Quantum groups and liberation of orthogonal matrix groups

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joint work with Teodor Banica

Liberation

group $\longrightarrow$ quantum group

Liberation

group	$\longrightarrow$	quantum group
orthogonal group	$\longrightarrow$	quantum orthogonal group
permutation group	$\longrightarrow$	quantum permutation group

Liberation

group	$\longrightarrow$	quantum group
orthogonal group	$\longrightarrow$	quantum orthogonal group
???	$\longrightarrow$	quantum ???
permutation group	$\longrightarrow$	quantum permutation group

Orthogonal Hopf Algebra

is a C^*-algebra A, given with a system of n^2 self-adjoint generators $u_{ij} \in A$ $(i, j = 1, \ldots, n)$, subject to the following conditions:

- The inverse of $u = (u_{ij})$ is the transpose matrix $u^t = (u_{ji})$.
- $\Delta(u_{ij}) = \Sigma_k \, u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.
- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.
- $S(u_{ij}) = u_{ji}$ defines a morphism $S : A \to A^{op}$.

These are compact quantum groups in the sense of Woronowicz.

Quantum Orthogonal Group (Wang 1995)

The quantum orthogonal group $A_o(n)$ is the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \ldots, n)$ subject to the relation

- $u = (u_{ij})_{i,j=1}^n$ is an orthogonal matrix

This means: for all i, j we have

$$\sum_{k=1}^{n} u_{ik} u_{jk} = \delta_{ij} \qquad \text{and} \qquad \sum_{k=1}^{n} u_{ki} u_{kj} = \delta_{ij}$$

Quantum Permutation Group (Wang 1998)

The quantum permutation group $A_s(n)$ is the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \dots, n)$ subject to the relations

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i, j = 1, \dots, n$
- each row and column of $u = (u_{ij})_{i,j=1}^n$ is a partition of unity:

$$\sum_{j=1}^{n} u_{ij} = 1 \qquad \sum_{i=1}^{n} u_{ij} = 1$$

(this will feature prominently in the talk of Claus Koestler!)

How can we describe and understand intermediate quantum groups, sitting between these two cases:

$$A_o(n) \to \mathbf{A} \to A_s(n)$$

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Deal with quantum groups by looking on their representations!!!

Spaces of Intertwiners

Associated to an orthogonal Hopf algebra $(A,(u_{ij})_{i,j=1}^n)$ are the spaces of intertwiners:

$$C_a(k,l)=\{T:(\mathbb{C}^n)^{\otimes k}\to(\mathbb{C}^n)^{\otimes l}\mid Tu^{\otimes k}=u^{\otimes l}T\}$$

where $u^{\otimes k}$ is the $n^k\times n^k$ matrix $(u_{i_1j_1}\dots u_{i_kj_k})_{i_1\dots i_k,j_1\dots j_k}$.

$$u\in M_n(A)\qquad u:\mathbb{C}^n\to\mathbb{C}^n\otimes A$$

$$u^{\otimes k}:(\mathbb{C}^n)^{\otimes k}\to(\mathbb{C}^n)^{\otimes k}\otimes A$$

Tensor Category with Duals

The collection of vector spaces $C_a(k,l)$ has the following properties:

- $T, T' \in C_a$ implies $T \otimes T' \in C_a$.
- If $T, T' \in C_a$ are composable, then $TT' \in C_a$.
- $T \in C_a$ implies $T^* \in C_a$.
- $id(x) = x$ is in $C_a(1,1)$.
- $\xi = \sum e_i \otimes e_i$ is in $C_a(0,2)$.

Quantum Groups $\leftrightarrow$ Intertwiners

The compact quantum group A can actually be rediscovered from its space of interwiners:

There is a one-to-one correspondence between:

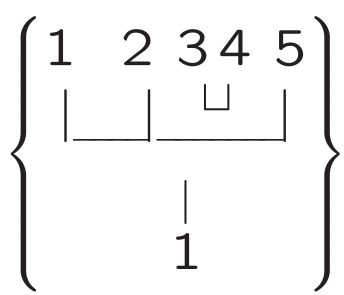
- orthogonal Hopf algebras $A_o(n) \to \mathbf{A} \to A_s(n)$
- tensor categories with duals $C_{ao} \subset \mathbf{C_a} \subset C_{as}$.

We denote by $P(k,l)$ the set of partitions of the set with repetitions $\{1,\ldots,k,1,\ldots,l\}$. Such a partition will be pictured as

$$p = \begin{Bmatrix} 1\ldots k \\ \mathcal{P} \\ 1\ldots l \end{Bmatrix}$$

where $\mathcal{P}$ is a diagram joining the elements in the same block of the partition.

Example in $P(5,1)$:



Associated to any partition $p \in P(k,l)$ is the linear map

$$T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$$

given by

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \cdots j_l} \delta_p(i,j)\, e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$.

$$T_{-}\left\{\,\Big|\;\;\Big|\,\right\}(e_a \otimes e_b) = e_a \otimes e_b$$

$$T_{-}\left\{\Big|\!\!-\!\!\Big|\right\}(e_a \otimes e_b) = \delta_{ab}\, e_a \otimes e_a$$

$$T_{-}\left\{\begin{matrix}\sqcup \\ |\;\;|\end{matrix}\right\}(e_a \otimes e_b) = \delta_{ab} \sum_{cd} e_c \otimes e_d$$

Intertwiners of Quantum Permutation and of Quantum Orthogonal Group

Let $NC(k,l) \subset P(k,l)$ be the subset of noncrossing partitions.

The tensor category of $A_s(n)$ is given by:

$$C_{as}(k,l) = \text{span}(T_p | p \in NC(k,l))$$

The tensor category of $A_o(n)$ is given by:

$$C_{ao}(k,l) = \text{span}(T_p | p \in NC_2(k,l))$$

Free Quantum Groups

A quantum group $A_o(n) \to \mathbf{A} \to A_s(n)$ is called **free** when its associated tensor category is of the form

$$\begin{array}{c} C_{as} = \mathrm{span}(T_p \mid p \in NC) \\ \cup \\ \mathbf{C_a} \\ \cup \\ C_{ao} = \mathrm{span}(T_p \mid p \in NC_2) \end{array},$$

Free Quantum Groups

A quantum group $A_o(n) \to \mathbf{A} \to A_s(n)$ is called **free** when its associated tensor category is of the form

$$C_{as} = \mathrm{span}(T_p \mid p \in NC)$$
$$\cup$$
$$\mathbf{C_a = span(T_p \mid p \in NC_a)},$$
$$\cup$$
$$C_{ao} = \mathrm{span}(T_p \mid p \in NC_2)$$

for a certain collection of subsets $NC_a \subset NC$.

Category of Noncrossing Partitions

A category of noncrossing partitions is a collection of subsets $NC_x(k,l) \subset NC(k,l)$, subject to the following conditions:

- NC_x is stable by tensor product.
- NC_x is stable by composition.
- NC_x is stable by involution.
- NC_x contains the "unit" partition $|$.
- NC_x contains the "duality" partition $\sqcap$.

Category of Noncrossing Partitions $\leftrightarrow$ Free Quantum Groups

Let NC_x be a category of noncrossing partitions, and $n \in \mathbb{N}$.

- $C_x = \text{span}(T_p | p \in NC_x)$ is a tensor category with duals.
- The associated quantum group $A_o(n) \to A \to A_s(n)$ is free.
- Any free quantum group appears in this way.

There are 6 Categories of Noncrossing Partitions:

$$
\begin{array}{ccccc}
\left\{\begin{matrix}\text{singletons and}\\ \text{pairings}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons and}\\ \text{pairings (even part)}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all}\\ \text{pairings}\end{matrix}\right\} \\
\cap & & \cap & & \cap \\
\left\{\begin{matrix}\text{all}\\ \text{partitions}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all partitions}\\ \text{(even part)}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{with blocks of}\\ \text{even size}\end{matrix}\right\}
\end{array}
$$

... and thus 6 free Quantum Groups:

$$\begin{array}{ccccc} A_b(n) & \leftarrow & A_{b'}(n) & \leftarrow & A_o(n) \\ \downarrow & & \downarrow & & \downarrow \\ A_s(n) & \leftarrow & A_{s'}(n) & \leftarrow & A_h(n) \end{array}$$

- *Orthogonal*, if its entries are self-adjoint, and $uu^t = u^t u = 1$.

- *Magic*, if it is orthogonal, and its entries are projections.

- *Cubic*, if it is orthogonal, and $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$, for $j \neq k$.

- *Bistochastic*, if it is orthogonal, and $\Sigma_j u_{ij} = \Sigma_j u_{ji} = 1$.

- *Magic'*, if it is cubic, with the same sum on rows and columns.

- *Bistochastic'*, if it is orthogonal, with the same sum on rows and columns.

More General Classification

$$\begin{array}{ccc} A_o(n) & \longrightarrow & A_s(n) \\ \downarrow & & \downarrow \\ C(O_n) & \longrightarrow & C(S_n) \end{array}$$

More General Classification

$$\begin{array}{ccccc} A_o(n) & \rightarrow & A_{free} & \rightarrow & A_s(n) \\ \downarrow & & & & \downarrow \\ C(O_n) & & \rightarrow & & C(S_n) \end{array}$$

- There are exactly six free quantum groups A_{free}!

More General Classification

$$\begin{array}{ccccc} A_o(n) & \rightarrow & A_{free} & \rightarrow & A_s(n) \\ \downarrow & & & & \downarrow \\ C(O_n) & \rightarrow & C(G_{easy}) & \rightarrow & C(S_n) \end{array}$$

- There are exactly six free quantum groups A_{free}!
- There are exactly six classical easy groups G_{easy}!

More General Classification

$$
\begin{array}{ccccc}
A_o(n) & \rightarrow & A_{free} & \rightarrow & A_s(n) \\
 & \searrow & & & \\
\downarrow & & A_{easy} & & \downarrow \\
 & & & \searrow & \\
C(O_n) & \rightarrow & C(G_{easy}) & \rightarrow & C(S_n)
\end{array}
$$

- There are exactly six free quantum groups A_{free}!
- There are exactly six classical easy groups G_{easy}!
- Can we have more easy quantum groups A_{easy}???

A quantum group satisfying $A_o(n) \to A \to C(S_n)$ is called **easy** when its associated tensor category of intertwiners is spanned by partitions. The corresponding **full category of partitions** $P_x \subset P$ must satisfy:

- P_x is stable by tensor product.
- P_x is stable by composition.
- P_x is stable by involution.
- P_x contains the "unit" partition $|$.
- P_x contains the "duality" partition $\sqcap$.

There are at least 3 more easy quantum groups

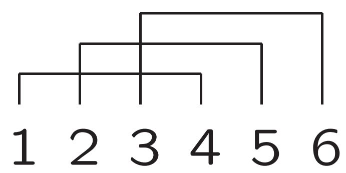
The following are full categories of partitions:

- P_o^*: the set of pairings having the property that each string has an even number of crossings.

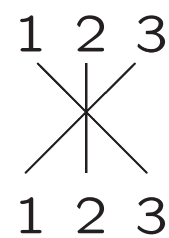
- P_b^*: the set of singletons and pairings having the property that when removing the singletons, each string has an even number of crossings.

- $P_{b'}^*$: the even part of P_b^*, consisting of pairings having an even number of crossings, completed with an even number of singletons.

P_o^* is generated by



or



The algebras $A_o^*(n), A_b^*(n), A_{b'}^*(n)$ are respectively the quotients of the algebras $A_o(n), A_b(n), A_{b'}(n)$ by the collection of relations

$$abc = cba$$

one for each choice of a, b, c in the set $\{u_{ij} | i, j = 1, \ldots, n\}$.

Literature

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