The Asymptotics of Young Diagrams and Free Probability Theory

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Section 1

Young Diagrams and Representations of Symmetric Groups

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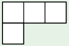
Irreducible Representations of S_n are Given in Terms of Young Diagrams

Example ($n = 4$)

$4 = 4$ $\qquad (4) =$ 

$4 = 3 + 1$ $\qquad (3, 1) =$ 

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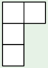
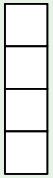
$4 = 2 + 1 + 1$ $\qquad (2, 1, 1) =$ 

$4 = 1 + 1 + 1 + 1$ $\qquad (1, 1, 1, 1) =$ 

Induction of representation of subgroup

- Let $H \subset G$ be subgroup of G; write G as disjoint union of cosets:

$$G = g_1 H \cup \cdots \cup g_m H$$

- Let π be representation of H
- Define induced representation

$$\mathrm{Ind}_H^G \pi(g) = \big(\pi(g_i^{-1} g g_j)\big)_{i,j=1}^{m}$$

where

$$\pi(g) = 0 \qquad \text{for } g \notin H$$

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- Even if π is irreducible, the induction $\mathrm{Ind}_H^G \pi$ is in general reducible, hence decomposes into irreducible components

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Inducing a representation of S_m and of S_n from $S_m \times S_n \subset S_{m+n}$ to a (reducible) representation of S_{m+n}

- Decomposition into irreducible components is given by Littlewood-Richardson rule
- λ appears in $\mu \times \nu$ as often as there are tableaux of skew shape $\lambda \backslash \nu$ of weight μ such that ...

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- this rule is algorithmic, decomposition can be calculated quite effectively by computer algebra: Littlewood-Richardson Calculator `lrcalc` by Anders Buch
- for m, n large there are many terms; there is no apparent structure

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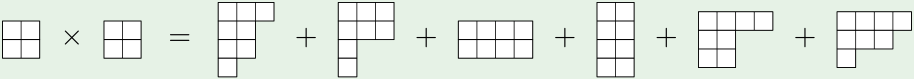
Inducing a representation of S_4 and of S_4 to S_8

Example

$$\mathrm{Ind}_{S_4 \times S_4}^{S_8} \pi_{(2,2)} \otimes \pi_{(2,2)} = \pi_{(3,2,2,1)} \oplus \pi_{(3,3,1,1)} \oplus \pi_{(4,4)} \oplus \pi_{(2,2,2,2)} \oplus \pi_{(4,2,2)} \oplus \pi_{(4,3,1)},$$

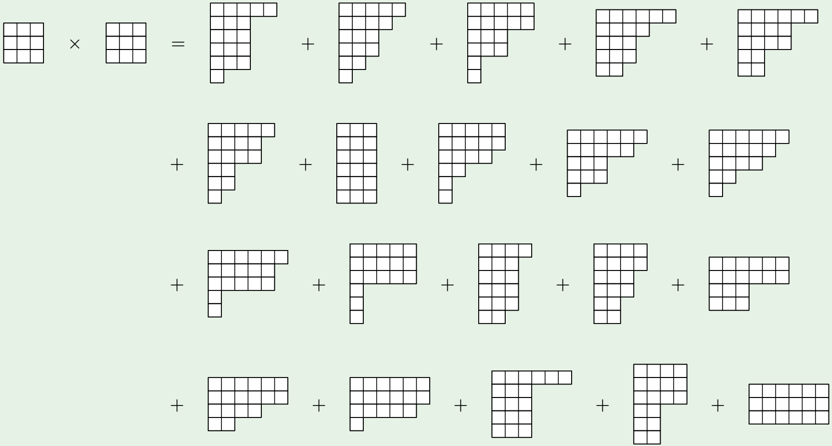
or in terms of Young diagrams



Inducing a representation of S_9 and of S_9 to S_{18}

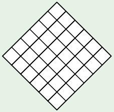
Example

$\mathrm{Ind}_{S_9 \times S_9}^{S_{18}} \pi_{(3,3,3)} \otimes \pi_{(3,3,3)}$:

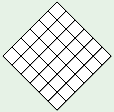


Typical Components in Such Induced Representations

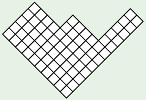
Example



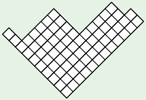
$\times$



$= \cdots +$



$+$



$+ \cdots$

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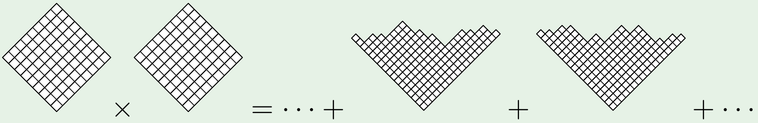

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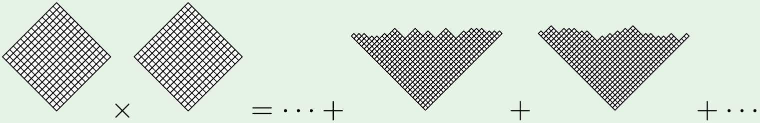

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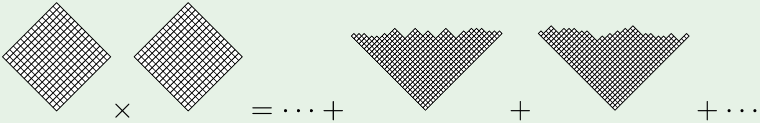
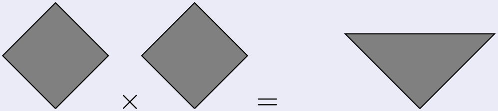
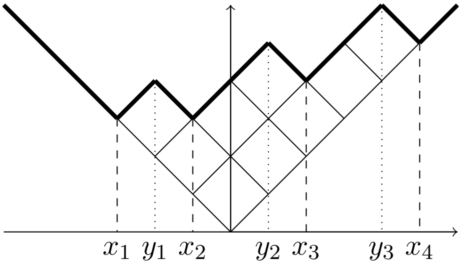
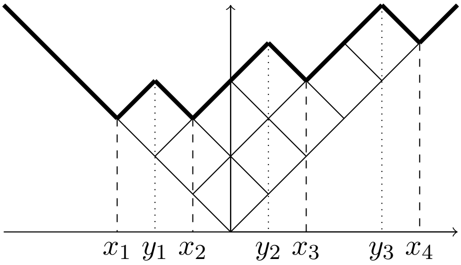


$$\downarrow$$

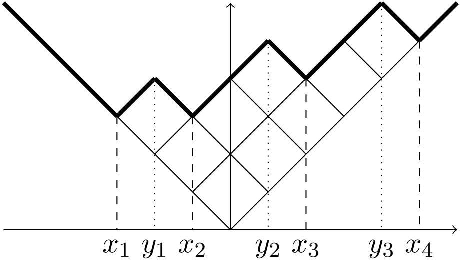



x_1 y_1 x_2 y_2 x_3 y_3 x_4



$$\mu_\lambda = \sum_{k=1}^{m} \alpha_k \delta_{x_k}$$

$$\alpha_k = \frac{\prod_{i=1}^{m-1}(x_k - y_i)}{\prod_{i\neq k}(x_k - x_i)}$$



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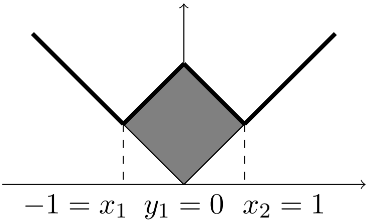

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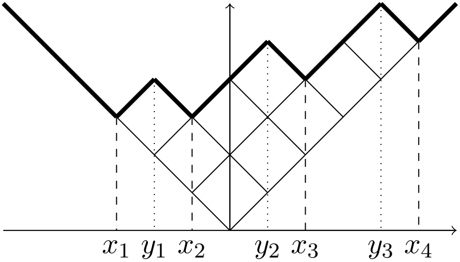

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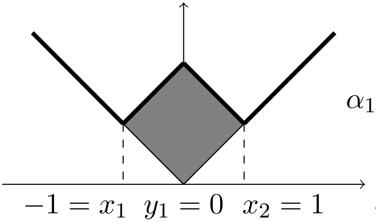

$$\alpha_1 = \frac{x_1 - y_1}{x_1 - x_2} = \frac{1}{2}, \quad \alpha_2 = \frac{x_2 - y_1}{x_2 - x_1} = \frac{1}{2}.$$

$$\mu_\diamond = \mu_{(1)} = \mu_{(2,2)} = \cdots = \frac{1}{2}(\delta_{-1} + \delta_{+1}).$$

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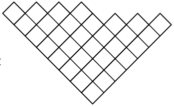
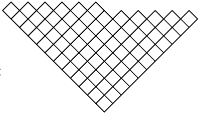
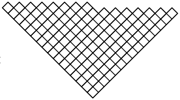
Measure Associated to limiting Young Shape $\blacktriangledown$

$$\lambda_m = \left(m-1, m-2, \ldots, \frac{m}{2}+2, \frac{m}{2}+1, \frac{m}{2}, \frac{m}{2}, \frac{m}{2}-1, \frac{m}{2}-2, \ldots, 4, 3, 2, 1\right)$$

$\lambda_8 =$  $\lambda_{12} =$  $\lambda_{16} =$ 

For $m \to \infty$ this converges to our triangular limit shape, thus we can assign to the latter continuous Young diagram the measure

$$\mu_{\blacktriangledown} = \lim_{m\to\infty} \mu_{\lambda_m}.$$

Measure Associated to limiting Young Shape $\triangledown$

The interlacing sequences for these λ_m are given by

$$\begin{array}{ccccccccccccc} -2 & & -2+\frac{4}{m} & & \dots & -\frac{4}{m} & & \frac{2}{m} & \dots & 2-\frac{6}{m} & & 2-\frac{2}{m} \\ & -2+\frac{2}{m} & & -2+\frac{6}{m} & \dots & & -\frac{2}{m} & \frac{4}{m} & \dots & & 2-\frac{4}{m} & \end{array}$$

It is quite straightforward to check that the corresponding measure μ_{λ_m} converges, for $m \to \infty$, to the arcsine distribution on $[-2, 2]$, which has the density

$$\mu_{\triangledown} = \begin{cases} \frac{1}{\pi\sqrt{4-t^2}}, & |t| \leq 2 \\ 0, & |t| > 2. \end{cases}$$

Free Convolution

We say

$$\mu_{\blacktriangledown} = \mu_{\blacklozenge} \boxplus \mu_{\blacklozenge}$$

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$$\mu_{\blacktriangledown} = \mu_{\blacklozenge} \boxplus \mu_{\blacklozenge}$$

Can we understand (and calculate) this?

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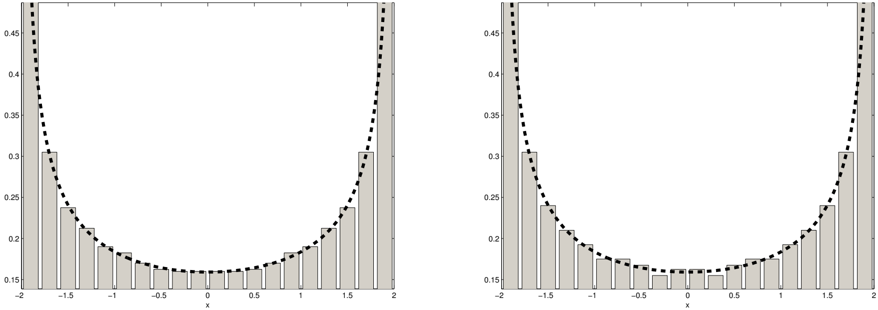
Another Instance of $\mu_{\blacktriangledown} = \mu_{\blacklozenge} \boxplus \mu_{\blacklozenge}$



Histogram of the n eigenvalues of $X + Y$ for X diagonal and two random choices of the eigenspace of Y; both X and Y have eigenvalue distribution $\mu_{\blacklozenge}$; the eigenvalue histogram of $X + Y$ is in both cases compared to $\mu_{\blacktriangledown} = \mu_{\blacklozenge} \boxplus \mu_{\blacklozenge}$; $n = 2000$

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How to Capture Information About $\mu_\diamond$

X has eigenvalue distribution $\mu_\diamond = \frac{1}{2}(\delta_{-1} + \delta_{+1})$ can be described by

- $X^2 = 1$
- X has moments

$$\tau(X^{\text{even}}) = \tau(1) = 1, \qquad \tau(X^{\text{odd}}) = \tau(X) = 0$$

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- Embedd G_1 and G_2 in free product $G := G_1 \star G_2$
- Hence both x and y, and thus also $x + y$ live in $\mathbb{C}G$

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- Canonical extension of this is

$$\tau\Big(\sum_{g \in G} \alpha_g g\Big) = \alpha_1$$

Transfering Freeness from G to $\mathbb{C}G$

$G = G_1 * G_2$ free product of groups

$\downarrow$

$\mathbb{C}G = \mathbb{C}G_1 * \mathbb{C}G_2$ free product of algebras

G_1, G_2 are free in G (as subgroups) means:

$$g_i \in G_{j(i)}$$

$$g_i \neq e \quad \forall i$$

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Reformulation in Terms of the Trace

We can reformulate the freeness of the subgroups also in terms of τ:

$$\left.\begin{array}{c} g_i \in G_{j(i)} \\ \\ \tau(g_i) = 0 \quad \forall i \\ \\ j(1) \neq \cdots \neq j(k) \end{array}\right\} \implies \tau(g_1 \cdots g_k) = 0$$

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This characterisation goes over to finite (as well as to infinite) sums.

This motivated Voiculescu to make the following definition.

The Fundamental Notion: Freeness

Definition (Voiculescu 1985)

Let $\mathcal{A}$ be a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a unital linear functional.
Subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{A}$ are **free (w.r.t. φ)**, if:

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- Freeness is a special structure of the mixed moments in elements from $\mathcal{A}_1, \ldots, \mathcal{A}_n$.
- **This structure should be seen and investigated in analogy to the classical concept of "independence".**

Section 2

Freeness

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Some History

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1994 Speicher develops combinatorial theory of freeness, based on "free cumulants"

1998 Biane describes operations on representations asymptotically by free probability

later ... many new results on operator algebras, eigenvalue distribution of random matrices, and much more ....

Definition (Voiculescu 1985)

Let $(\mathcal{A}, \varphi)$ be a **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$).

Example (Commutative Probability Space)

For a classical probability space (Ω, P) take

- $\mathcal{A} = L^\infty(\Omega, P)$
- $\varphi(x) = \int_\Omega x(\omega) dP(\omega)$ for $x \in \mathcal{A}$

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Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever

- $a_i \in \mathcal{A}_{j(i)} \qquad j(i) \in I \quad \forall i$
- $j(1) \neq j(2) \neq \cdots \neq j(n)$
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Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever

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- $j(1) \neq j(2) \neq \cdots \neq j(n)$
- $\varphi(a_i) = 0 \quad \forall i$

Random variables $x_1, \dots, x_n \in \mathcal{A}$ are freely independent, if their generated unital subalgebras $\mathcal{A}_i :=$ algebra$(1, x_i)$ are so.

What is Freeness?

Freeness between x and y is an infinite set of equations relating various moments in x and y:

$$\varphi\Big(p_1(x)q_1(y)p_2(x)q_2(y)\cdots\Big) = 0$$

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Basic observation: freeness between x and y is actually a **rule for calculating mixed moments** in x and y from the moments of x and the moments of y:

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Example

If x and y are free, then we have

$$\varphi(x^m y^n) = \varphi(x^m)\cdot\varphi(y^n)$$

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which results in

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If x and y are free, then we have

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Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables. This is the reason that it is also called "free independence".

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Free independence is a rule for calculating mixed moments, analogous to the concept of independence for random variables.
Note: free independence is a different rule from classical independence; free independence occurs typically for non-commuting random variables, like operators on Hilbert spaces or (random) matrices.

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Where Does Freeness Show Up?

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$

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- asymptotically for operations on representations of "large groups"

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Section 3

How can we calculate the free convolution

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Definition

For any probability measure μ on $\mathbb{R}$ we define its *Cauchy transform* G by

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Theorem (Voiculescu, 1986)

For a compactly supported probability measure we define its $\mathcal{R}$-transform $\mathcal{R}_\mu$ as the analytic function in some disk about 0, which is uniquely determined by

$$G_\mu[1/z + \mathcal{R}(z)] = z,$$

where G_μ is the Cauchy transform of μ.
Then we have the additivity of the $\mathcal{R}$-transform under free convolution, i.e., for compactly supported probability measures μ and ν, we have

$$\mathcal{R}_{\mu \boxplus \nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z).$$

Example (Calculation of $\mu_\diamond \boxplus \mu_\diamond$ via $\mathcal{R}$-transform)

We set $\mu := \mu_\diamond = 1/2(\delta_{-1} + \delta_{+1})$ and $\nu := \mu \boxplus \mu$.

Then $$G_\mu(z) = \int \frac{1}{z-t} d\mu(t) = \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right) = \frac{z}{z^2-1}.$$

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Hence $$z = \frac{\mathcal{R}_\mu(z) + 1/z}{(\mathcal{R}_\mu(z) + 1/z)^2 - 1},$$ thus $$\mathcal{R}_\mu(z) = \frac{\sqrt{1+4z^2} - 1}{2z}$$

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$$\text{So} \quad \mathcal{R}_\nu(z) = 2\mathcal{R}_\mu(z) = \frac{\sqrt{1+4z^2} - 1}{z}, \qquad G_\nu(z) = \frac{1}{\sqrt{z^2-4}}$$

$$\text{Thus} \quad d\nu(t) = -\frac{1}{\pi} \Im \frac{1}{\sqrt{t^2-4}} dt = \begin{cases} \frac{1}{\pi\sqrt{4-t^2}}, & |t| \le 2 \\ 0, & \text{otherwise} \end{cases},$$

i.e., we have indeed that $\mu_\diamond \boxplus \mu_\diamond = \nu$ is the arcsine distribution $\mu_\blacktriangledown$.