Free Probability Theory and Non-crossing Partitions

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Freeness

Definition [Voiculescu 1985]: Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, i.e. $\mathcal{A}$ is unital algebra, and $\varphi : \mathcal{A} \to \mathbb{C}$ is linear functional with $\varphi(1) = 1$.
Unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_r \in \mathcal{A}$ are called **free** if we have

$$\varphi(a_1 \cdots a_k) = 0$$

whenever

- $a_j \in \mathcal{A}_{i(j)}$ for all $j = 1, \dots, k$
- $\varphi(a_j) = 0$ for all $j = 1, \dots, k$
- $i(1) \neq i(2), i(2) \neq i(3), \dots, i(k-1) \neq i(k)$

Freeness of random variables

Elements ('random variables') a and b in $\mathcal{A}$ are called **free** if their generated unital subalgebras are free, i.e.

$$\varphi\Big(p_1(a)q_1(b)p_2(a)q_2(b)\cdots\Big) = 0$$

$$\varphi\Big(q_1(b)p_1(a)q_2(b)p_2(a)\cdots\Big) = 0$$

whenever

- p_i, q_j are polynomials
- $\varphi\big(p_i(a)\big) = 0 = \varphi\big(q_j(b)\big)$ for all i, j

Examples

Canonical examples for free random variables appear in the context of

- **operator algebras:**
  creation and annihilation operators on full Fock spaces
  von Neumann algebras of free groups

- **random matrices**

Operators on full Fock spaces

For a Hilbert space $\mathcal{H}$ we define **full Fock space**

$$\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$$

Operators on full Fock spaces

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For each $g \in \mathcal{H}$ we have corresponding **creation operator** $l(g)$ and **annihilation operator** $l^*(g)$

$$l(g) h_1 \otimes \cdots \otimes h_n = g \otimes h_1 \otimes \cdots \otimes h_n$$

and

$$l^*(g)\Omega = 0$$

$$l^*(g) h_1 \otimes \cdots \otimes h_n = \langle h_1, g \rangle h_2 \otimes \cdots \otimes h_n$$

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We have **vacuum expectation**

$$\varphi(a) = \langle \Omega, a\Omega\rangle$$

Freeness for operators on full Fock spaces

If $\mathcal{H}_1, \dots, \mathcal{H}_r$ are **orthogonal** sub-Hilbert spaces in $\mathcal{H}$ and

$$\mathcal{A} := B(\mathcal{F}(\mathcal{H})), \qquad \varphi(\cdot) := \langle \Omega, \cdot \Omega \rangle$$

$$\mathcal{A}_i := \text{unital } *\text{-algebra generated by } l(f) \; (f \in \mathcal{H}_i)$$

then $\mathcal{A}_1, \dots, \mathcal{A}_r$ are **free** in $(\mathcal{A}, \varphi)$.

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then $\mathcal{A}_1, \dots, \mathcal{A}_r$ are **free** in $(\mathcal{A}, \varphi)$.

Reason: If $a_j \in \mathcal{A}_{i(j)}$ with $i(1) \neq i(2) \neq \cdots \neq i(k)$ and $\varphi(a_j) = 0$ for all j then

$$a_1 a_2 \cdots a_k \Omega = a_1 \Omega \otimes a_2 \Omega \otimes \cdots \otimes a_k \Omega$$

Gaussian random matrices

$$A_N = (a_{ij})_{i,j=1}^N : \Omega \to M_N(\mathbb{C})$$

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- $a_{ij} = \bar{a}_{ji}$ (i.e. $A_N = A_N^*$)

- $\{a_{ij}\}_{1 \leq i \leq j \leq N}$ are independent Gaussian random variables with

$$E[a_{ij}] = 0$$
$$E[|a_{ij}|^2] = \frac{1}{N}$$

Freeness for Gaussian random matrices

If A_N and B_N are **independent** Gaussian random matrices, i.e.,

- A_N is Gaussian random matrix, and B_N is Gaussian random matrix

- entries of A_N are independent from entries of B_N

Then A_N and B_N are **asymptotically free**

Asymptotic freeness

asymptotic freeness $\hat{=}$ freeness relations hold in the large N-limit

$$\lim_{N\to\infty} \varphi\Big(p_1(A_N)q_1(B_N)p_2(A_N)q_2(B_N)\cdots\Big) = 0$$

whenever

- p_i, q_j polynomials
- $\lim_{N\to\infty} \varphi\big(p_i(A_N)\big) = 0 = \lim_{N\to\infty} \varphi\big(q_j(A_N)\big)$ for all i, j

What is state φ for random matrices

$$\lim_{N\to\infty} \varphi\Big(p_1(A_N)q_1(B_N)p_2(A_N)q_2(B_N)\cdots\Big) = 0$$

Two possibilities:

- (averaged) asymptotic freeness:

$$\varphi = E \circ \text{tr}$$

- almost sure asymptotic freeness:

$\varphi = \text{tr}$, and lim-equations hold almost surely

What is Freeness?

Freeness between a and b is an infinite set of equations relating various moments in a and b:

$$\varphi\Big(p_1(a)q_1(b)p_2(a)q_2(b)\cdots\Big) = 0$$

What is Freeness?

Freeness between a and b is an infinite set of equations relating various moments in a and b:

$$\varphi\Big(p_1(a)q_1(b)p_2(a)q_2(b)\cdots\Big) = 0$$

Basic observation: freeness between a and b is actually a **rule for calculating mixed moments** in a and b from the moments of a and the moments of b:

$$\varphi\Big(a^{n_1}b^{m_1}a^{n_2}b^{m_2}\cdots\Big) = \text{polynomial}\Big(\varphi(a^i), \varphi(b^j)\Big)$$

Example:

$$\varphi\Big(\big(a^n - \varphi(a^n)1\big)\big(b^m - \varphi(b^m)1\big)\Big) = 0,$$

thus

$$\varphi(a^n b^m) - \varphi(a^n \cdot 1)\varphi(b^m) - \varphi(a^n)\varphi(1 \cdot b^m) + \varphi(a^n)\varphi(b^m)\varphi(1 \cdot 1) = 0,$$

and hence

$$\boldsymbol{\varphi(a^n b^m) = \varphi(a^n) \cdot \varphi(b^m)}$$

Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables.

Thus freeness is also called **free independence**

Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables.

Note: free independence is a different rule from classical independence; free independence occurs typically for **non-commuting random variables**, like operators on Hilbert spaces or (random) matrices

Example:

$$\varphi\Big(\big(a-\varphi(a)1\big)\cdot\big(b-\varphi(b)1\big)\cdot\big(a-\varphi(a)1\big)\cdot\big(b-\varphi(b)1\big)\Big)=0,$$

which results in

$$\boldsymbol{\varphi(abab)=\varphi(aa)\cdot\varphi(b)\cdot\varphi(b)+\varphi(a)\cdot\varphi(a)\cdot\varphi(bb)} \\ \boldsymbol{-\varphi(a)\cdot\varphi(b)\cdot\varphi(a)\cdot\varphi(b)}$$

Consider independent Gaussian random matrices A_N and B_N. Then one has

$$\lim_{N\to\infty} \varphi(A_N) = 0, \qquad \lim_{N\to\infty} \varphi(B_N) = 0$$

$$\lim_{N\to\infty} \varphi(A_N^2) = 1, \qquad \lim_{N\to\infty} \varphi(B_N^2) = 1$$

$$\lim_{N\to\infty} \varphi(A_N^3) = 0, \qquad \lim_{N\to\infty} \varphi(B_N^3) = 0$$

$$\lim_{N\to\infty} \varphi(A_N^4) = 2, \qquad \lim_{N\to\infty} \varphi(B_N^4) = 2$$

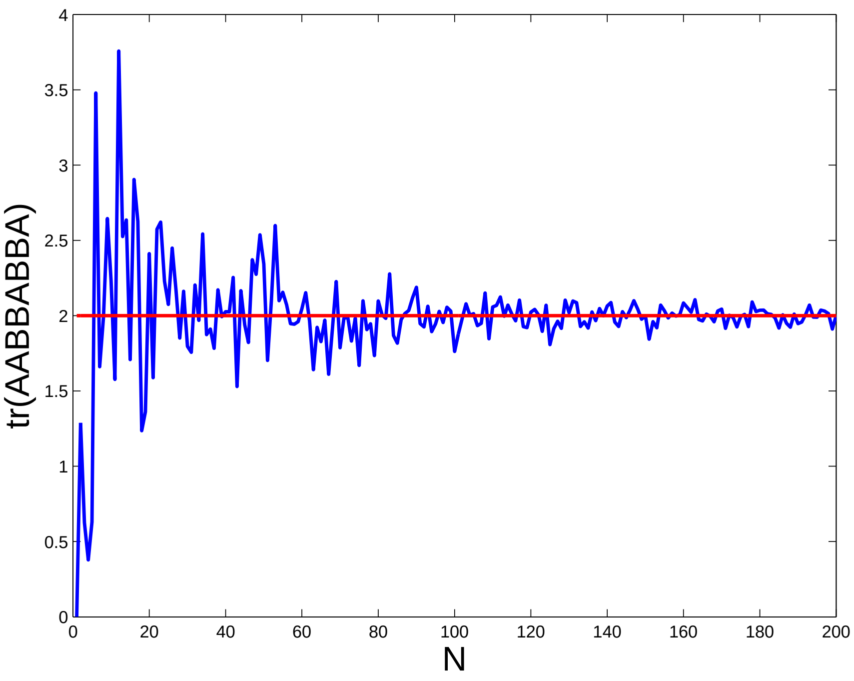
Asymptotic freeness between A_N and B_N implies then for example:

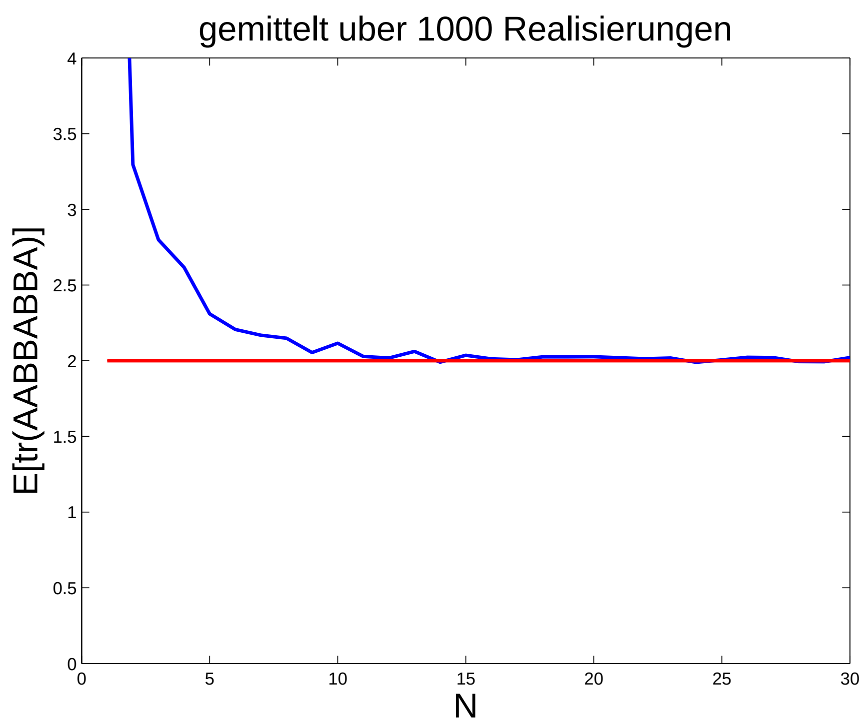
$$\lim_{N\to\infty} \varphi(A_N A_N B_N B_N A_N B_N B_N A_N) = 2$$

$$\text{tr}(A_N A_N B_N B_N A_N B_N B_N A_N)$$

one realization

averaged over 1000 realizations





Understanding the freeness rule: the idea of cumulants

- write moments in terms of other quantities, which we call **free cumulants**

- freeness is much easier to describe on the level of free cumulants: **vanishing of mixed cumulants**

- relation between moments and cumulants is given by summing over **non-crossing or planar partitions**

Example: independent Gaussian random matrices

Consider two independent Gaussian random matrices A and B

Then, in the limit $N \to \infty$, the moments

$$\varphi\left(A^{n_1} B^{m_1} A^{n_2} B^{m_2} \cdots\right)$$

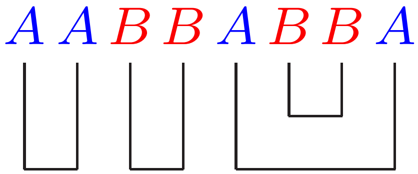
are given by

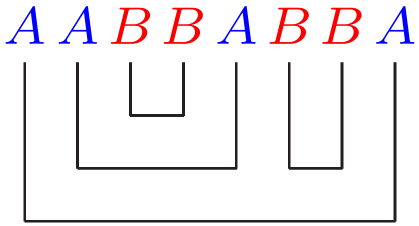
$$\#\Big\{\text{non-crossing/planar pairings of pattern}$$

$$\underbrace{A \cdot A \cdots A}_{n_1\text{-times}} \cdot \underbrace{B \cdot B \cdots B}_{m_1\text{-times}} \cdot \underbrace{A \cdot A \cdots A}_{n_2\text{-times}} \cdot \underbrace{B \cdot B \cdots B}_{m_2\text{-times}} \cdots,$$

$$\text{which do not pair } A \text{ with } B\Big\}$$

Example: $\varphi(AABBABBA) = 2$





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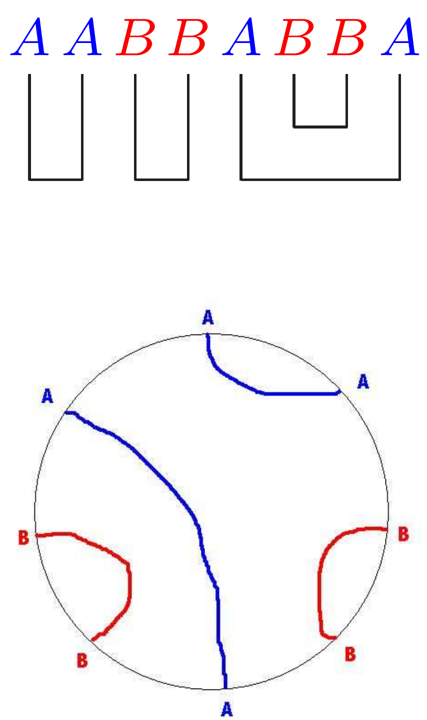
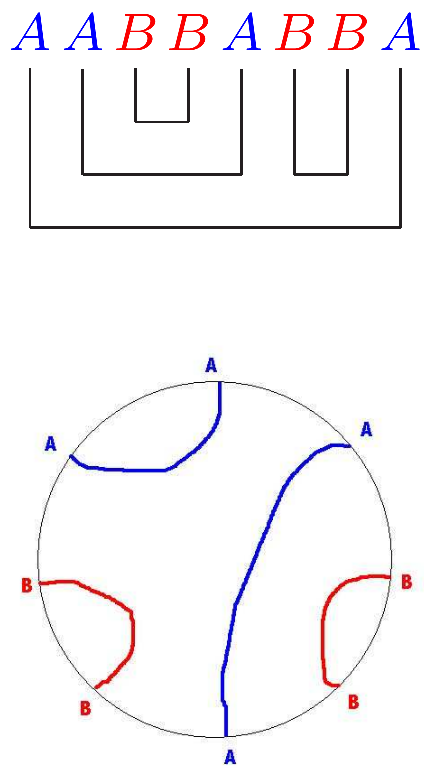




Moments and cumulants

For

$$\varphi : \mathcal{A} \to \mathbb{C}$$

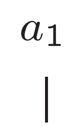
we define **cumulant functionals** κ_n (for all $n \geq 1$)

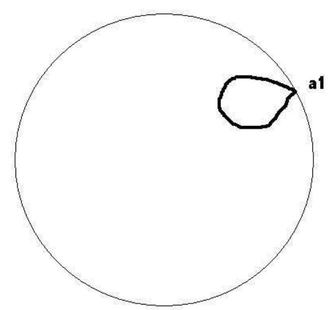
$$\kappa_n : \mathcal{A}^n \to \mathbb{C}$$

by moment-cumulant relation

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi[a_1, \dots, a_n]$$

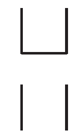
$$\varphi(a_1) = \kappa_1(a_1)$$

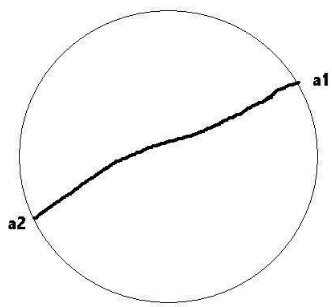


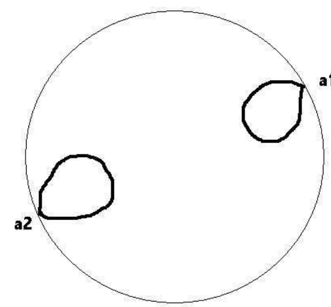


$$\varphi(a_1 a_2) = \kappa_2(a_1, a_2) + \kappa_1(a_1)\kappa_1(a_2)$$

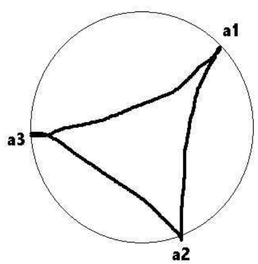
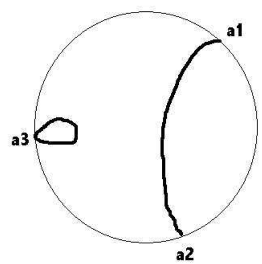
$a_1 a_2$







$$
\begin{aligned}
\varphi(a_1a_2a_3) = {} & \kappa_3(a_1, a_2, a_3) \\
& + \kappa_1(a_1)\kappa_2(a_2, a_3) \\
& + \kappa_2(a_1, a_2)\kappa_1(a_3) \\
& + \kappa_2(a_1, a_3)\kappa_1(a_2) \\
& + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3)
\end{aligned}
$$


a1
a3
a2

a1
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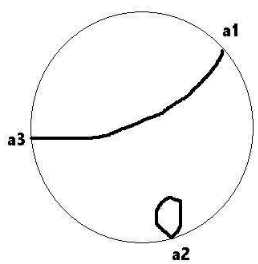
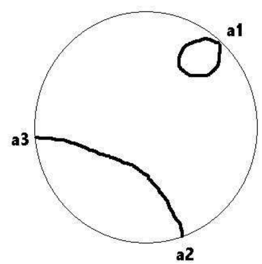

a1
a3
a2

a1
a3
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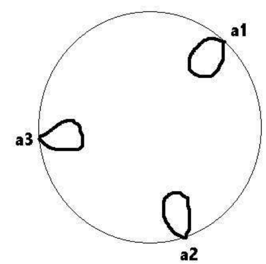
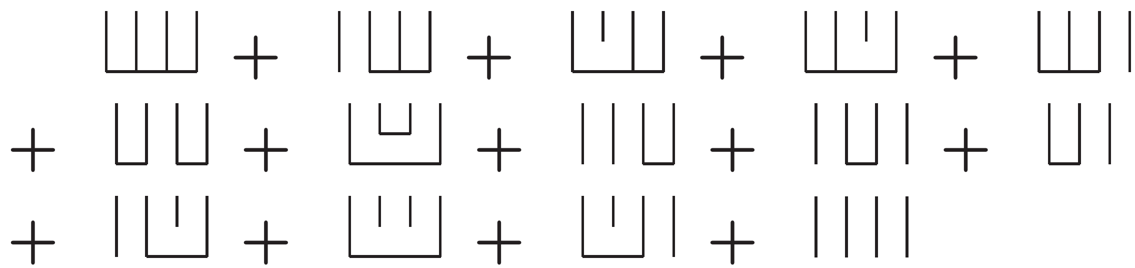

a1
a3
a2

$$\varphi(a_1 a_2 a_3 a_4) =$$



$$\begin{aligned}
\varphi(a_1 a_2 a_3 a_4) = \;& \kappa_4(a_1, a_2, a_3, a_4) + \kappa_1(a_1)\kappa_3(a_2, a_3, a_4) \\
& + \kappa_1(a_2)\kappa_3(a_1, a_3, a_4) + \kappa_1(a_3)\kappa_3(a_1, a_2, a_4) \\
& + \kappa_3(a_1, a_2, a_3)\kappa_1(a_4) + \kappa_2(a_1, a_2)\kappa_2(a_3, a_4) \\
& + \kappa_2(a_1, a_4)\kappa_2(a_2, a_3) + \kappa_1(a_1)\kappa_1(a_2)\kappa_2(a_3, a_4) \\
& + \kappa_1(a_1)\kappa_2(a_2, a_3)\kappa_1(a_4) + \kappa_2(a_1, a_2)\kappa_1(a_3)\kappa_1(a_4) \\
& + \kappa_1(a_1)\kappa_2(a_2, a_4)\kappa_1(a_3) + \kappa_2(a_1, a_4)\kappa_1(a_2)\kappa_1(a_3) \\
& + \kappa_2(a_1, a_3)\kappa_1(a_2)\kappa_1(a_4) + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3)\kappa_1(a_4)
\end{aligned}$$

Freeness $\hat{=}$ vanishing of mixed cumulants

free product $\hat{=}$ direct sum of cumulants

$\varphi(\boldsymbol{a^n})$ given by sum over **blue** planar diagrams

$\varphi(\boldsymbol{b^m})$ given by sum over **red** planar diagrams

then: for a and b free, $\varphi(\boldsymbol{a^{n_1}}\boldsymbol{b^{m_1}}\cdots)$ is given by sum over planar diagrams with monochromatic (blue or red) blocks

Freeness $\hat{=}$ vanishing of mixed cumulants

free product $\hat{=}$ direct sum of cumulants

We have: a and b free is equivalent to

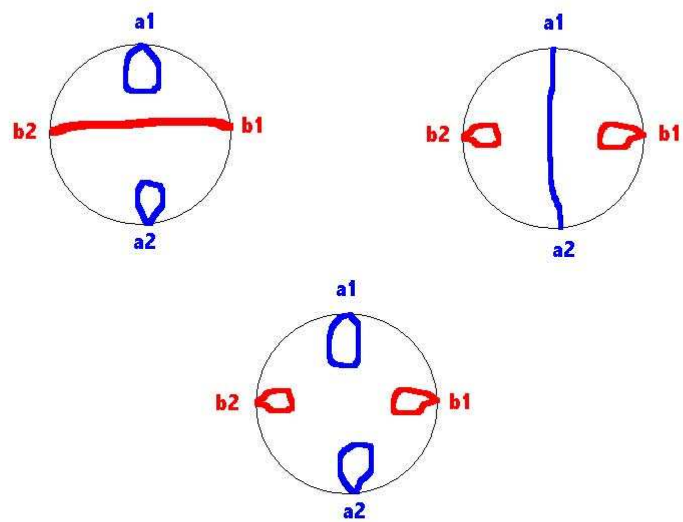
$$\kappa_n(c_1, \dots, c_n) = 0$$

whenever

- $n \geq 2$
- $c_i \in \{a, b\}$ for all i
- there are i, j such that $c_i = a$, $c_j = b$

$$\varphi(a_1 b_1 a_2 b_2) =$$

$$\kappa_1(a_1)\kappa_1(a_2)\kappa_2(b_1, b_2) + \kappa_2(a_1, a_2)\kappa_1(b_1)\kappa_1(b_2)$$



$$+\kappa_1(a_1)\kappa_1(b_1)\kappa_1(a_2)\kappa_1(b_2)$$

One random variable and free convolution

Consider one random variable $a \in \mathcal{A}$ and define their **Cauchy transform G** and their **$\mathcal{R}$-transform $\mathcal{R}$** by

$$G(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\varphi(a^m)}{z^{m+1}}, \qquad \mathcal{R}(z) = \sum_{m=1}^{\infty} \kappa_m(a,\dots,a) z^{m-1}$$

Then we have

- $\frac{1}{G(z)} + \mathcal{R}(G(z)) = z$
- $\mathcal{R}^{a+b}(z) = \mathcal{R}^a(z) + \mathcal{R}^b(z)$ if a and b are free

Random matrices: a route from classical to free probability

Consider unitarily invariant random matrices $A = (a_{ij})_{i,j=1}^{N}$

entries	$\longrightarrow$	**matrix**
independence between $\{a_{ij}\}$ and $\{b_{kl}\}$		freeness between A and B

Random matrices: a route from classical to free probability

Consider unitarily invariant random matrices $A = (a_{ij})_{i,j=1}^{N}$

entries	$\longrightarrow$	**matrix**
independence between $\{a_{ij}\}$ and $\{b_{kl}\}$		freeness between A and B
classical cumulants c_n of a_{ij}		free cumulants κ_n of A

Free cumulants as classical cumulants of cycles

Let

$$A = (a_{ij})_{i,j=1}^{N}$$

be a unitarily invariant random matrix. Then we have

$$\kappa_n(A, \dots, A) = \lim_{N \to \infty} N^{n-1} c_n(a_{12}, a_{23}, a_{34}, \dots, a_{n1})$$