars with $\varphi(c_{ij}c_{ij}^*) = \sigma_{ij}/N$

that actually, by [Nica, Shlyakhtenko, Speicher, JFA 2002],

- A_N and S_N are free over $D_N(\mathbb{C})$
- S_N is semicircular over $D_N(\mathbb{C})$


$M_N(\mathcal{C})$
E_M
$M_N(\mathbb{C})$
tr
$\mathbb{C}$
with correlation
$M_N(\mathcal{C})$
E_D
$D_N(\mathbb{C})$
tr
$\mathbb{C}$
free entries
varying variance
$M_N(\mathcal{C})$
$\mathrm{tr} \otimes \varphi$
$\mathbb{C}$
free entries
constant variance

Let us now treat more relevant non-selfadjoint situation

$$H_N = A_N + C_N$$

A_N ↑ deterministic matrix of means

C_N ↑ free circulars, no symmetry condition, $\varphi(c_{ij}c_{ij}^*) = \sigma_{ij}/N$

Calculate the distribution of HH^*!

HH^* has the same distribution as square of

$$T := \begin{pmatrix} 0 & H \\ H^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}$$

These are $2N \times 2N$ selfadjoint matrices of the type considered before.

We have

- $\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}$ are free over $D_{2N}(\mathbb{C})$

- $\begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}$ is semicircular over $D_{2N}(\mathbb{C})$ with

$$\eta \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} \eta_1(B_2) & 0 \\ 0 & \eta_2(B_1) \end{pmatrix},$$

  where

$$\eta_1(B_2) = E_{D_N}[CD_2C^*]$$

$$\eta_2(B_1) = E_{D_N}[C^*D_1C]$$

We have

$$G_T(z) = zG_{T^2}(z^2)$$

and

$$G_{T^2}(z) = \begin{pmatrix} G_1(z) & 0 \\ 0 & G_2(z) \end{pmatrix}$$

Thus

$$\begin{aligned} zG_{T^2}(z^2) = G_T(z) &= G_{\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}} \left[z - R_{\begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}}(G_T(z)) \right] \\ &= E_{D_{2N}} \left[\left(z - z\eta \begin{pmatrix} G_1(z^2) & 0 \\ 0 & G_2(z^2) \end{pmatrix} - \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \right)^{-1} \right] \\ &= E_{D_{2N}} \left[\begin{pmatrix} z - z\eta_1(G_2(z^2)) & -A \\ -A^* & z - z\eta_2(G_1(z^2)) \end{pmatrix}^{-1} \right] \end{aligned}$$

This yields

$$zG_1(z) = E_{D_N}\left[\left(1 - \eta_1(G_2(z)) + A_N \frac{1}{z - z\eta_2(G_1(z))} A_N^*\right)^{-1}\right]$$

and

$$zG_2(z) = E_{D_N}\left[\left(1 - \eta_2(G_1(z)) + A_N^* \frac{1}{z - z\eta_1(G_2(z))} A_N\right)^{-1}\right]$$

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and

$$zG_2(z) = E_{D_N}\left[\left(1 - \eta_2(G_1(z)) + A_N^* \frac{1}{z - z\eta_1(G_2(z))} A_N\right)^{-1}\right]$$

These are actually the fixed point equations of

[Hachem, Loubaton, Najim, Ann. Appl. Prob. 2007]

for a **deterministic equivalent** (a la Girko) of square of random matrix with non-centered, independent Gaussians with non-constant variance as entries.

Conclusion

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Conclusion

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- operator-valued free probability allows conceptual and streamlined treatment of those

- also convergence questions might be treated more uniformly by relying on asymptotic freeness results

- this approach also allows to treat classes of random matrices with correlation between entries