Introduction to Free Probability Theory: Combinatorics, Random Matrices, and Quantum Groups

Roland Speicher

Saarland University
Saarbrücken
Germany

European Research Council
Established by the European Commission

Section 1

Free Probability: The Basics

European Research Council
Established by the European Commission

Some History

1985 Voiculescu introduces "freeness" in the context of isomorphism problem of free group factors

1991 Voiculescu discovers relation with random matrices (which leads, among others, to deep results on free group factors)

1994 Speicher develops combinatorial theory of freeness, based on "free cumulants"

2009 Köstler and Speicher discover free de Finetti theorem

.... many new results on operator algebras, eigenvalue distribution of random matrices, and much more ....

European Research Council
Established by the European Commission

Definition of Freeness

Definition (Voiculescu 1985)

Let $(\mathcal{A}, \varphi)$ be **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$)

Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever

- $a_i \in \mathcal{A}_{j(i)}, \qquad j(i) \in I \quad \forall i, \qquad j(1) \neq j(2) \neq \cdots \neq j(n)$
- $\varphi(a_i) = 0 \quad \forall i$

Random variables $x_1, \ldots, x_n \in \mathcal{A}$ are free, if their generated unital subalgebras $\mathcal{A}_i :=$ algebra$(1, x_i)$ are so.

What is Freeness?

Freeness between A and B is an infinite set of equations relating various moments in A and B:

$$\varphi\Big(p_1(A)q_1(B)p_2(A)q_2(B)\cdots\Big) = 0$$

Basic Observation

Freeness between A and B is actually a **rule for calculating mixed moments** in A and B from the moments of A and the moments of B:

$$\varphi\Big(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\Big) = \text{polynomial}\big(\varphi(A^i), \varphi(B^j)\big)$$

European Research Council
Established by the European Commission

Freeness is a Rule for Calculating Mixed Moments

Example

$$\varphi\Big(\big(A^n - \varphi(A^n)1\big)\big(B^m - \varphi(B^m)1\big)\Big) = 0,$$

thus

$$\varphi(A^nB^m) - \varphi(A^n \cdot 1)\varphi(B^m) - \varphi(A^n)\varphi(1 \cdot B^m) + \varphi(A^n)\varphi(B^m)\varphi(1 \cdot 1) = 0,$$

and hence

$$\boldsymbol{\varphi(A^nB^m) = \varphi(A^n) \cdot \varphi(B^m)}$$

Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables.
Thus freeness is also called **free independence**

Freeness is a Rule for Calculating Mixed Moments

Freeness is analogous to the concept of classical independence for random variables.
Note: free independence is a different rule from classical independence; free independence occurs typically for **non-commuting random variables**, like for operators on Hilbert spaces or (random) matrices

Example

$$\varphi\Big((A-\varphi(A)1)\cdot(B-\varphi(B)1)\cdot(A-\varphi(A)1)\cdot(B-\varphi(B)1)\Big)=0,$$

which results in

$$\begin{aligned}\varphi(ABAB) = \varphi(AA)\cdot\varphi(B)\cdot\varphi(B) + \varphi(A)\cdot\varphi(A)\cdot\varphi(BB)\\ - \varphi(A)\cdot\varphi(B)\cdot\varphi(A)\cdot\varphi(B)\end{aligned}$$

Section 2

Free Probability and Random Matrices

Where Does Freeness Show Up?

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices

European Research Council
Established by the European Commission

Asymptotic Freeness of Random Matrices

Theorem (Voiculescu 1991)

Large classes of independent random matrices (like Gaussian or Wishart matrices) become asymptoticially freely independent, with respect to $\varphi = tr := \frac{1}{N} Tr$, if $N \to \infty$.

Example

This means, for example: if X_N and Y_N are independent $N \times N$ Wigner or Wishart matrices, respectively, then we have almost surely:

$$\lim_{N\to\infty} \text{tr}(X_N Y_N X_N Y_N) = \lim_{N\to\infty} \text{tr}(X_N^2) \cdot \lim_{N\to\infty} \text{tr}(Y_N)^2 + \lim_{N\to\infty} \text{tr}(X_N)^2 \cdot \lim_{N\to\infty} \text{tr}(Y_N^2) - \lim_{N\to\infty} \text{tr}(X_N)^2 \cdot \lim_{N\to\infty} \text{tr}(Y_N)^2$$

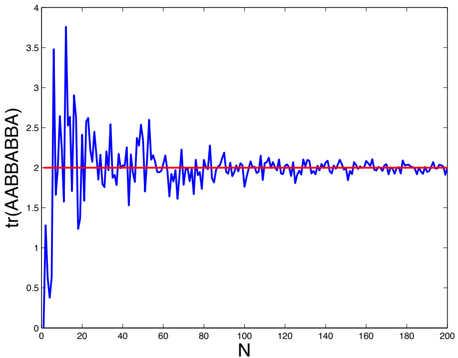
European Research Council
Established by the European Commission

Asymptotic Freeness of Random Matrices

Theorem (Voiculescu 1991)

Large classes of independent random matrices (like Gaussian or Wishart matrices) become asymptoticially freely independent, with respect to $\varphi = tr := \frac{1}{N} Tr$, if $N \to \infty$.

Example ($tr(X_N X_N Y_N Y_N X_N Y_N Y_N X_N) \to 2$ for X_N, Y_N Gaussian)



Asymptotic Freeness of Random Matrices

Theorem (Voiculescu 1991)

Large classes of independent random matrices (like Gaussian or Wishart matrices) become asymptoticially freely independent, with respect to $\varphi = tr := \frac{1}{N} Tr$, if $N \to \infty$.

Example

This means, for example: if X_N and Y_N are independent $N \times N$ Wigner or Wishart matrices, respectively, then we have almost surely:

$$\lim_{N\to\infty} \mathrm{tr}(X_N Y_N X_N Y_N) = \lim_{N\to\infty} \mathrm{tr}(X_N^2) \cdot \lim_{N\to\infty} \mathrm{tr}(Y_N)^2 + \lim_{N\to\infty} \mathrm{tr}(X_N)^2 \cdot \lim_{N\to\infty} \mathrm{tr}(Y_N^2) - \lim_{N\to\infty} \mathrm{tr}(X_N)^2 \cdot \lim_{N\to\infty} \mathrm{tr}(Y_N)^2$$

Hence we have a rule for calculating asymptotically mixed moments of our matrices with respect to the normalized trace tr.

Asymptotic Freeness of Random Matrices

Theorem (Voiculescu 1991)

Large classes of independent random matrices (like Gaussian or Wishart matrices) become asymptoticially freely independent, with respect to $\varphi = tr := \frac{1}{N} Tr$, *if* $N \to \infty$.

Note that moments with respect to tr determine the eigenvalue distribution of a matrix.
For an $N \times N$ matrix $X = X^*$ with eigenvalues $\lambda_1, \dots, \lambda_N$ its eigenvalue distribution

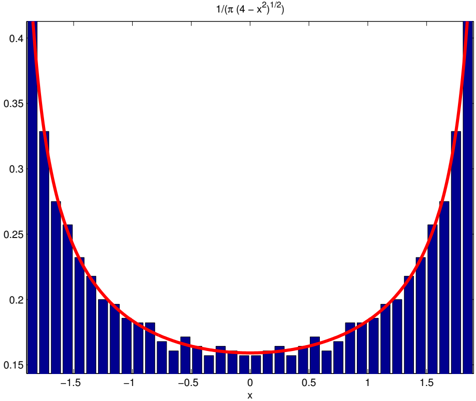
$$\mu_X := \frac{1}{N}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_N})$$

is determined by

$$\int_{\mathbb{R}} t^k d\mu_X(t) = \mathrm{tr}(X^k) \qquad \text{for all } k = 0, 1, 2, \dots$$

.

Randomly Rotated Matrices are Asymptotically Free



2800 eigenvalues of $A + UBU^*$, where A and B are diagonal matrices with 1400 eigenvalues +1 and 1400 eigenvalues -1, and U is a randomly chosen (with respect to the Haar measure) unitary matrix

Section 3

The Combinatorics of Freeness: Non-Crossing Partitions and Free Cumulants

European Research Council
Established by the European Commission

Understanding the Freeness Rule: the Idea of Cumulants

- write moments in terms of other quantities, which we call **free cumulants**
- freeness is much easier to describe on the level of free cumulants: **vanishing of mixed cumulants**
- relation between moments and cumulants is given by summing over **non-crossing or planar partitions**

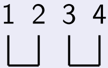
Non-Crossing Partitions

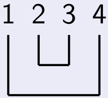
Definition

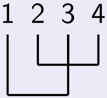
A **partition** of $\{1, \ldots, n\}$ is a decomposition $\pi = \{V_1, \ldots, V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq y), \qquad \bigcup_i V_i = \{1, \ldots, n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.







Non-Crossing Partitions

Definition

A **partition** of $\{1, \dots, n\}$ is a decomposition $\pi = \{V_1, \dots, V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq y), \qquad \bigcup_i V_i = \{1, \dots, n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.
A partition π is **non-crossing** if we do not have

$$p_1 < q_1 < p_2 < q_2$$

such that p_1, p_2 are in same block, q_1, q_2 are in same block, but those two blocks are different.

$$\mathbf{NC(n)} := \{\text{non-crossing partitions of } \{1,\dots,n\}\}$$

Moments and Cumulants

Definition

For unital linear functional

$$\varphi : \mathcal{A} \to \mathbb{C}$$

we define **cumulant functionals** k_n (for all $n \geq 1$)

$$k_n : \mathcal{A}^n \to \mathbb{C}$$

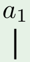
as multi-linear functionals by moment-cumulant relation

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} k_\pi[a_1, \ldots, a_n]$$

Note: classical cumulants are defined by a similar formula, where only $NC(n)$ is replaced by $\mathcal{P}(n)$

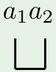
Example ($n = 1$)

$$\varphi(a_1) = k_1(a_1)$$



Example ($n = 2$)

$$\varphi(a_1 a_2) = k_2(a_1, a_2)$$



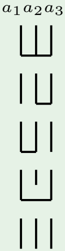
$$+ k_1(a_1) k_1(a_2)$$

and thus

$$k_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$$

Example ($n = 3$)

$$
\begin{aligned}
\varphi(a_1 a_2 a_3) = k_3(a_1, a_2, a_3) \\
+ k_1(a_1) k_2(a_2, a_3) \\
+ k_2(a_1, a_2) k_1(a_3) \\
+ k_2(a_1, a_3) k_1(a_2) \\
+ k_1(a_1) k_1(a_2) k_1(a_3)
\end{aligned}
$$

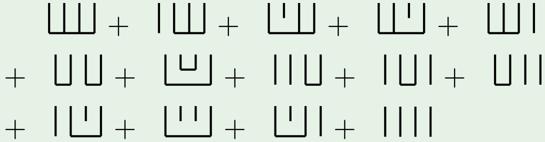


and thus

$$
\begin{aligned}
k_3(a_1, a_2, a_3) = \varphi(a_1 a_2 a_3) - \varphi(a_1)\varphi(a_2 a_3) - \varphi(a_2)\varphi(a_1 a_3) \\
- \varphi(a_3)\varphi(a_1 a_2) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3)
\end{aligned}
$$

Example ($n = 4$)

$\varphi(a_1 a_2 a_3 a_4) =$



$$
\begin{aligned}
= \; & k_4(a_1, a_2, a_3, a_4) + k_1(a_1)k_3(a_2, a_3, a_4) \\
& + k_1(a_2)k_3(a_1, a_3, a_4) + k_1(a_3)k_3(a_1, a_2, a_4) \\
& + k_3(a_1, a_2, a_3)k_1(a_4) + k_2(a_1, a_2)k_2(a_3, a_4) \\
& + k_2(a_1, a_4)k_2(a_2, a_3) + k_1(a_1)k_1(a_2)k_2(a_3, a_4) \\
& + k_1(a_1)k_2(a_2, a_3)k_1(a_4) + k_2(a_1, a_2)k_1(a_3)k_1(a_4) \\
& + k_1(a_1)k_2(a_2, a_4)k_1(a_3) + k_2(a_1, a_4)k_1(a_2)k_1(a_3) \\
& + k_2(a_1, a_3)k_1(a_2)k_1(a_4) + k_1(a_1)k_1(a_2)k_1(a_3)k_1(a_4)
\end{aligned}
$$

Freeness $\hat{=}$ Vanishing of Mixed Cumulants

Theorem (Speicher 1994)

The fact that $x_1, \dots, x_m$ are free is equivalent to the fact that

$$k_n(x_{i(1)}, \dots, x_{i(n)}) = 0$$

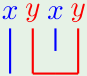
whenever

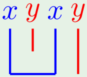
- $1 \leq i(1), \dots, i(n) \leq m$
- *there are p, q such that $i(p) \neq i(q)$ (in particular, $n \geq 2$)*

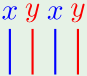
Example

If x and y are free then: $\varphi(xyxy) =$

$$k_1(x)k_1(x)k_2(y,y) + k_2(x,x)k_1(y)k_1(y) + k_1(x)k_1(y)k_1(x)k_1(y)$$







Sum of Free Variables: Description via R-Transform

Definition

Consider a random variable $x \in \mathcal{A}$. We define its **Cauchy transform** $\boldsymbol{G = G_x}$ and its **$\mathcal{R}$-transform $\boldsymbol{\mathcal{R} = \mathcal{R}_x}$** by

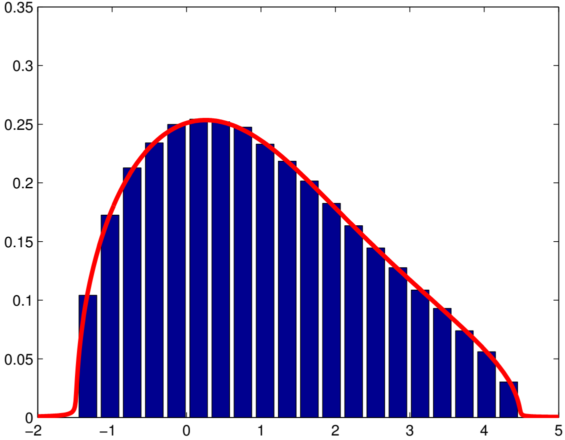
$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi(x^n)}{z^{n+1}}, \qquad \mathcal{R}(z) = \sum_{n=1}^{\infty} k_n(x, \dots, x) z^{n-1}$$

Theorem (Voiculescu 1986, Speicher 1994)

Then we have

- $\frac{1}{G(z)} + \mathcal{R}(G(z)) = z$
- $\mathcal{R}_{x+y}(z) = \mathcal{R}_x(z) + \mathcal{R}_y(z)$ *if x and y are free*

Eigenvalues of the Sum of Independent Gaussian and Wishart 3000×3000 Random Matrices



Product of Free Variables: Description via S-Transform

Theorem (Voiculescu 1987; Haagerup 1997; Nica, Speicher 1997)

Put

$$M_x(z) := \sum_{m=1}^{\infty} \varphi(x^m) z^m$$

and define

$$S_x(z) := \frac{1+z}{z} M_x^{<-1>}(z)$$

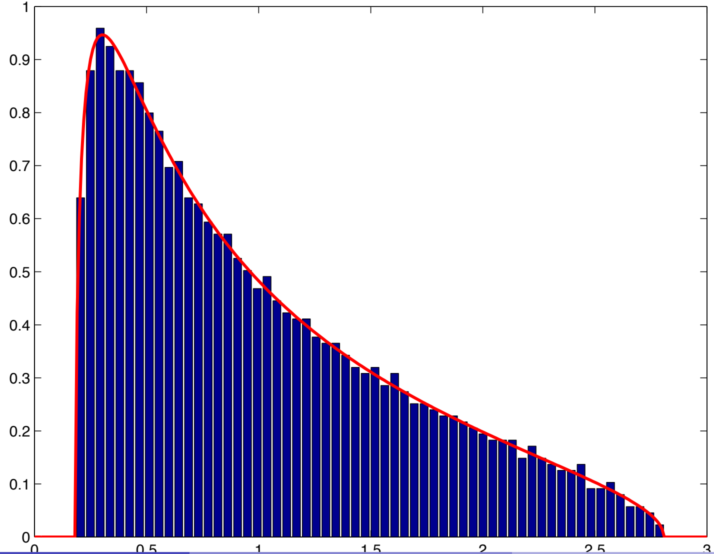
S-transform of x

Then: If x and y are free, we have

$$S_{xy}(z) = S_x(z) \cdot S_y(z).$$

European Research Council
Established by the European Commission

Eigenvalues of the Product of Two Independent Wishart 2000×2000 Random Matrices



Polynomials in Free Variables: Linearisation and Operator-Valued Free Probability

Theorem (Belinschi, Mai, Speicher 2013)

The following algorithm allows the calculation of the distribution of any selfadjoint polynomial $p(x, y)$ in two non-commuting variables x and y, given the distribution of x and the distribution of y:

- *Linearize $p(x, y)$ to $\hat{p} = \hat{x} + \hat{y}$.*
- *Calculate $G_{\hat{x}}(b)$ out of $G_x(z)$ and $G_{\hat{y}}(b)$ out of $G_y(z)$*
- *Get $w_1(b)$ as the fixed point of the iteration*

$$w \mapsto G_{\hat{y}}(b + G_{\hat{x}}(w)^{-1} - w)^{-1} - (G_{\hat{x}}(w)^{-1} - w)$$

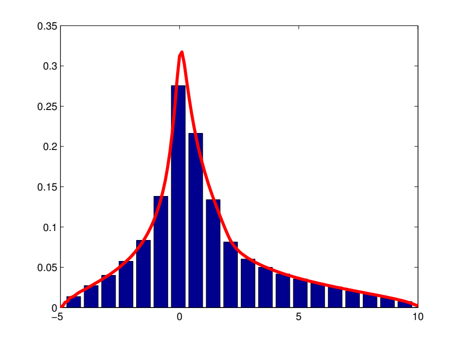
- *Calculate $G_{\hat{p}}(b) = G_{\hat{x}}(\omega_1(b))$ and recover $G_p(z)$ as one entry of $G_{\hat{p}}(b)$ for $b = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$*

Polynomials in Free Variables: Linearisation and Operator-Valued Free Probability

Example

$P(X,Y) = XY + YX + X^2$
for independent X, Y; X is Gaussian and Y is Wishart

$$\hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$



Section 4

Freeness and Quantum Groups: Non-Commutative de Finetti Theorem

Classical Exchangeable Random Variables

Consider probability space $(\Omega, \mathfrak{A}, P)$. Denote expectation by φ,

$$\varphi(Y) = \int_\Omega Y(\omega) dP(\omega).$$

Definition

We say that random variables $X_1, X_2, \dots$ are **exchangeable** if their joint distribution is invariant under finite permutations, i.e. if

$$\varphi(X_{i(1)} \cdots X_{i(n)}) = \varphi(X_{\pi(i(1))} \cdots X_{\pi(i(n))})$$

for all $n \in \mathbb{N}$, all $i(1), \dots, i(n) \in \mathbb{N}$, and all permutations π

Example

$$\varphi(X_1^n) = \varphi(X_7^n), \qquad \varphi(X_1^3 X_3^7 X_4) = \varphi(X_8^3 X_2^7 X_9)$$

Classical de Finetti Theorem

Definition

$$\mathfrak{A}_{\text{tail}} := \bigcap_{i \in \mathbb{N}} \sigma(X_k \mid k \geq i)$$

$$E : L^\infty(\Omega, \mathfrak{A}, P) \to L^\infty(\Omega, \mathfrak{A}_{\text{tail}}, P)$$

Theorem (de Finetti 1931, Hewitt, Savage 1955)

The following are equivalent for an infinite sequence of random variables:

- *the sequence is exchangeable*
- *the sequence is independent and identically distributed with respect to the conditional expectation E onto the tail σ-algebra of the sequence*

$$E[X_1^{m(1)} X_2^{m(2)} \cdots X_n^{m(n)}] = E[X_1^{m(1)}] \cdot E[X_2^{m(2)}] \cdots E[X_n^{m(n)}]$$

Quantum Permutation Group

Definition (Wang 1998)

The quantum permutation group $A_s(k)$ is given by the universal unital C^*-algebra generated by u_{ij} $(i,j=1,\dots,k)$ subject to the relations

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i,j=1,\dots,k$
- each row and column of $u=(u_{ij})_{i,j=1}^k$ is a partition of unity:

$$\sum_{j=1}^{k} u_{ij} = 1 \quad \forall i \qquad \text{and} \qquad \sum_{i=1}^{k} u_{ij} = 1 \quad \forall j$$

(note: elements within a row or within a column are orthogonal)

$A_s(k)$ is a compact quantum group in the sense of Woronowicz.

European Research Council
Established by the European Commission

Quantum Exchangeability

Definition

A sequence $x_1, \dots, x_k$ in $(\mathcal{A}, \varphi)$ is **quantum exchangeable** if its distribution does not change under the action of quantum permutations S_k^+, i.e., if we have:
Let a quantum permutation $u = (u_{ij}) \in C(S_k^+)$ act on $(x_1, \dots, x_k)$ by

$$y_i := \sum_j u_{ij} \otimes x_j \qquad \in \qquad C(S_k^+) \otimes \mathcal{A}$$

Then

- $(x_1, \dots, x_k) \in (\mathcal{A}, \varphi)$
- $(y_1, \dots, y_k) \in (C(S_k^+) \otimes \mathcal{A}, \mathrm{id} \otimes \varphi)$

have the same distribution, i.e.,

$$\varphi(x_{i(1)} \cdots x_{i(n)}) \cdot 1_{C(S_k^+)} = \mathrm{id} \otimes \varphi(y_{i(1)} \cdots y_{i(n)})$$

Non-commutative de Finetti Theorem

Definition

Define the **tail algebra** of the sequence:

$$\mathcal{A}_{\text{tail}} := \bigcap_{i \in \mathbb{N}} \mathsf{vN}(x_k \mid k \geq i),$$

then there exists **conditional expectation** $E : \mathsf{vN}(x_i \mid i \in \mathbb{N}) \to \mathcal{A}_{\text{tail}}$.

Theorem (Köstler, Speicher 2009)

The following are equivalent for an infinite sequence of non-commutative random variables:

- *the sequence is quantum exchangeable*
- *the sequence is free and identically distributed with respect to the conditional expectation E onto the tail-algebra of the sequence*

Section 5

The End

Summary

- Free independence is the basic probabilistic structure in a maximal non-commutative world
- It shows up in quite different contextes; like operator algebras, but ...
- ... in particular, it describes also the asymptotic large N-regime of random matrices
- Its combinatorial structure is governed by non-crossing partitions and free cumulants
- The symmetries of freeness are given by liberated quantum groups, like quantum permutation and quantum orthogonal groups

European Research Council
Established by the European Commission