The Unreasonable Effectiveness of Mathematics

Roland Speicher

Queen's University, Kingston
and
Universität des Saarlandes
Saarbrücken, Germany

Eugene Wigner, Nobel Laureate in Physics

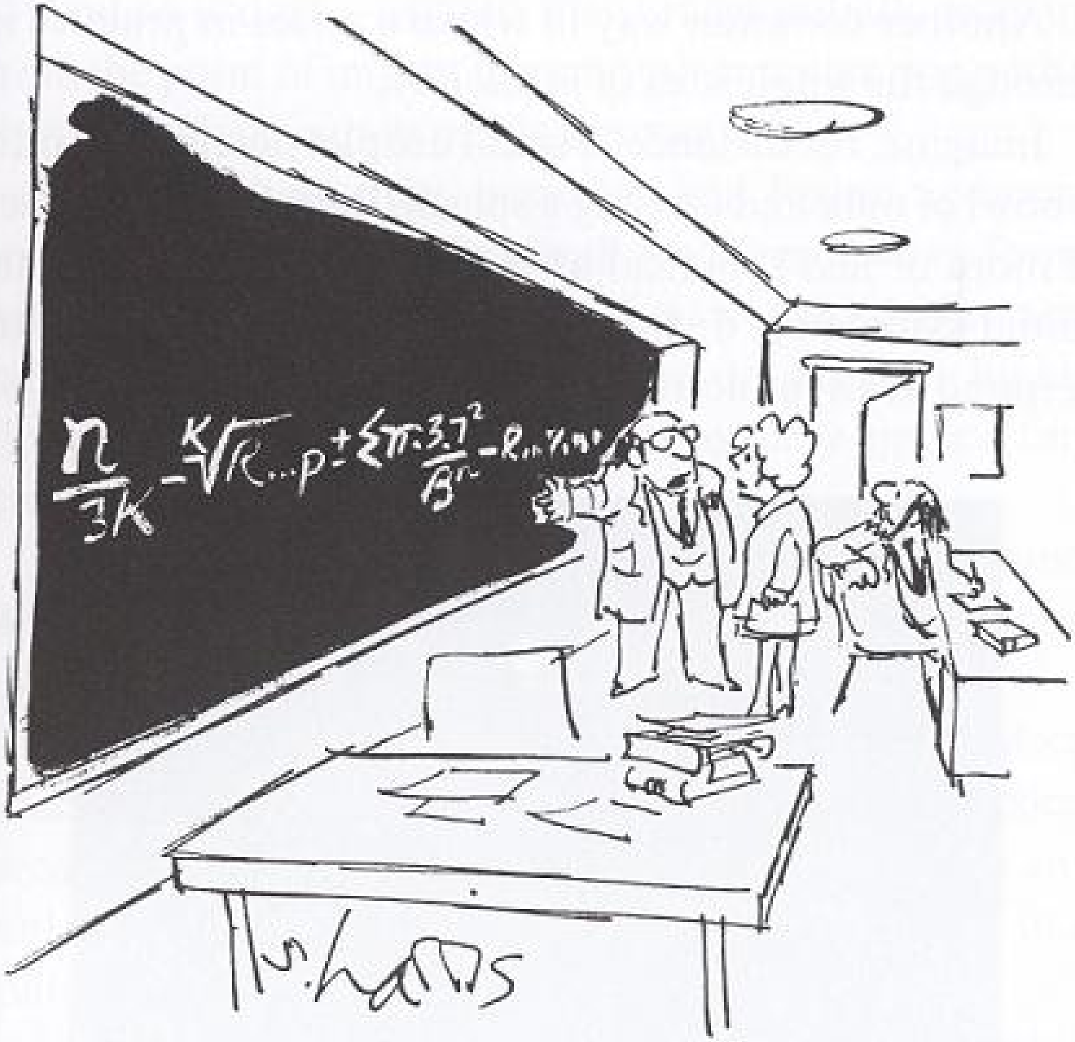
The Unreasonable Effectiveness of Mathematics in the Natural Sciences, 1960

”The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and there is no rational explanation for it.“

Aim of the Talk

- show two examples how pure mathematics can have very unexpected and powerful applications, within and outside of mathematics

- show along the way
  - what mathematics is about
  - what drives mathematicians and mathematics forward



"BUT THIS IS THE SIMPLIFIED VERSION FOR THE GENERAL PUBLIC."

What is Mathematics About

"So, what do you actually do as a mathematician? Isn't more or less everything known, all numbers multiplied, all integrals calculated?"

$$
\begin{aligned}
1 \times 8 + 1 &= 9 \\
12 \times 8 + 2 &= 98 \\
123 \times 8 + 3 &= 987 \\
1234 \times 8 + 4 &= 9876 \\
12345 \times 8 + 5 &= 98765 \\
123456 \times 8 + 6 &= 987654 \\
1234567 \times 8 + 7 &= 9876543 \\
12345678 \times 8 + 8 &= 98765432 \\
123456789 \times 8 + 9 &= 987654321
\end{aligned}
$$

$$
\begin{aligned}
1 \times 9 + 2 &= 11 \\
12 \times 9 + 3 &= 111 \\
123 \times 9 + 4 &= 1111 \\
1234 \times 9 + 5 &= 11111 \\
12345 \times 9 + 6 &= 111111 \\
123456 \times 9 + 7 &= 1111111 \\
1234567 \times 9 + 8 &= 11111111 \\
12345678 \times 9 + 9 &= 111111111 \\
123456789 \times 9 + 10 &= 1111111111
\end{aligned}
$$

What is Mathematics About

"So, what do you actually do as a mathematician? Isn't more or less everything known, all numbers multiplied, all integrals calculated?"

Mathematics is not really about numbers or integrals, but about

$$
\begin{aligned}
1 \times 8 + 1 &= 9\\
12 \times 8 + 2 &= 98\\
123 \times 8 + 3 &= 987\\
1234 \times 8 + 4 &= 9876\\
12345 \times 8 + 5 &= 98765\\
123456 \times 8 + 6 &= 987654\\
1234567 \times 8 + 7 &= 9876543\\
12345678 \times 8 + 8 &= 98765432\\
123456789 \times 8 + 9 &= 987654321
\end{aligned}
$$

$$
\begin{aligned}
1 \times 9 + 2 &= 11\\
12 \times 9 + 3 &= 111\\
123 \times 9 + 4 &= 1111\\
1234 \times 9 + 5 &= 11111\\
12345 \times 9 + 6 &= 111111\\
123456 \times 9 + 7 &= 1111111\\
1234567 \times 9 + 8 &= 11111111\\
12345678 \times 9 + 9 &= 111111111\\
123456789 \times 9 + 10 &= 1111111111
\end{aligned}
$$

structure, pattern, and beauty.

The Beauty of Music

… can be appreciated by just listening to it.

The Beauty of Mathematics

506 ANDREW WILES

defined by

$$Z_p = \begin{cases} X_p^2 - a_p(g)X_p + p\chi_g(p) & \text{if } p \mid M,\ p \nmid \operatorname{level}(g) \\ X_p - a_p(g) & \text{if } p \nmid M \\ X_p - a_p(g) & \text{if } p \mid \operatorname{level}(g), \end{cases}$$

where the Euler factor of g at p is $(1 - a_p(g)p^{-s} + \chi_g(p)\,p^{1-2s})$ in the first two cases and $(1 - a_p(g)p^{-s})$ in the third case. We then have a commutative diagram

$$\begin{array}{lccc} & \mathbf{T}'_H(M) & \hookrightarrow & \prod_g \mathcal{O}_g \\ (2.27) & \downarrow & & \downarrow \\ & \mathbf{T}_H(M) & \hookrightarrow & \prod_g S_g = \prod_g \mathcal{O}_g[X_{q_1}, \dots, X_{q_r}, X_p]/\{Y_i, Z_p\}_{i=1}^r \end{array}$$

where the lower map is given on $\{U_{q_i},\ U_p \text{ or } T_p\}$ by $U_{q_i} \longrightarrow X_{q_i}$, U_p or $T_p \longrightarrow X_p$ (according as $p \mid M$ or $p \nmid M$). To verify the existence of such a homomorphism one considers the action of $\mathbf{T}_H(M)$ on the space of forms of weight 2 invariant under $\Gamma_H(M)$ and uses that $\sum_{j=1}^r g_j(m_j z)$ is a free generator as a $\mathbf{T}_H(M) \otimes \mathbf{C}$-module where $\{g_j\}$ runs over the set of newforms and $m_j = M/\operatorname{level}(g_j)$.

Now we tensor all the rings in (2.27) with $\mathbf{Z}_p$. Then completing the top row of (2.27) with respect to $\mathfrak{m}'$ and the bottom row with respect to $\mathfrak{m}$ we get a commutative diagram

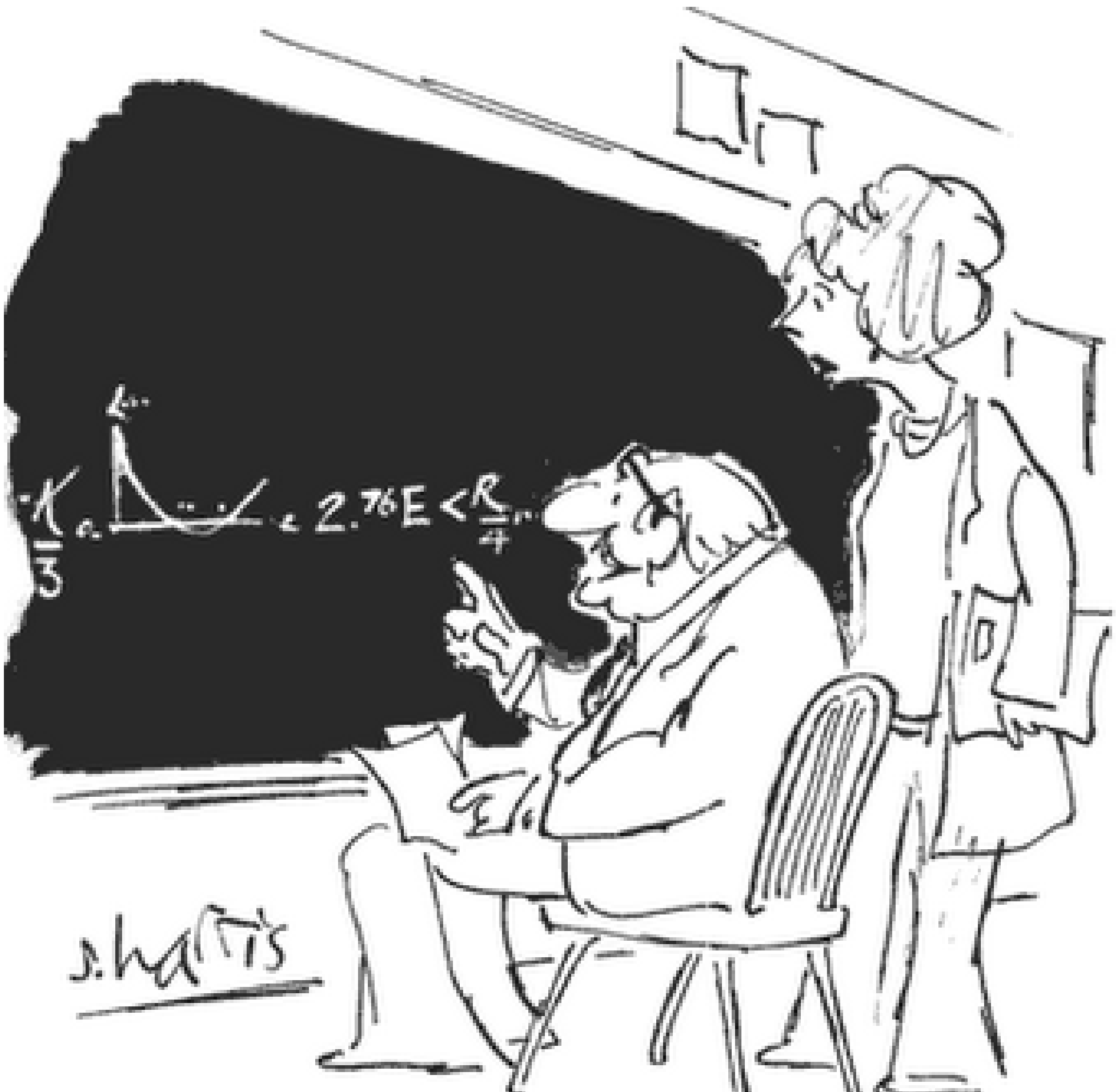
$$\begin{array}{lccccc} & \mathbf{T}'_H(M)_{\mathfrak{m}'} & \hookrightarrow & \left(\prod_g \mathcal{O}_g\right)_{\mathfrak{m}'} & \simeq & \prod_{\substack{g \\ \mathfrak{m}' \to \mu}} \mathcal{O}_{g,\mu} \\ (2.28) & \downarrow & & \downarrow & & \downarrow \\ & \mathbf{T}_H(M)_{\mathfrak{m}} & \hookrightarrow & \left(\prod_g S_g\right)_{\mathfrak{m}} & \simeq & \prod (S_g)_{\mathfrak{m}}. \end{array}$$

Here μ runs through the primes above p in each $\mathcal{O}_g$ for which $\mathfrak{m}' \to \mu$ under $\mathbf{T}_{H'}(M) \to \mathcal{O}_g$. Now $(S_g)_{\mathfrak{m}}$ is given by

$$\begin{aligned} (2.29)\quad (S_g \otimes \mathbf{Z}_p)_{\mathfrak{m}} &\simeq \left((\mathcal{O}_g \otimes \mathbf{Z}_p)[X_{q_1}, \dots, X_{q_r}, X_p]\big/\{Y_i, Z_p\}_{i=1}^r\right)_{\mathfrak{m}} \\ &\simeq \left(\prod_{\mu \mid p} \mathcal{O}_{g,\mu}[X_{q_1}, \dots, X_{q_r}, X_p]\big/\{Y_i, Z_p\}_{i=1}^r\right)_{\mathfrak{m}} \\ &\simeq \left(\prod_{\mu \mid p} A_{g,\mu}\right)_{\mathfrak{m}} \end{aligned}$$

where $A_{g,\mu}$ denotes the product of the factors of the complete semi-local ring $\mathcal{O}_{g,\mu}[X_{q_1}, \dots, X_{q_r}, X_p]\big/\{Y_i, Z_p\}_{i=1}^r$ in which X_{q_i} is topologically nilpotent for

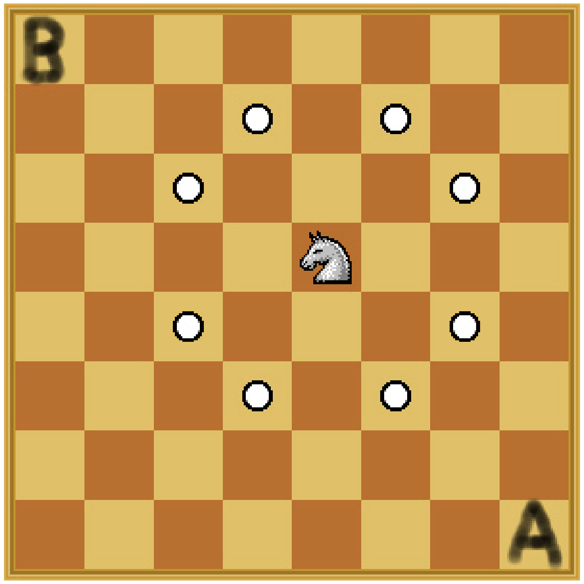
... is harder to appreciate.



"The beauty of this is that it is only of theoretical importance, and there is no way it can be of any practical use whatsoever."

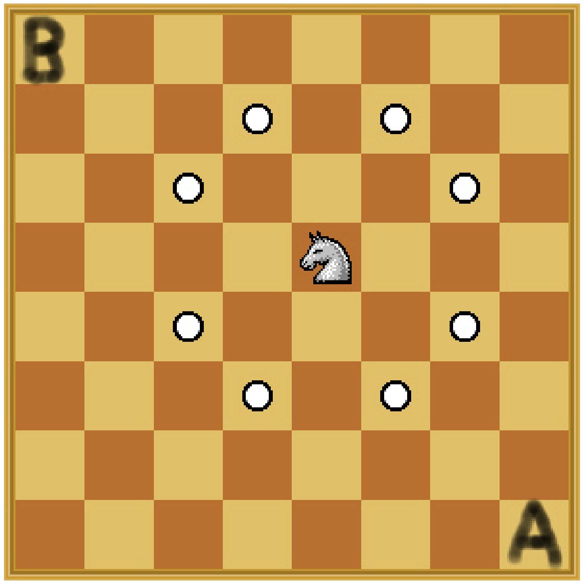
How Patterns Can Help to Understand

Is there a knight's tour, starting in A, visiting every square exactly once and ending in B?



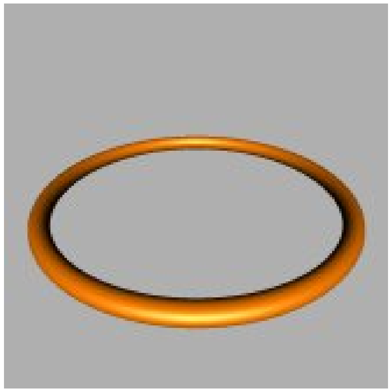
How Patterns Can Help to Understand

Is there a knight's tour, starting in A, visiting every square exactly once and ending in B?

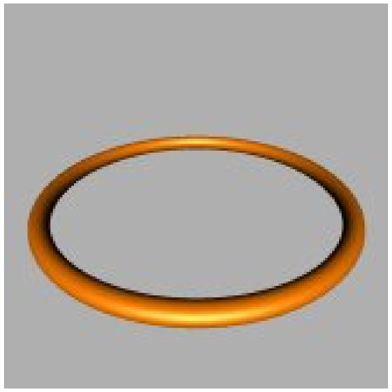


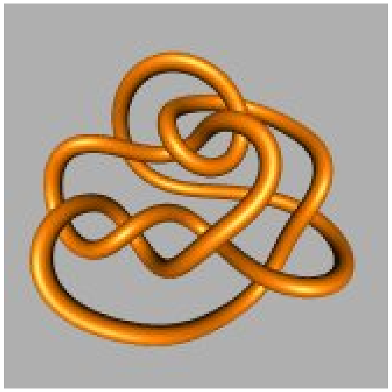
Answer: No, because in each step we have to change the color and in total we have to visit an even number of squares.

One of Our Most Beloved Questions:

This is the unknot: 

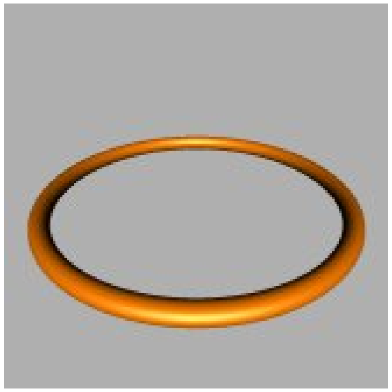
One of Our Most Beloved Questions:

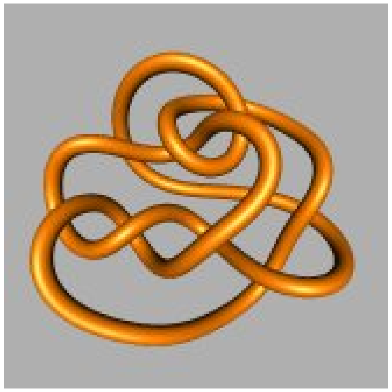
This is the unknot: 

How about this one .... 

One of Our Most Beloved Questions:

Are They the Same?

This is the unknot: 

How about this one .... 

..or that one ...

Figure 3.5. Wolfgang Haken's "Gordian knot."

... indeed ...

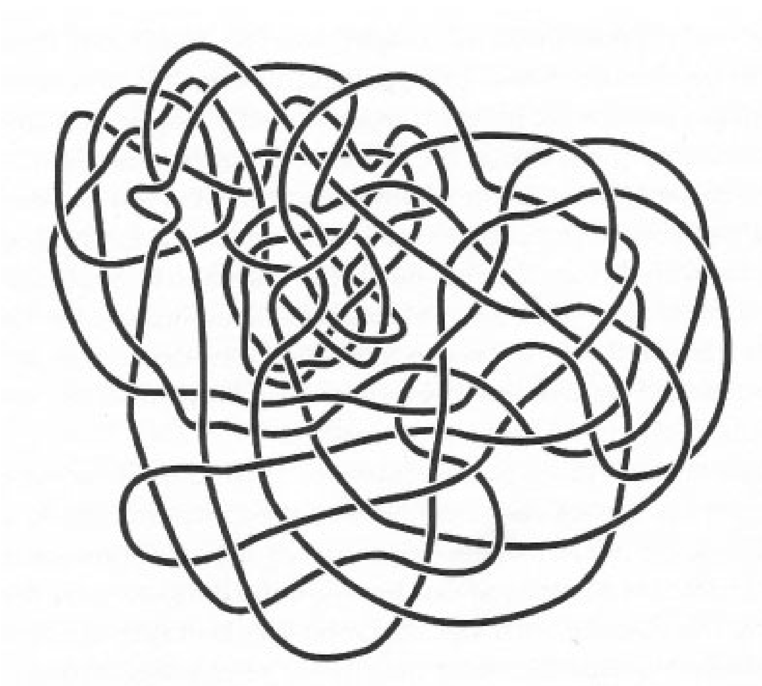
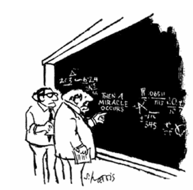
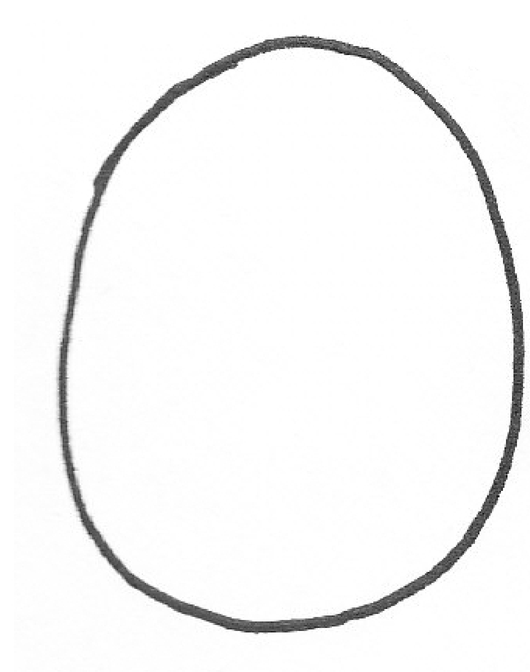
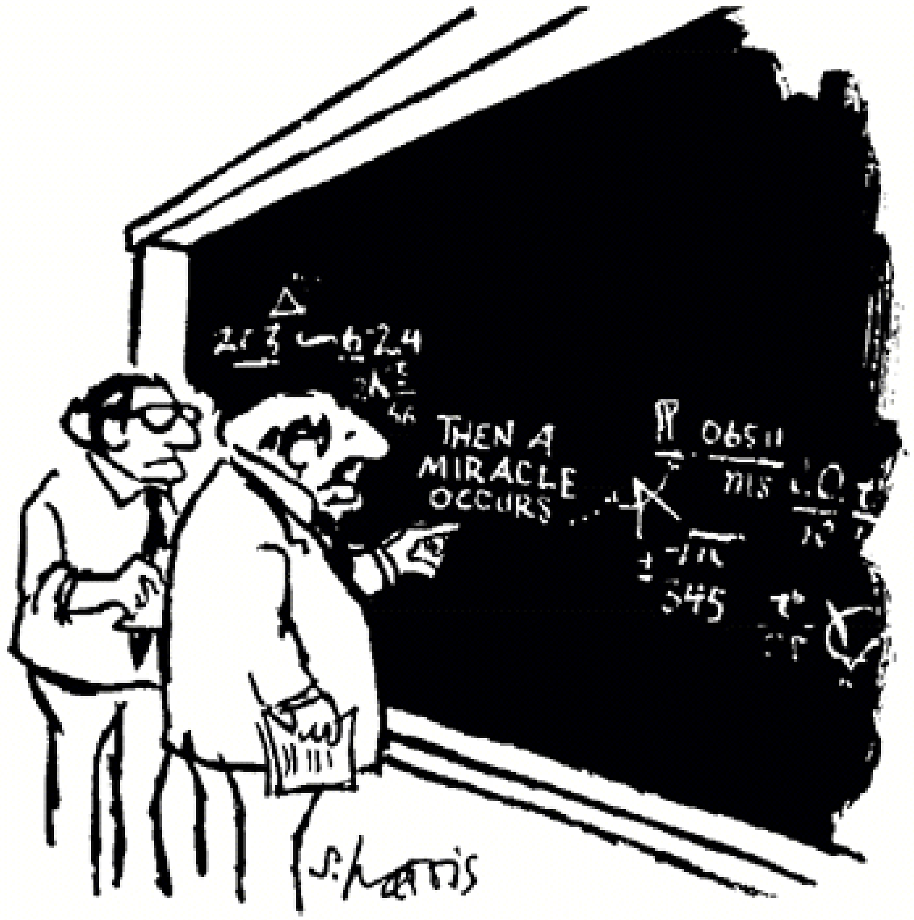


Figure 3.5. Wolfgang Haken's "Gordian knot."







"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Are the Two Trefoils the Same?

Today's Special Guest: von Neumann Algebras

von Neumann  algebras

- were introduced by von Neumann in 1929

- are quite non-trivial versions of $\infty \times \infty$ matrices

- are used to describe geometries with non-integer dimensions

- provide a mathematical description of quantum mechanics

Vaughan Jones  (Berkeley) compares von Neumann algebras with the deepest thing he can think of, the Pacific Ocean.

Why do we care about them?

Vaughan Jones  (Berkeley) compares von Neumann algebras with the deepest thing he can think of, the Pacific Ocean.

Why do we care about them?

Because we find them beautiful and we want to understand them!

Typical Questions About von Neumann Algebras

- **The same or not - the question about isomorphism**

  Two von Neumann algebas might look quite different on first look, but often they are just the same in disguise - how can we find this out?

  Here are 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  and here they are again, disguised as 4×4 matrices

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

Typical Questions About von Neumann Algebras

- **Building bigger from smaller objects**

  Often small von Neumann algebras are sitting inside bigger ones - can we understand the possibilities?

  Here, the (disguised) 2×2 matrices are sitting inside the 4×4 matrices

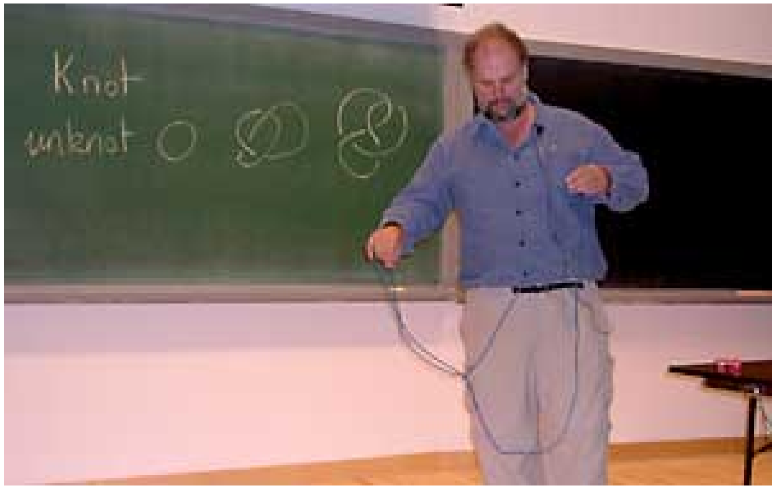
$$\left\{\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}\right\} \subset \left\{\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \epsilon & \zeta & \eta & \theta \\ \iota & \kappa & \lambda & \mu \\ \nu & \xi & o & \pi \end{pmatrix}\right\}$$

How Can Small von Neumann Algebras Sit inside Bigger Ones?

How Can Small von Neumann Algebras Sit inside Bigger Ones?

In 1984, Vaughan Jones discovered that

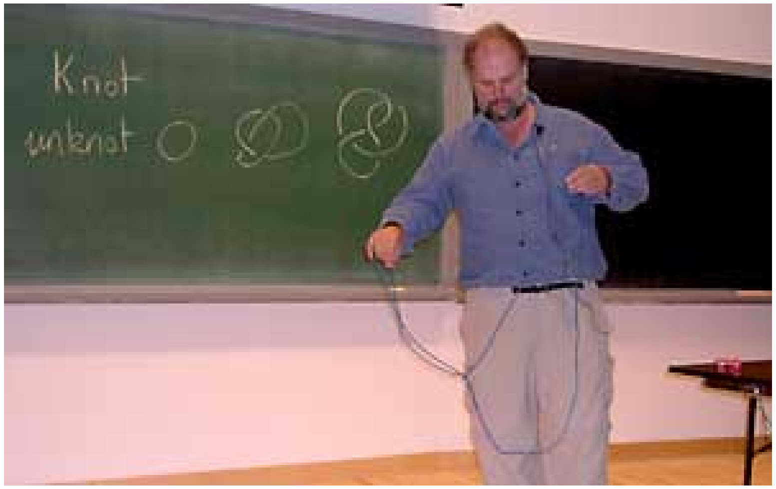
the small ones are knotted in the bigger ones



How Can Small von Neumann Algebras Sit inside Bigger Ones?

In 1984, Vaughan Jones discovered that

the small ones are knotted in the bigger ones



and thus, the other way round, our understanding of von Neumann algebras gives, quite unexpectedly, new results on knots!

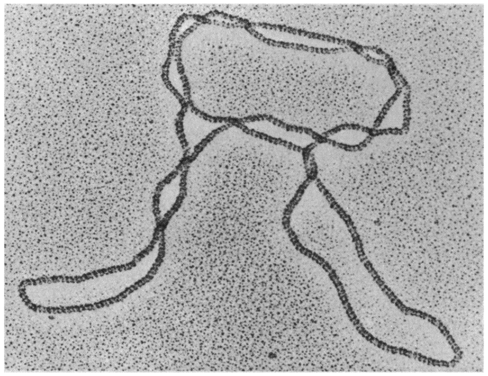
Are the Two Trefoils the Same?

Are the Two Trefoils the Same?

Easy to decide with Jones polynomial: No!

Unreasonable Usefulness in Studying the Chemical Processes of Life

”This DNA from the bacterial virus lambda is knotted and coded. Before it can replicate, it must straighten itselfout. Using new mathematics results, biologists can determine how knots and chain-like links of DNA are made and unmade.”



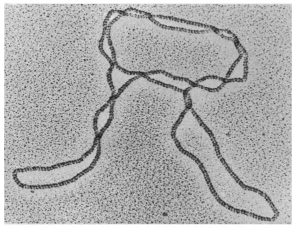
Solving Knotty Problems in Math and Biology, News Report, Science 1986

Unreasonable?

$$\left\{\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}\right\} \subset \left\{\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \epsilon & \zeta & \eta & \theta \\ \iota & \kappa & \lambda & \mu \\ \nu & \xi & o & \pi \end{pmatrix}\right\}$$

$$\Downarrow$$

$$\Downarrow$$



Are Two (Different Looking) von Neumann Algebras the Same?

Starting in the 1980's, Dan Voiculescu (Berkeley) developed a new tool to answer such questions

Free Probability Theory

and in 1991 he observed that some von Neumann algebras can be described by

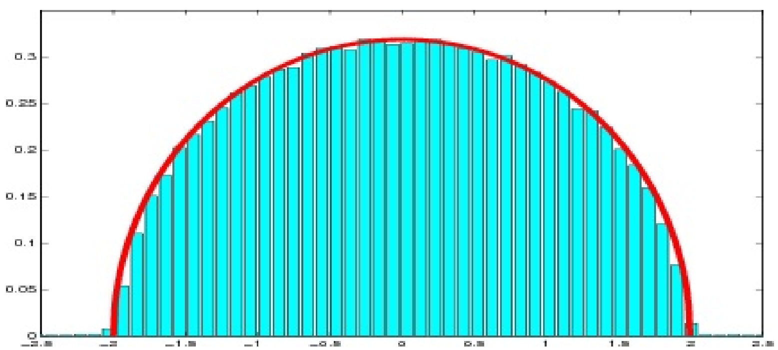
Random Matrices

Random Matrices

Consider large random matrix

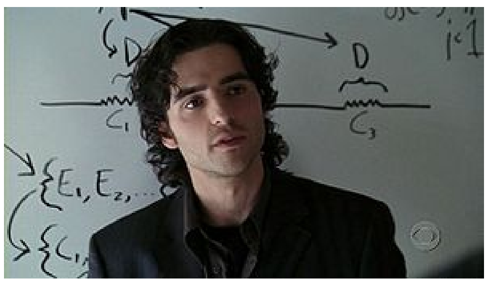
$$\begin{pmatrix} -1 & -1 & +1 & +1 & -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & +1 & +1 & +1 & +1 & -1 \\ +1 & +1 & +1 & +1 & +1 & -1 & -1 & +1 \\ +1 & +1 & +1 & +1 & -1 & -1 & +1 & -1 \\ -1 & +1 & +1 & -1 & +1 & +1 & -1 & -1 \\ +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\ -1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 \\ +1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 \end{pmatrix}$$

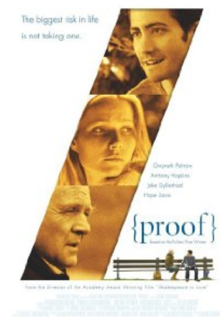
Eigenvalues converge to



Random Matrices are Fashionable

Random matrix theory is cited as one of the ”modern tools” used in Catherine’s proof of an important result in prime number theory in the 2005 film Proof.





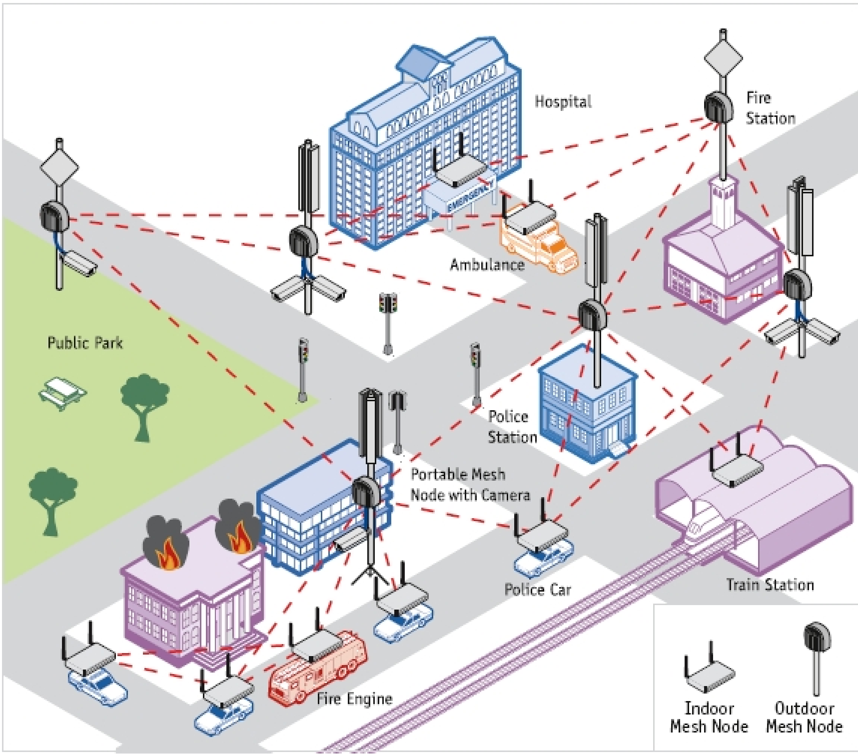
Pop-mathematician Charlie Eppes, main character of the contemporary US TV series Numb3rs, is portrayed as having published his first paper in the American Journal of Mathematics at the age of 14 with the title “Asymptotics of Hermitian Random Matrices”.

Unexpected Relation between Free Probability and Random Matrices

- realizing von Neumann algebras as random matrices solved some longstanding problems about von Neumann algebras

- but also the other way round: results from free probability give new techniques for calculating eigenvalue distributions of random matrices

Unreasonable Usefulness in Studying Wireless Communication Networks



Unreasonable Usefulness in Studying Wireless Communication Networks

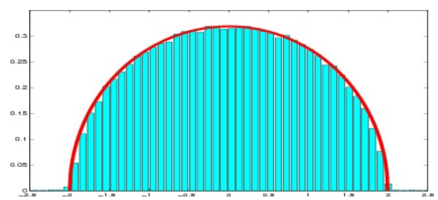
"There are often curious and deep connections between engineering and mathematics ... **Using free probability as a new tool, we are in turn able to analyze more complex models for the original wireless communications problem.**"

David Tse, 1999

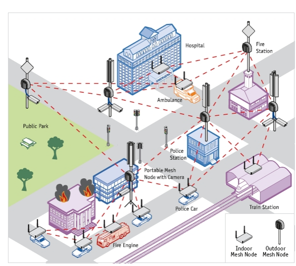
Unreasonable?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

$$\Downarrow$$



$$\Downarrow$$



Eugene Wigner, Nobel Laureate in Physics

The Unreasonable Effectiveness of Mathematics in the Natural Sciences, 1960

”The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research...”

In this sense I am looking forward to more examples of unreasonable – unexpected and unpredictable – consequences and applications of our quest to reveal the beauty of mathematical structures.

Thank You!