Free Probability Theory

and

Random Matrices

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$N \times N$ random matrices for $N \to \infty$.

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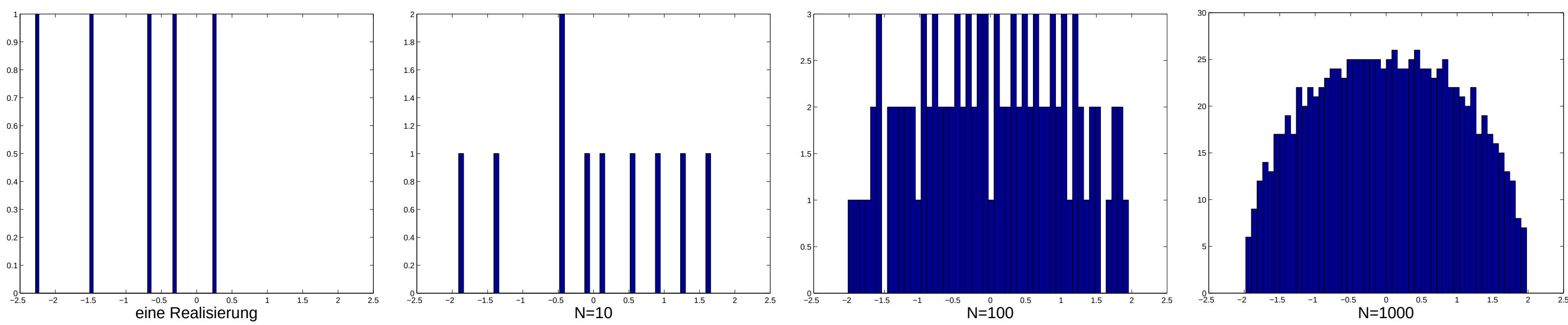
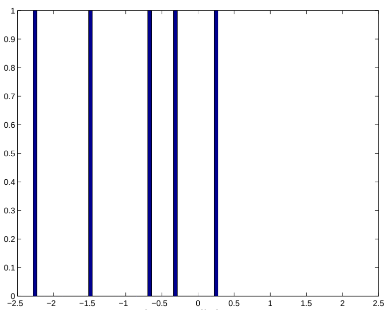
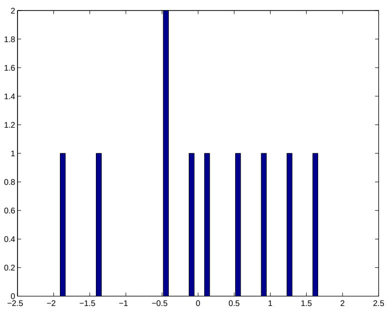
Typical phenomena for basic random matrix ensembles:

- almost sure convergence to a deterministic limit eigenvalue distribution
- this limit distribution can be effectively calculated


eine Realisierung
N=10
N=100
N=1000



eine Realisierung



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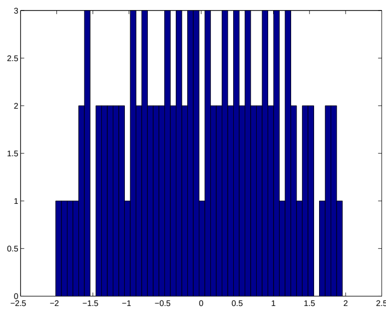

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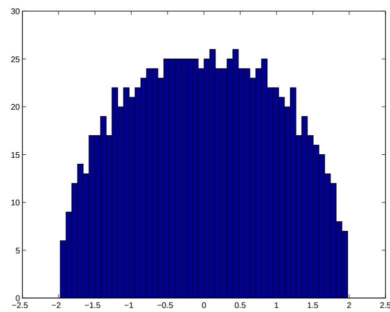
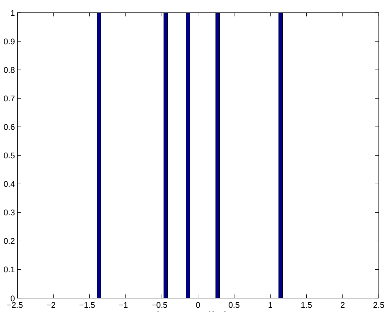
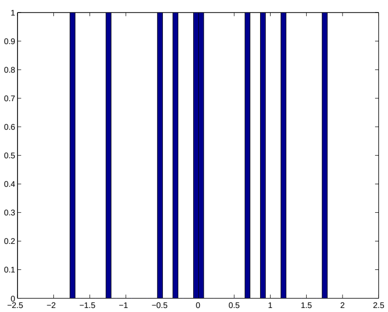


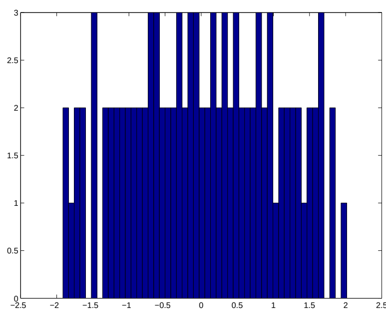
N=1000



zweite Realisierung



N=10



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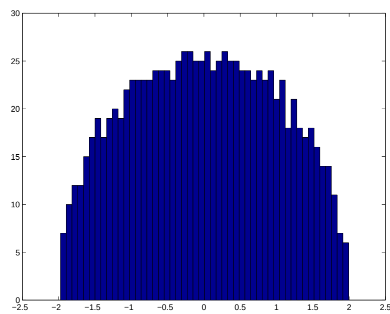
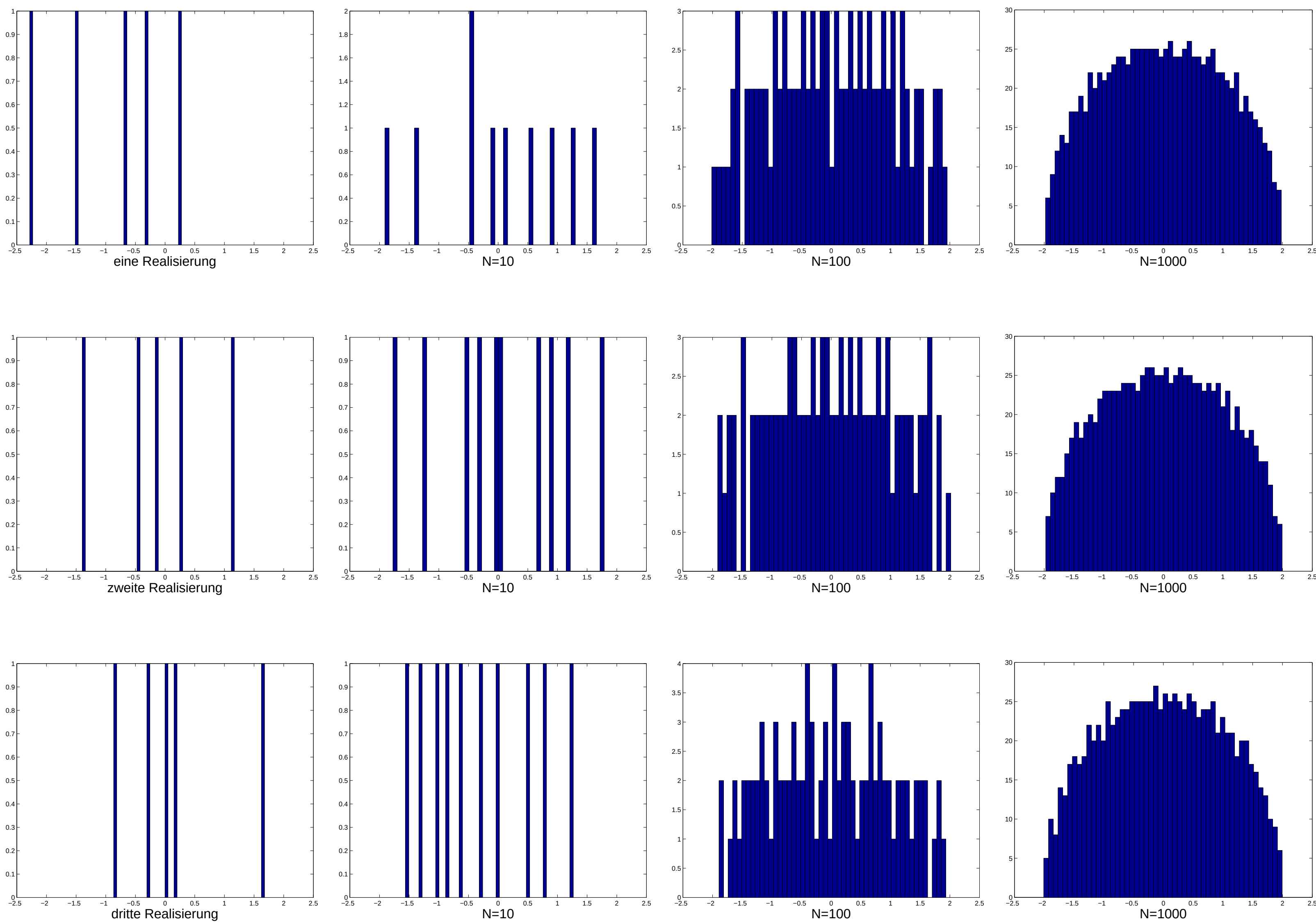


N=1000



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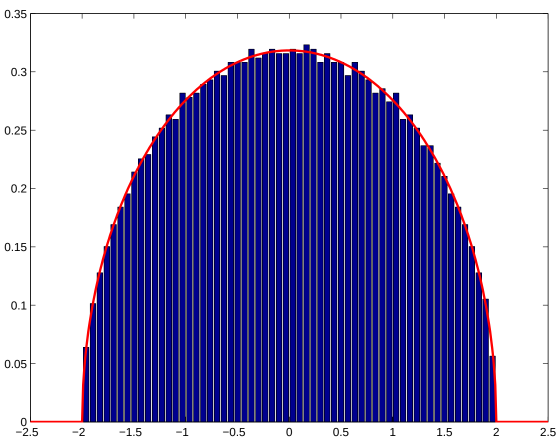
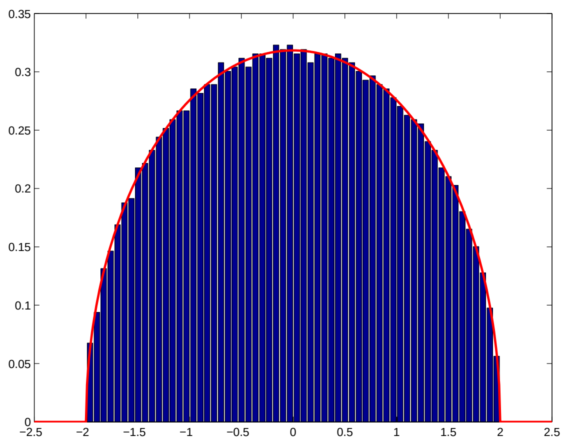
Consider selfadjoint **Gaussian $N \times N$ random matrix**.

We have almost sure convergence (convergence of "typical" realization) of its eigenvalue distribution to

Wigner's semicircle.



... one realization ...



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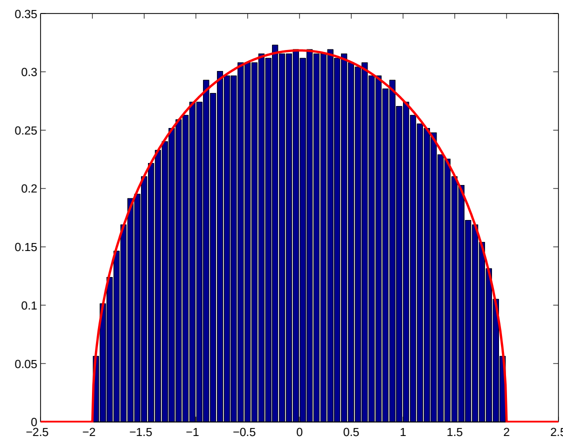
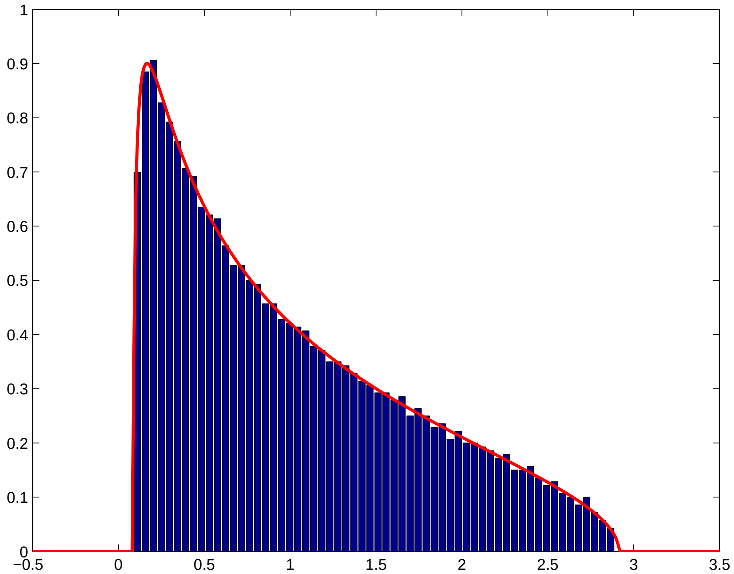


... yet another one ...

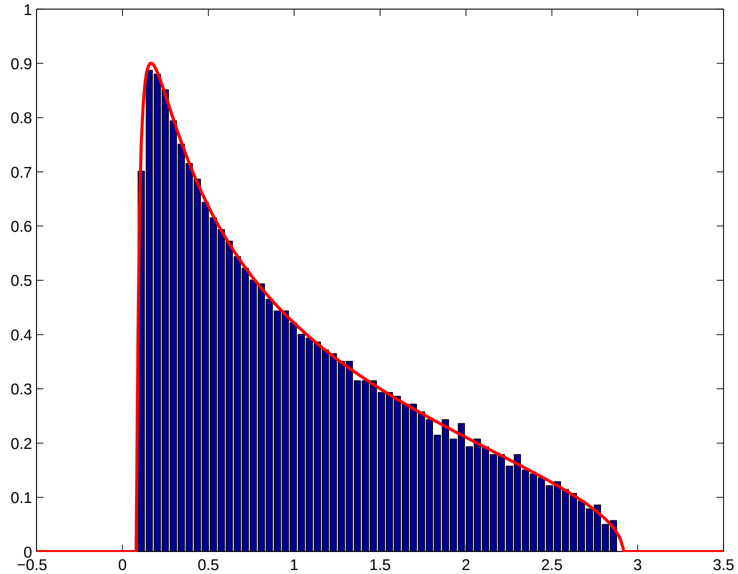
$$N = 4000$$

Consider **Wishart random matrix** $A = XX^*$, where X is $N \times M$ random matrix with independent Gaussian entries.
Its eigenvalue distribution converges almost surely to

Marchenko-Pastur distribution.



... one realization ...



... another realization ...

$$N = 3000, M = 6000$$

We want to consider more complicated situations, built out of simple cases (like Gaussian or Wishart) by doing operations like

- taking the sum of two matrices
- taking the product of two matrices
- taking corners of matrices

Note: If several $N \times N$ random matrices A and B are involved then the eigenvalue distribution of non-trivial functions $f(A, B)$ (like $A+B$ or AB) will of course depend on the relation between the eigenspaces of A and of B.

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- if $N \to \infty$ and

- if the eigenspaces are almost surely in a **"typical" or "generic" position**.

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However: we might expect that we have almost sure convergence to a deterministic result

- if $\boldsymbol{N \to \infty}$ and
- if the eigenspaces are almost surely in a **"typical" or "generic" position**.

This is the realm of **free probability theory**.

Consider $N \times N$ random matrices A and B such that

- A has an asymptotic eigenvalue distribution for $N \to \infty$
  B has an asymptotic eigenvalue distribution for $N \to \infty$

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Then, almost surely, eigenspaces of A and of B are in generic position.

In such a generic case we expect that the asymptotic eigenvalue distribution of functions of A and B should almost surely depend in a deterministic way on the asymptotic eigenvalue distribution of A and of B the asymptotic eigenvalue distribution.

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Basic examples for such functions:

- the sum

$$A + B$$

- the product

$$AB$$

- corners of the unitarily invariant matrix B

Example: sum of independent Gaussian and Wishart ($M = 2N$) random matrices, for $N = 3000$

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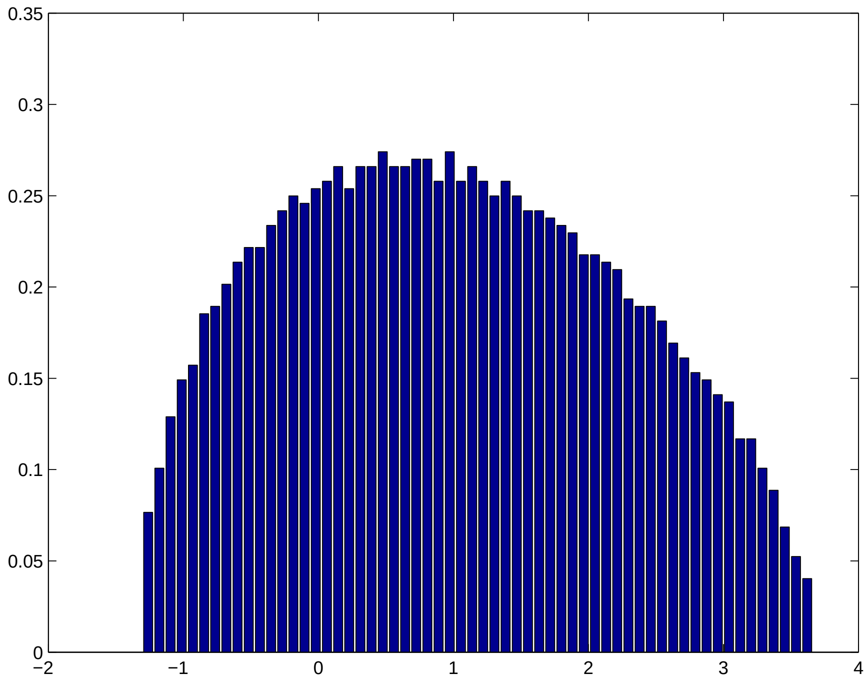

... one realization ...

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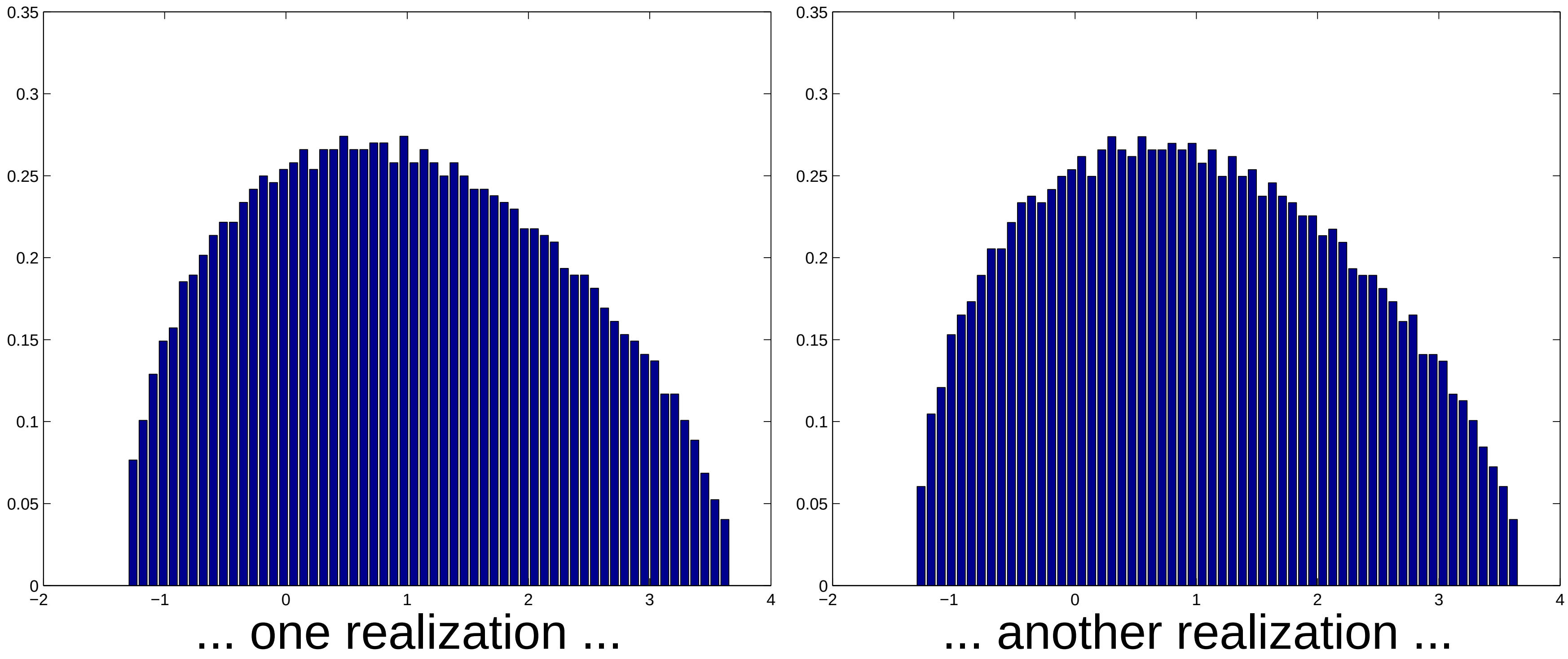

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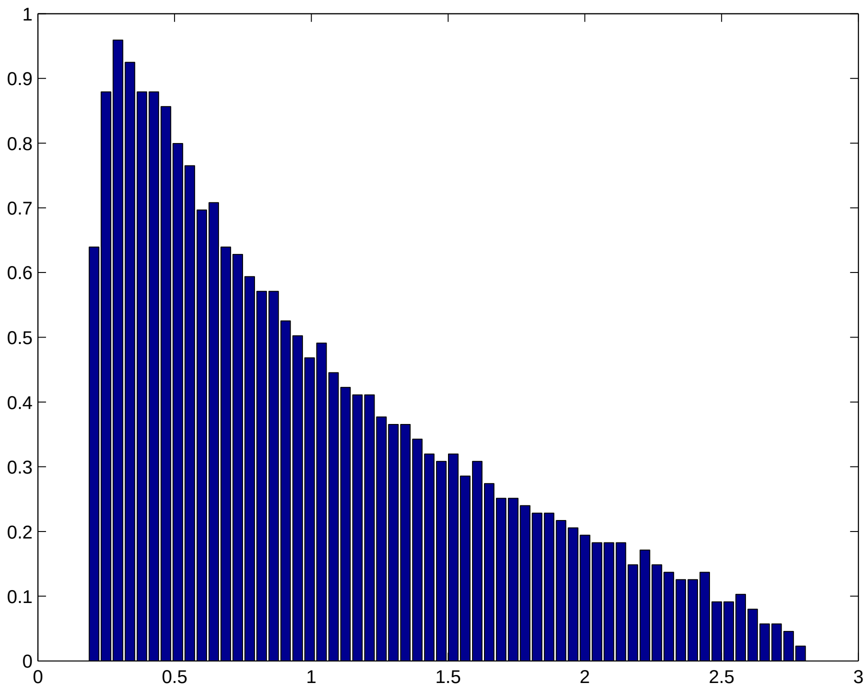

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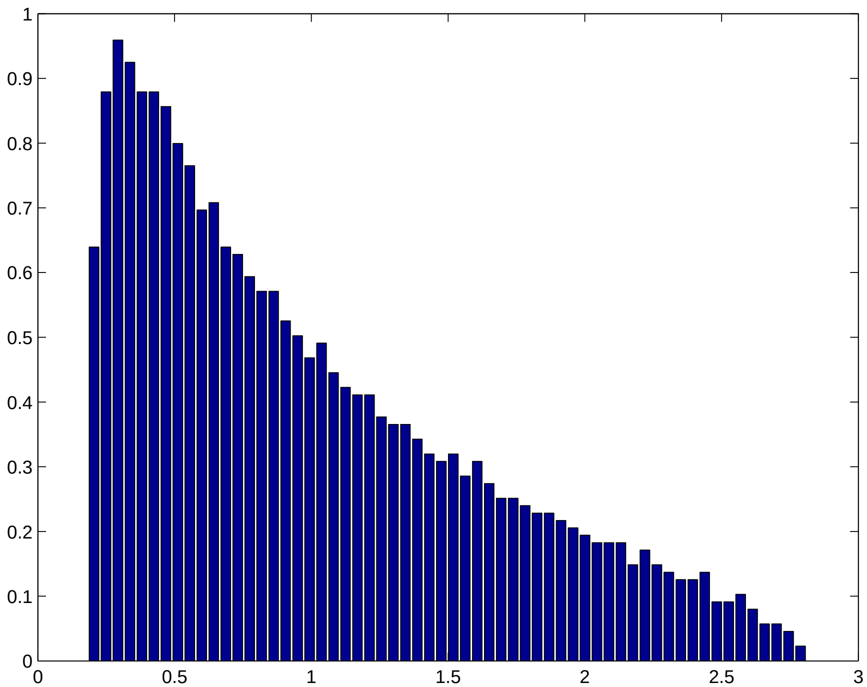

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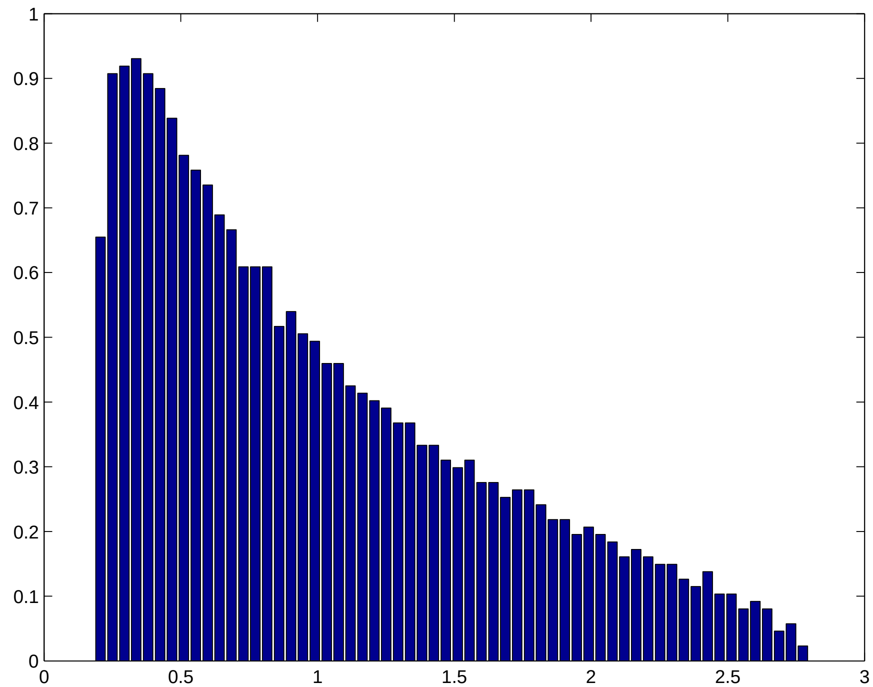

... another realization ...

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$N = 2048$

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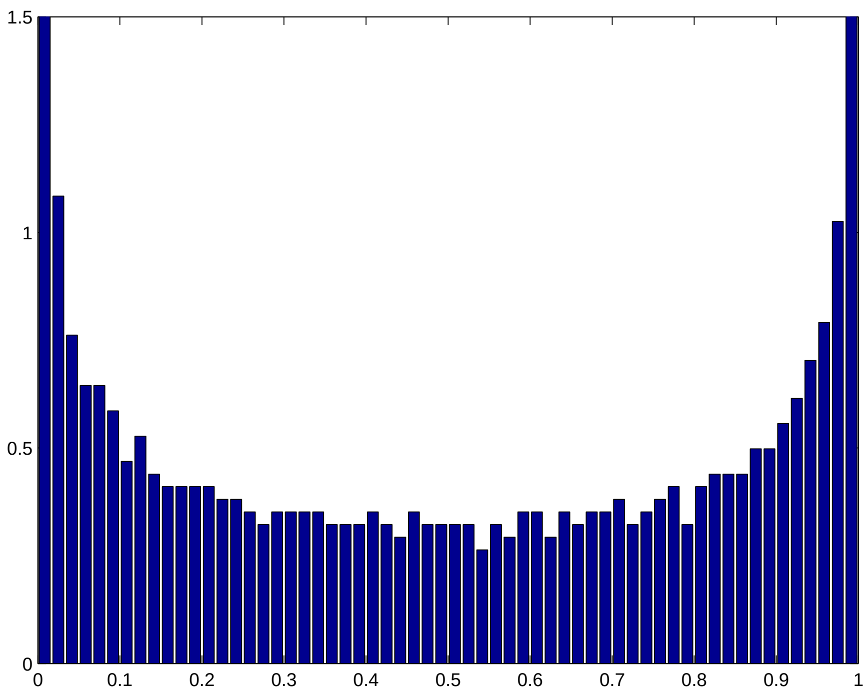

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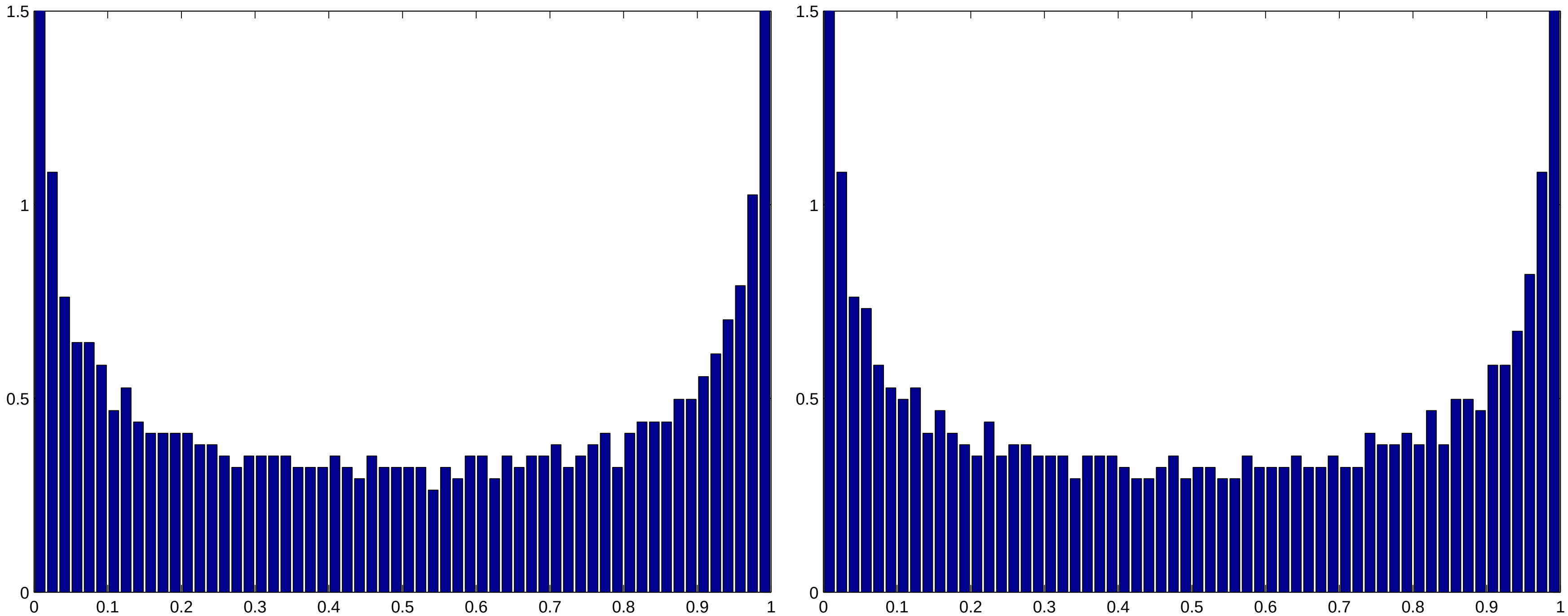

... one realization ... ... another realization ...

Problems:

- Do we have a **conceptual way of understanding** the asymptotic eigenvalue distributions in such cases?

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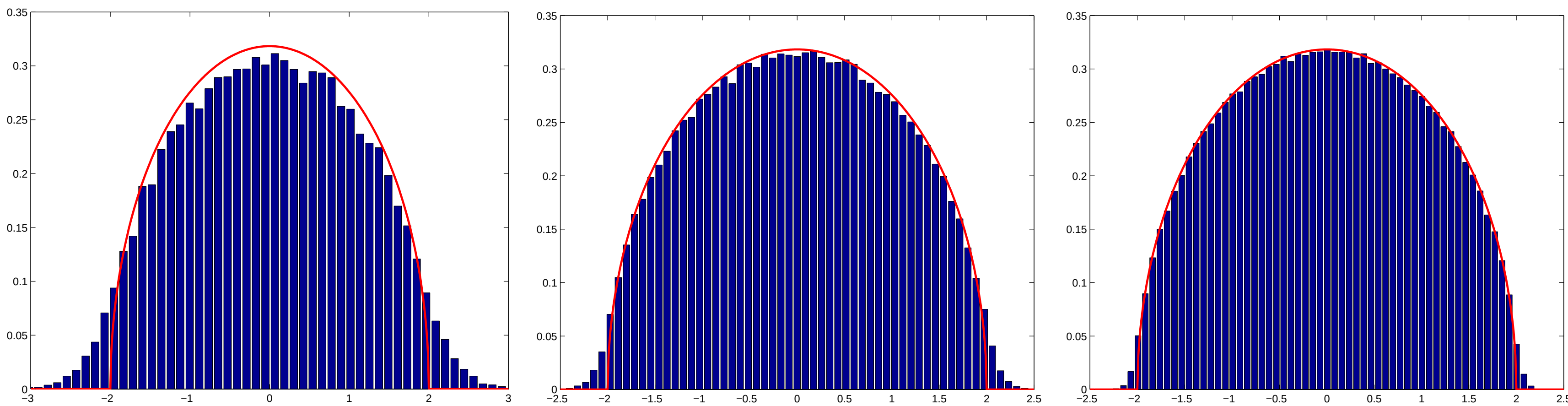
- Is there an **algorithm for actually calculating** the corresponding asymptotic eigenvalue distributions?

Instead of eigenvalue distribution of typical realization we will now look at eigenvalue distribution averaged over ensemble.

This has the advantages:

- convergence to asymptotic eigenvalue distribution happens much faster; very good agreement with asymptotic limit for moderate N
- theoretically easier to deal with averaged situation than with almost sure one (note however, this is just for convenience; the following can also be justified for typical realizations)

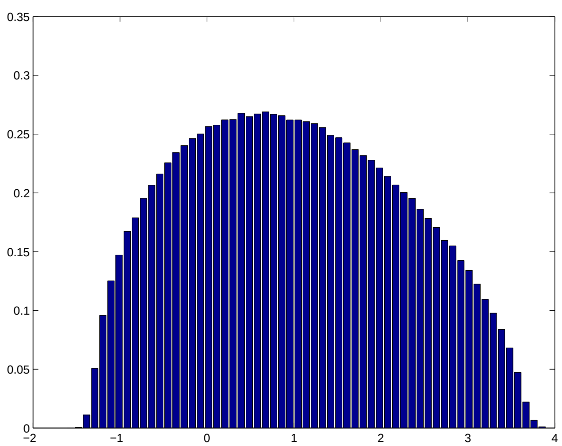
Example: Convergence of averaged eigenvalue distribution of $N \times N$ Gaussian random matrix to **semicircle**



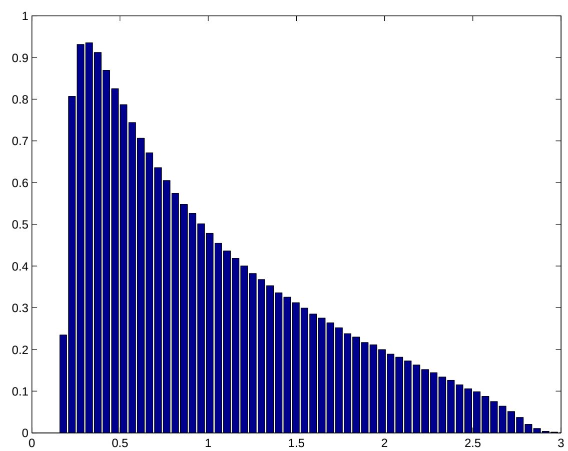
N=5 N=20 N=50

trials=10000

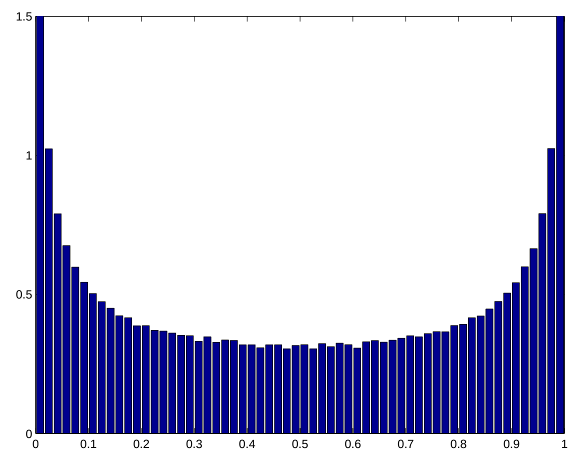
Examples: averaged sums, products, corners for moderate N



averaged Wigner + Wishart; N=100



averaged Wishart x Wishart; N=100



averaged upper corner; N=64

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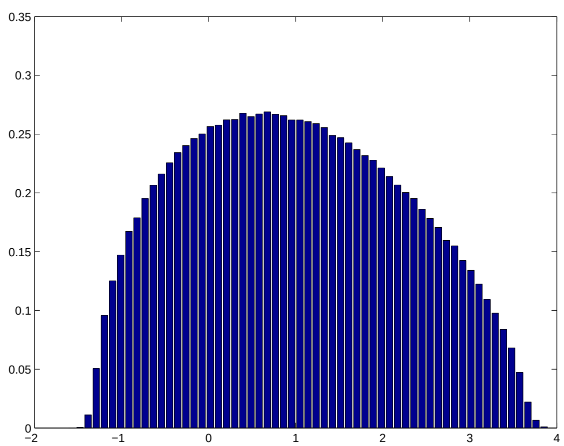
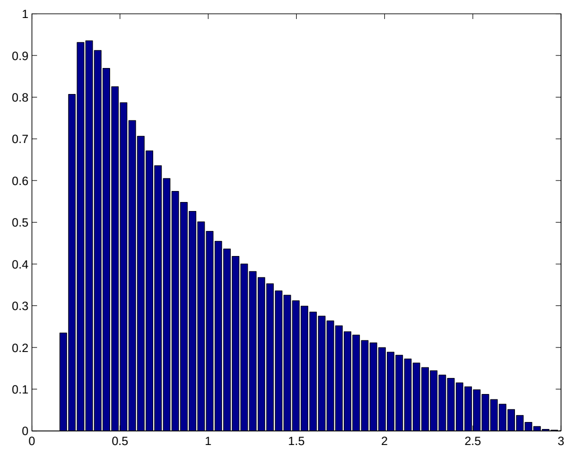


averaged Wigner + Wishart; N=100



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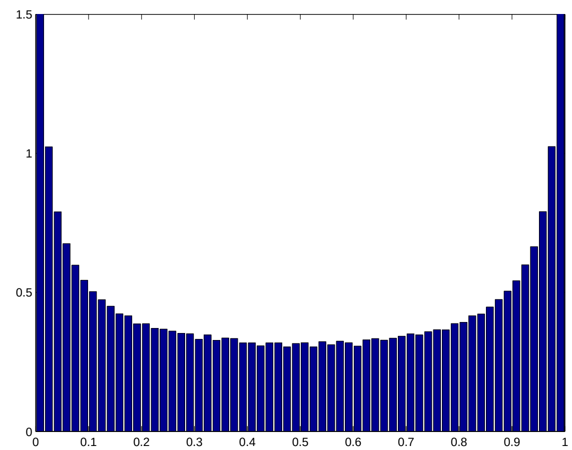

averaged upper corner; N=64

What is the asymptotic eigenvalue distribution in these cases?

How does one analyze asymptotic eigenvalue distributions?

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- analytical

- combinatorial

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- combinatorial: moment method
  try to calculate moments of the limit distribution
  advantage: can, in principle, deal directly with several matrices A, B; by looking on **mixed moments**

Moment Method

eigenvalue distribution of matrix A	$\widehat{=}$	knowledge of traces of powers, $\text{tr}(A^k)$
$\frac{1}{N}\left(\lambda_1^k + \cdots + \lambda_N^k\right)$	$=$	$\text{tr}(A^k)$
averaged eigenvalue distribution of random matrix A	$\widehat{=}$	knowledge of expectations of traces of powers, $E[\text{tr}(A^k)]$

Moment Method

Consider random matrices A and B in generic position.

We want to understand $f(A, B)$ in a uniform way for many f!

We have to understand for all $k \in \mathbb{N}$ the moments

$$E\left[\mathrm{tr}\Big(f(A,B)^k\Big)\right].$$

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$$E\left[\text{tr}\Big((A+B)^k\Big)\right], \quad E\left[\text{tr}\Big((AB)^k\Big)\right], \quad E\left[\text{tr}\Big((AB-BA)^k\Big)\right], \quad \text{etc.}$$

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Thus we need to understand as basic objects

mixed moments $\qquad \varphi\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)$

Use following notation:

$$\varphi(A) := \lim_{N\to\infty} E[\text{tr}(A)].$$

Question: If A and B are in generic position, can we understand

$$\varphi\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)$$

in terms of

$$\Big(\varphi(A^k)\Big)_{k\in\mathbb{N}} \qquad \text{and} \qquad \Big(\varphi(B^k)\Big)_{k\in\mathbb{N}}$$

Example: independent Gaussian random matrices

Consider two independent Gaussian random matrices A and B

Then, in the limit $N \to \infty$, the moments

$$\varphi\left(A^{n_1} B^{m_1} A^{n_2} B^{m_2} \cdots\right)$$

are given by

$$\#\Big\{\text{non-crossing/planar pairings of pattern}$$

$$\underbrace{A \cdot A \cdots A}_{n_1\text{-times}} \cdot \underbrace{B \cdot B \cdots B}_{m_1\text{-times}} \cdot \underbrace{A \cdot A \cdots A}_{n_2\text{-times}} \cdot \underbrace{B \cdot B \cdots B}_{m_2\text{-times}} \cdots,$$

$$\text{which do not pair } A \text{ with } B\Big\}$$

Example: $\varphi(AABBABBA) = 2$

because there are two such non-crossing pairings:

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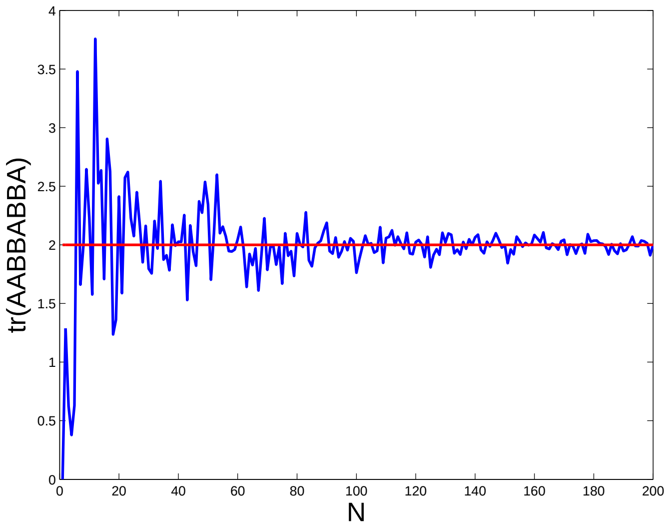
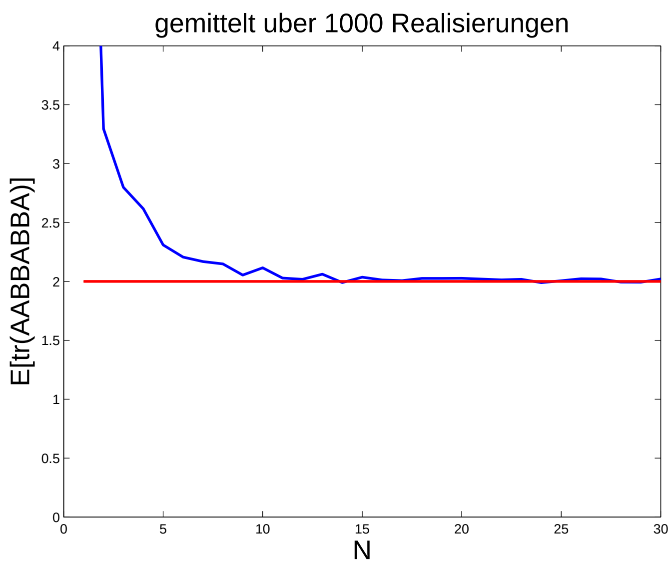
one realization



averaged over 1000 realizations



$$\varphi\left(A^{n_1} B^{m_1} A^{n_2} B^{m_2} \cdots\right)$$

$$= \#\left\{\text{non-crossing pairings which do not pair } A \text{ with } B\right\}$$

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implies

$$\varphi\Big(\big(A^{n_1} - \varphi(A^{n_1}) \cdot 1\big) \cdot \big(B^{m_1} - \varphi(B^{m_1}) \cdot 1\big) \cdot \big(A^{n_2} - \varphi(A^{n_2}) \cdot 1\big) \cdots\Big)$$

$$= \#\Big\{\text{non-crossing pairings which do not pair } A \text{ with } B, \text{ and for which each blue group and each red group is connected with some other group}\Big\}$$

$$\varphi\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)$$

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$$= 0$$

Actual equation for the calculation of the mixed moments

$$\varphi_1\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)$$

is different for different random matrix ensembles.

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However, the relation between the mixed moments,

$$\varphi\Big(\big(A^{n_1} - \varphi(A^{n_1}) \cdot 1\big) \cdot \big(B^{m_1} - \varphi(B^{m_1}) \cdot 1\big) \cdots \Big) = 0$$

remains the same for matrix ensembles in generic position and constitutes the **definition of freeness**.

Definition [Voiculescu 1985]: A and B are **free** (with respect to φ) if we have for all $n_1, m_1, n_2, \dots \geq 1$ that

$$\varphi\Big(\big(A^{n_1} - \varphi(A^{n_1}) \cdot 1\big) \cdot \big(B^{m_1} - \varphi(B^{m_1}) \cdot 1\big) \cdot \big(A^{n_2} - \varphi(A^{n_2}) \cdot 1\big) \cdots\Big) = 0$$

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$$\varphi\Big(\text{alternating product in centered words in } A \text{ and in } B\Big) = 0$$

Theorem [Voiculescu 1991]: Consider $N \times N$ random matrices A and B such that

- A has an asymptotic eigenvalue distribution for $N \to \infty$
  B has an asymptotic eigenvalue distribution for $N \to \infty$

- A and B are independent
  (i.e., entries of A are independent from entries of C)

- B is a unitarily invariant ensemble
  (i.e., the joint distribution of its entries does not change under unitary conjugation)

Then, for $N \to \infty$, A and B are free.

What is Freeness?

Freeness between A and B is an infinite set of equations relating various moments in A and B:

$$\varphi\Big(p_1(A)q_1(B)p_2(A)q_2(B)\cdots\Big) = 0$$

Basic observation: freeness between A and B is actually a **rule for calculating mixed moments** in A and B from the moments of A and the moments of B:

$$\varphi\Big(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\Big) = \text{polynomial}\Big(\varphi(A^i), \varphi(B^j)\Big)$$

Example:

$$\varphi\Big(\big(A^n - \varphi(A^n)1\big)\big(B^m - \varphi(B^m)1\big)\Big) = 0,$$

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$$\varphi\Big(\big(A^n - \varphi(A^n)1\big)\big(B^m - \varphi(B^m)1\big)\Big) = 0,$$

thus

$$\varphi(A^nB^m) - \varphi(A^n \cdot 1)\varphi(B^m) - \varphi(A^n)\varphi(1 \cdot B^m) + \varphi(A^n)\varphi(B^m)\varphi(1 \cdot 1) = 0,$$

Example:

$$\varphi\Big(\big(A^n - \varphi(A^n)1\big)\big(B^m - \varphi(B^m)1\big)\Big) = 0,$$

thus

$$\varphi(A^nB^m)-\varphi(A^n\cdot 1)\varphi(B^m)-\varphi(A^n)\varphi(1\cdot B^m)+\varphi(A^n)\varphi(B^m)\varphi(1\cdot 1) = 0,$$

and hence

$$\boldsymbol{\varphi(A^nB^m) = \varphi(A^n)\cdot\varphi(B^m)}$$

Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables.

Thus freeness is also called **free independence**

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Note: free independence is a different rule from classical independence; free independence occurs typically for **non-commuting random variables**, like operators on Hilbert spaces or (random) matrices

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Example:

$$\varphi\Big(\big(A-\varphi(A)1\big)\cdot\big(B-\varphi(B)1\big)\cdot\big(A-\varphi(A)1\big)\cdot\big(B-\varphi(B)1\big)\Big)=0,$$

which results in

$$\boldsymbol{\varphi(ABAB) = \varphi(AA)\cdot\varphi(B)\cdot\varphi(B) + \varphi(A)\cdot\varphi(A)\cdot\varphi(BB) - \varphi(A)\cdot\varphi(B)\cdot\varphi(A)\cdot\varphi(B)}$$

Understanding the freeness rule: the idea of cumulants

- write moments in terms of other quantities, which we call **free cumulants**

- freeness is much easier to describe on the level of free cumulants: **vanishing of mixed cumulants**

- relation between moments and cumulants is given by summing over **non-crossing or planar partitions**

Moments and cumulants

For

$$\varphi : \mathcal{A} \to \mathbb{C}$$

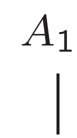
we define **cumulant functionals** κ_n (for all $n \geq 1$)

$$\kappa_n : \mathcal{A}^n \to \mathbb{C}$$

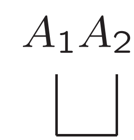
by moment-cumulant relation

$$\varphi(A_1 \cdots A_n) = \sum_{\pi \in NC(n)} \kappa_\pi[A_1, \dots, A_n]$$

$$\varphi(A_1) = \kappa_1(A_1)$$



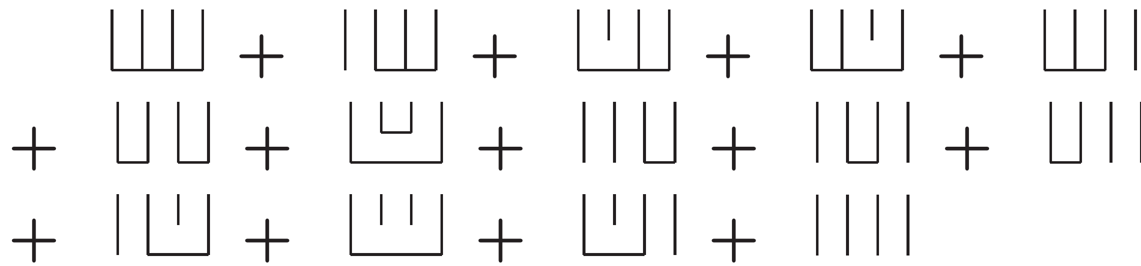
$$\varphi(A_1 A_2) = \kappa_2(A_1, A_2)$$



$$+ \kappa_1(A_1)\kappa_1(A_2)$$

$$
\begin{aligned}
\varphi(A_1A_2A_3) = {} & \kappa_3(A_1, A_2, A_3) \\
& + \kappa_1(A_1)\kappa_2(A_2, A_3) \\
& + \kappa_2(A_1, A_2)\kappa_1(A_3) \\
& + \kappa_2(A_1, A_3)\kappa_1(A_2) \\
& + \kappa_1(A_1)\kappa_1(A_2)\kappa_1(A_3)
\end{aligned}
$$

$$\varphi(A_1A_2A_3A_4) =$$



$$\begin{aligned}
= \quad & \kappa_4(A_1, A_2, A_3, A_4) + \kappa_1(A_1)\kappa_3(A_2, A_3, A_4) \\
& + \kappa_1(A_2)\kappa_3(A_1, A_3, A_4) + \kappa_1(A_3)\kappa_3(A_1, A_2, A_4) \\
& + \kappa_3(A_1, A_2, A_3)\kappa_1(A_4) + \kappa_2(A_1, A_2)\kappa_2(A_3, A_4) \\
& + \kappa_2(A_1, A_4)\kappa_2(A_2, A_3) + \kappa_1(A_1)\kappa_1(A_2)\kappa_2(A_3, A_4) \\
& + \kappa_1(A_1)\kappa_2(A_2, A_3)\kappa_1(A_4) + \kappa_2(A_1, A_2)\kappa_1(A_3)\kappa_1(A_4) \\
& + \kappa_1(A_1)\kappa_2(A_2, A_4)\kappa_1(A_3) + \kappa_2(A_1, A_4)\kappa_1(A_2)\kappa_1(A_3) \\
& + \kappa_2(A_1, A_3)\kappa_1(A_2)\kappa_1(A_4) + \kappa_1(A_1)\kappa_1(A_2)\kappa_1(A_3)\kappa_1(A_4)
\end{aligned}$$

Freeness $\hat{=}$ vanishing of mixed cumulants

free product $\hat{=}$ direct sum of cumulants

We have: A and B free is equivalent to

$$\kappa_n(C_1, \dots, C_n) = 0$$

whenever

- $n \geq 2$

- $C_i \in \{A, B\}$ for all i

- there are i, j such that $C_i = A$, $C_j = B$

Freeness $\hat{=}$ vanishing of mixed cumulants

free product $\hat{=}$ direct sum of cumulants

$\varphi(\boldsymbol{A^n})$ given by sum over **blue** planar diagrams

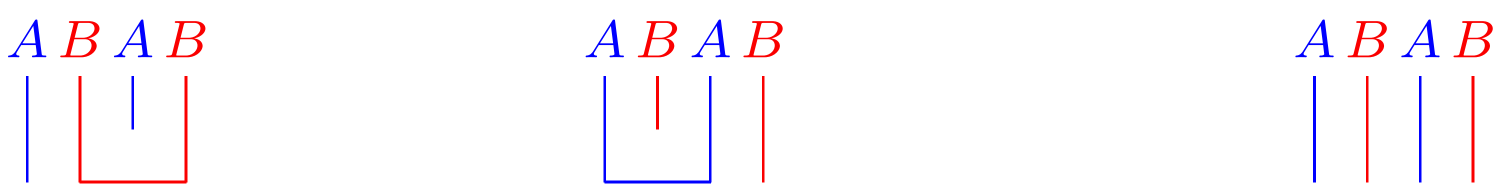
$\varphi(\boldsymbol{B^m})$ given by sum over **red** planar diagrams

then: for A and B free, $\varphi(\boldsymbol{A^{n_1}B^{m_1}A^{n_2}}\cdots)$ is given by sum over planar diagrams with monochromatic (blue or red) blocks

Vanishing of Mixed Cumulants

$$\varphi(ABAB) =$$

$$\kappa_1(A)\kappa_1(A)\kappa_2(B,B) + \kappa_2(A,A)\kappa_1(B)\kappa_1(B) + \kappa_1(A)\kappa_1(B)\kappa_1(A)\kappa_1(B)$$



Sum of Free Variables

Consider A, B free.

Then, by freeness, the moments of $A+B$ are uniquely determined by the moments of A and the moments of B.

Notation: We say the distribution of $A + B$ is the

free convolution

of the distribution of A and the distribution of B,

$$\mu_{A+B} = \mu_A \boxplus \mu_B.$$

Sum of Free Variables

In principle, freeness determines this, but the concrete nature of this rule on the level of moments is not apriori clear.

Example:

$$\varphi\Big((A+B)^1\Big) = \varphi(A) + \varphi(B)$$

$$\varphi\Big((A+B)^2\Big) = \varphi(A^2) + 2\varphi(A)\varphi(B) + \varphi(B^2)$$

$$\varphi\Big((A+B)^3\Big) = \varphi(A^3) + 3\varphi(A^2)\varphi(B) + 3\varphi(A)\varphi(B^2) + \varphi(B^3)$$

$$\begin{aligned}\varphi\Big((A+B)^4\Big) = \varphi(A^4) &+ 4\varphi(A^3)\varphi(B) + 4\varphi(A^2)\varphi(B^2) \\ &+ 2\Big(\varphi(A^2)\varphi(B)\varphi(B) + \varphi(A)\varphi(A)\varphi(B^2) \\ &\quad - \varphi(A)\varphi(B)\varphi(A)\varphi(B)\Big) + 4\varphi(A)\varphi(B^3) + \varphi(B^4)\end{aligned}$$

Sum of Free Variables

Corresponding rule on level of free cumulants is easy: If A and B are free then

$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)$$
$$+ \kappa_n(\dots, A, B, \dots) + \cdots$$

Sum of Free Variables

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Sum of Free Variables

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$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)$$

i.e., we have **additivity of cumulants for free variables**

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

Sum of Free Variables

Corresponding rule on level of free cumulants is easy: If A and B are free then

$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)$$

i.e., we have additivity of cumulants for free variables

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

Also: Combinatorial relation between moments and cumulants can be rewritten easily as a relation between corresponding formal power series.

Sum of Free Variables

Consider one random variable $A \in \mathcal{A}$ and define their **Cauchy transform G** and their **$\mathcal{R}$-transform $\mathcal{R}$** by

$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi(A^n)}{z^{n+1}}, \qquad \mathcal{R}(z) = \sum_{n=1}^{\infty} \kappa_n(A, \ldots, A) z^{n-1}$$

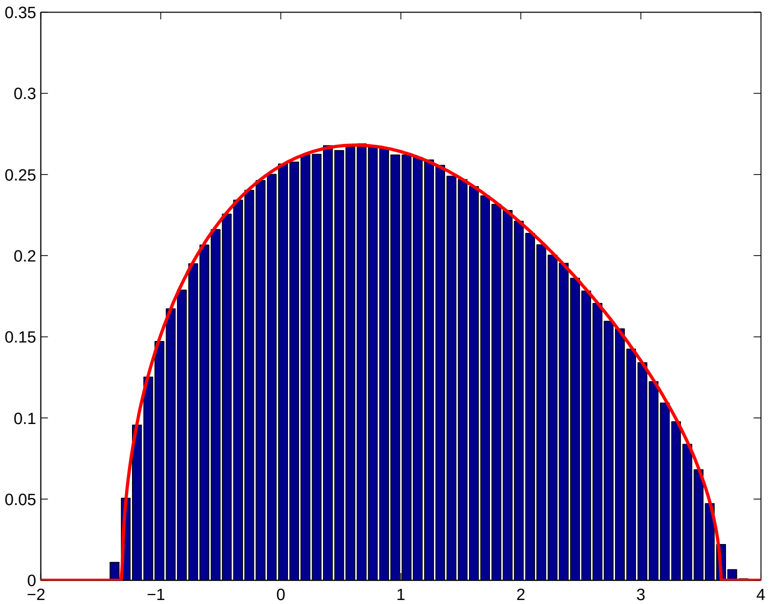
Theorem [Voiculescu 1986, Speicher 1994]: Then we have

- $\frac{1}{G(z)} + \mathcal{R}(G(z)) = z$
- $\mathcal{R}^{A+B}(z) = \mathcal{R}^A(z) + \mathcal{R}^B(z)$ if A and B are free

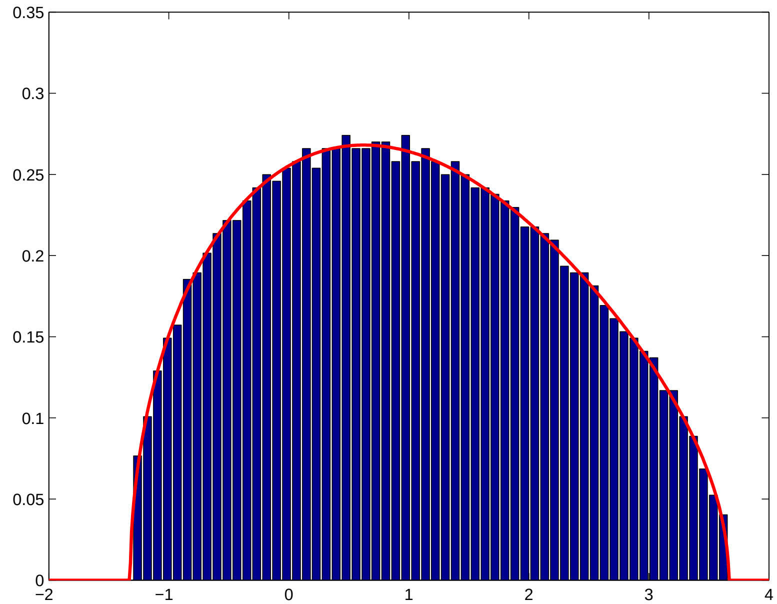
This, together with the relation between Cauchy transform and $\mathcal{R}$-transform and with the Stieltjes inversion formula, gives an effective algorithm for calculating free convolutions, i.e., the asymptotic eigenvalue distribution **of sums of random matrices in generic position**:

$$\begin{array}{ccccccccc}
A & \rightsquigarrow & G^A & \rightsquigarrow & R^A & & & & \\
 & & & & \downarrow & & & & \\
 & & & & R^A + R^B = R^{A+B} & \rightsquigarrow & G^{A+B} & \rightsquigarrow & A+B \\
 & & & & \uparrow & & & & \\
B & \rightsquigarrow & G^B & \rightsquigarrow & R^B & & & &
\end{array}$$

Example: Wigner + Wishart ($M = 2N$)



averaged Wigner + Wishart; N=100



... one realization ...

trials=4000

N=3000

Product of Free Variables

Consider A, B free.

Then, by freeness, the moments of AB are uniquely determined by the moments of A and the moments of B.

Notation: We say the distribution of AB is the

free multiplicative convolution

of the distribution of A and the distribution of B,

$$\mu_{AB} = \mu_A \boxtimes \mu_B.$$

In principle, freeness determines this, but the concrete nature of this rule on the level of moments is not apriori clear.

Examples: We have

$$\varphi\Big((AB)^1\Big) = \varphi(A)\varphi(B)$$

$$\varphi\Big((AB)^2\Big) = \varphi(A^2)\varphi(B)^2 + \varphi(A)^2\varphi(B^2) - \varphi(A)^2\varphi(B)^2$$

$$\begin{aligned}\varphi\Big((AB)^3\Big) = {} & \varphi(A^3)\varphi(B)^3 + \varphi(A)^3\varphi(B^3) + 3\varphi(A)\varphi(A^2)\varphi(B)\varphi(B^2) \\ & - 3\varphi(A)\varphi(A^2)\varphi(B)^3 - 3\varphi(A)^3\varphi(B)\varphi(B^2) \\ & + 2\varphi(A)^3\varphi(B)^3\end{aligned}$$

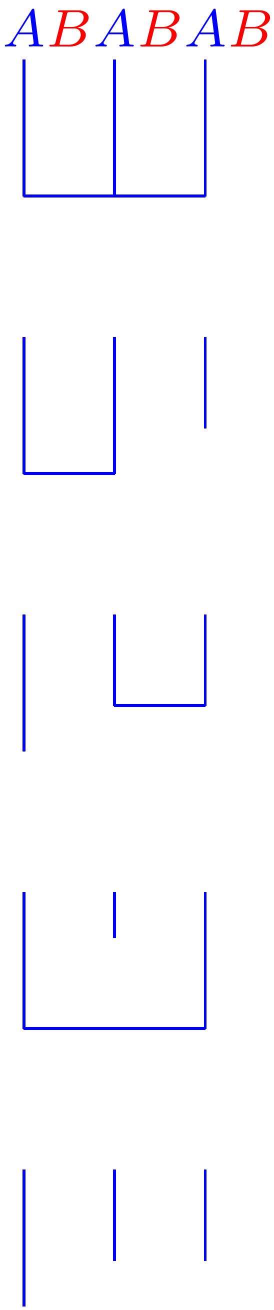
Product of Free Variables

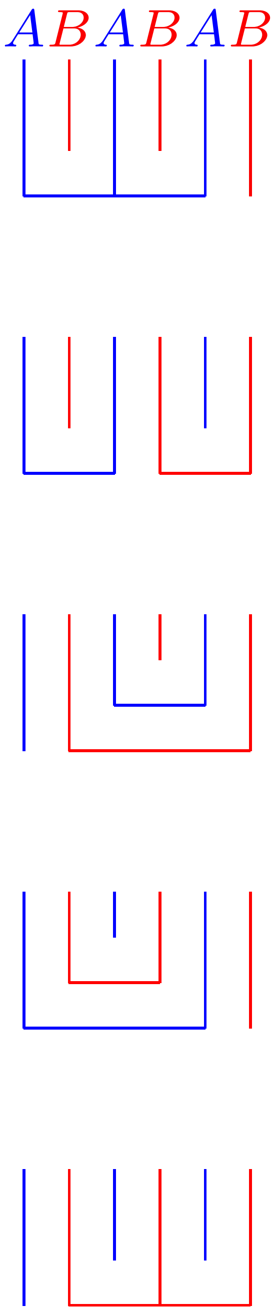
Corresponding rule on level of free cumulants is relatively easy (at least conceptually): If A and B are free then

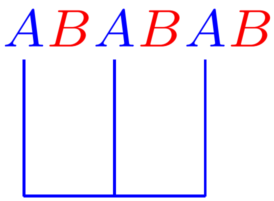
$$\kappa^{AB} = \kappa^{A} \boxed{\star} \kappa^{B},$$

i.e.,

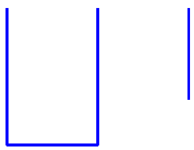
$$\kappa_n(AB, AB, \ldots, AB) = \sum_{\pi \in NC(n)} \kappa_\pi[A, A, \ldots, A] \cdot \kappa_{K(\pi)}[B, B, \ldots, B].$$


ABABAB

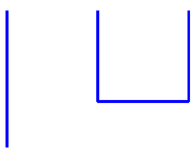

ABABAB



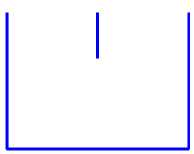
$$\kappa_3(A, A, A)$$



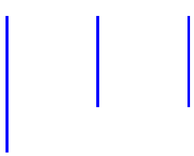
$$\kappa_2(A, A)\kappa_1(A)$$



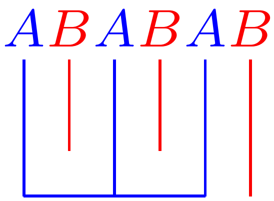
$$\kappa_2(A, A)\kappa_1(A)$$



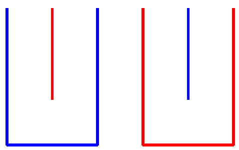
$$\kappa_2(A, A)\kappa_1(A)$$



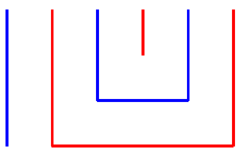
$$\kappa_1(A)\kappa_1(A)\kappa_1(A)$$



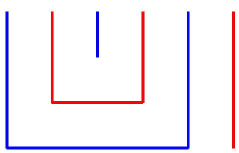
$$\kappa_3(A, A, A)\ \kappa_1(B)\kappa_1(B)\kappa_1(B)$$



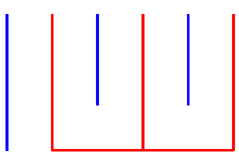
$$\kappa_2(A, A)\kappa_1(A)\ \kappa_2(B, B)\kappa_1(B)$$



$$\kappa_2(A, A)\kappa_1(A)\ \kappa_2(B, B)\kappa_1(B)$$



$$\kappa_2(A, A)\kappa_1(A)\ \kappa_2(B, B)\kappa_1(B)$$



$$\kappa_1(A)\kappa_1(A)\kappa_1(A)\ \kappa_3(B, B, B)$$

Product of Free Variables

Theorem [Voiculescu 1987; Haagerup 1997; Nica, Speicher 1997]:
Put

$$M_A(z) := \sum_{m=0}^{\infty} \varphi(A^m) z^m$$

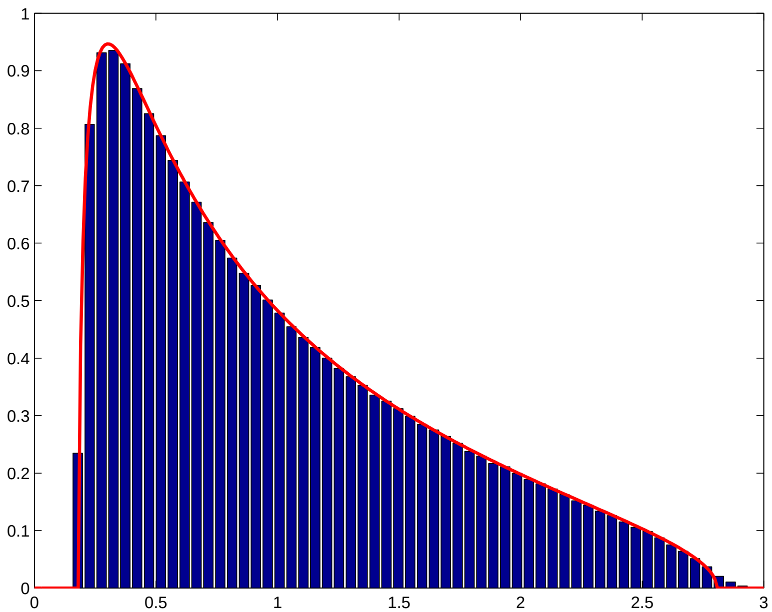
and define

$$S_A(z) := \frac{1+z}{z} M_A^{<-1>}(z) \qquad \textbf{\textit{S}-transform of \textit{A}}$$

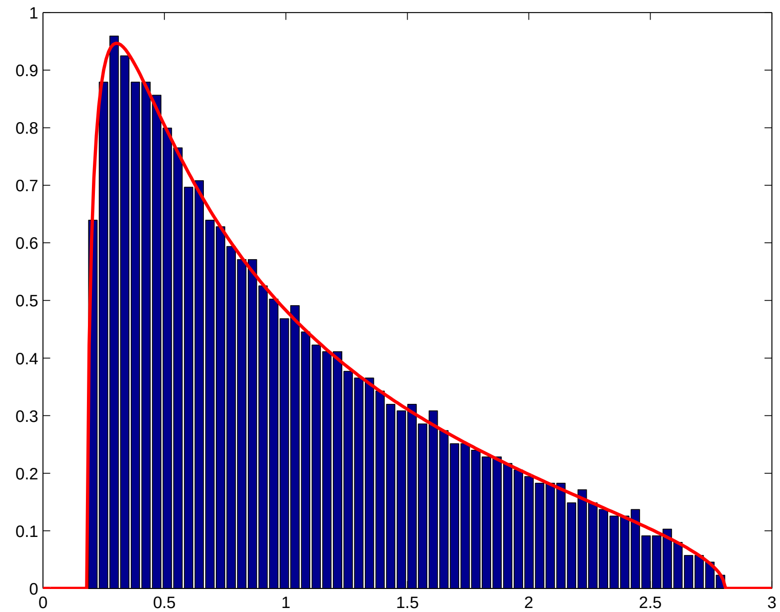
Then: If A and B are free, we have

$$S_{AB}(z) = S_A(z) \cdot S_B(z).$$

Example: Wishart x Wishart ($M = 5N$)



averaged Wishart x Wishart; N=100



... one realization ...

trials=10000

N=2000

Free cumulants as classical cumulants of cycles

Theorem [Collins, Mingo, Sniady, Speicher 2007]: Let

$$A = (a_{ij})_{i,j=1}^{N}$$

be a unitarily invariant random matrix. Then we have

$$\kappa_n(A, \ldots, A) = \lim_{N \to \infty} N^{n-1} c_n(a_{12}, a_{23}, a_{34}, \ldots, a_{n1})$$

Classical cumulants of entries without cycle structure vanish.

Corners of Random Matrices

Let

$$A = (a_{ij})_{i,j=1}^{N}$$

be a unitarily invariant random matrix. Fix $0 \leq \alpha \leq 1$ and let B be the $\alpha N \times \alpha N$ upper left corner of A.

$$A = \begin{pmatrix} B & * \\ * & * \end{pmatrix}$$

Then B is also unitarily invariant and we have

$$\begin{aligned}
\kappa_n(B, \dots, B) &= \lim_{N \to \infty} (\alpha N)^{n-1} c_n(a_{12}, a_{23}, a_{34}, \dots, a_{n1}) \\
&= \alpha^{n-1} \kappa_n(A, \dots, A) \\
&= \frac{1}{\alpha} \kappa_n(\alpha A, \dots, \alpha A)
\end{aligned}$$

Corners of Random Matrices

Theorem [Nica, Speicher 1996]: The asymptotic eigenvalue distribution of a corner B of ratio α of a unitarily invariant random matrix A is given by

$$\mu_B = \mu_{\alpha A}^{\boxplus 1/\alpha}$$

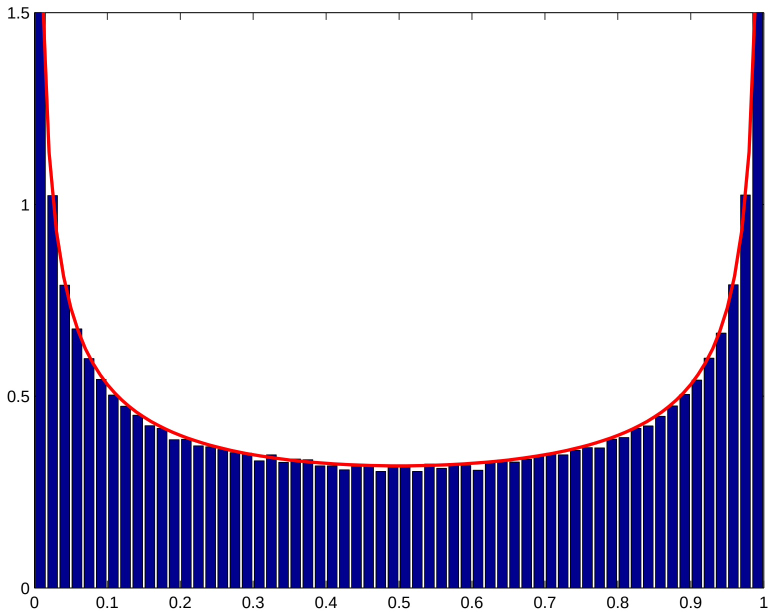
Corners of Random Matrices

Theorem [Nica, Speicher 1996]: The asymptotic eigenvalue distribution of a corner B of ratio α of a unitarily invariant random matrix A is given by

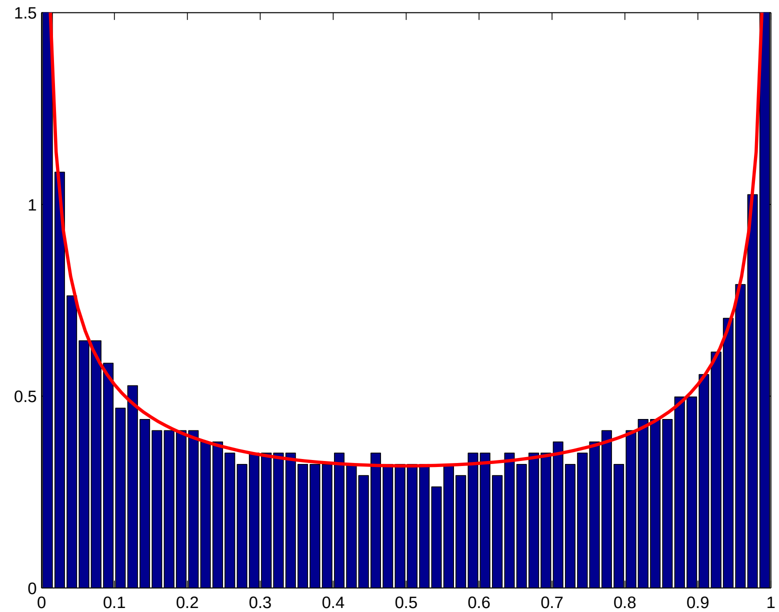
$$\mu_B = \mu_{\alpha A}^{\boxplus 1/\alpha}$$

In particular, a corner of size $\alpha = 1/2$, has up to rescaling the same distribution as $\mu_A \boxplus \mu_A$.

Example: upper left corner of size $N/2 \times N/2$ of a projection matrix, with $N/2$ eigenvalues 0 and $N/2$ eigenvalues 1



averaged upper corner; N=64



... one realization ...

trials=5000

N=2048

Literature

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- F. Hiai, D. Petz: The Semicircle Law, Free Random Variables and Entropy. Math. Surveys and Monogr. 77, AMS 2000

- A. Nica, R. Speicher: Lectures on the Combinatorics of Free Probability. *London Mathematical Society Lecture Note Series*, vol. 335, Cambridge University Press, 2006

- J. Mingo, R. Speicher: Free Probability and Random Matrices. Coming soon