Free Probability Theory and Random Matrices

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joint work with Serban Belinschi, Tobias Mai, John Treilhard, Carlos Vargas

ICM, August 20, 2014

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A Short History of Free Probability

History (with a strong bias towards our problem)

- 1983: Voiculescu introduces concept of freeness in the context of operator algebras
- 1991: Voiculescu discovers relation between freeness and random matrices
- 1993: Speicher develops combinatorial theory of freeness
- 1995: Voiculescu considers operator-valued versions of free probability theory
- 2005: Haagerup and Thorbjørnsen show the power of the linearization trick in free probability
- 2013: Belinschi, Mai and Speicher develop analytic theory of operator-valued free convolution and apply this to random matrix problem

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Asymptotic Eigenvalue Distribution

Problem

We are interested in the limiting eigenvalue distribution of an

$$N \times N \text{ random matrix for } N \to \infty.$$

Typical phenomena for basic random matrix ensembles:

- almost sure convergence to a deterministic limit eigenvalue distribution
- this limit distribution can be effectively calculated

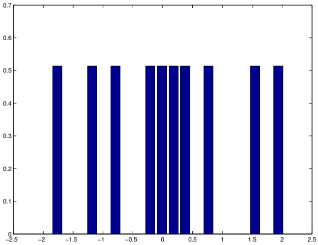
Gaussian Random Matrix (Wigner 1955)

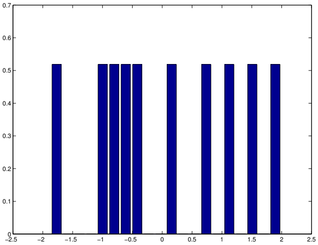
Definition

A **Gaussian random matrix** $X = \frac{1}{\sqrt{N}}(x_{ij})_{i,j=1}^{N}$

- is symmetric: $X^* = X$
- $\{x_{ij} \mid 1 \le i \le j \le N\}$ are independent and identically distributed, with a centered normal distribution of variance 1

Example (eigenvalue distribution for $N = 10$)





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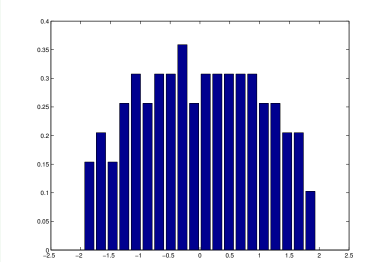
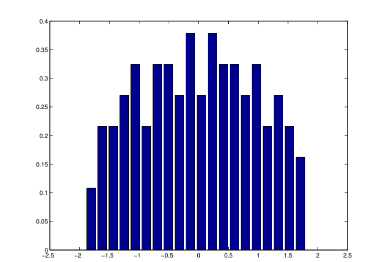
Example (eigenvalue distribution for $N = 100$)





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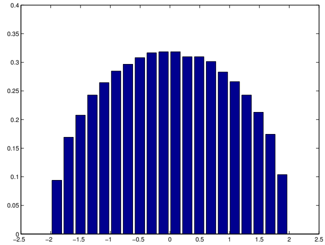
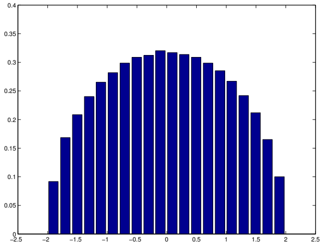
Example (eigenvalue distribution for $N = 3000$)





Wishart Random Matrix (Wishart 1928)

Definition

A **Wishart random matrix** X is of the form $X = \frac{1}{\sqrt{N}} AA^*$ where

- A is an $N \times M$ matrix $A = (a_{ij})_{\substack{i=1,\dots,N \\ j=1,\dots,M}}$
- $\{a_{ij} \mid 1 \leq i \leq N, 1 \leq j \leq M\}$ are independent and identically distributed, with a centered normal distribution of variance 1

For $N \to \infty$, one keeps the ratio

$$\lambda := \frac{N}{M}$$

fixed.

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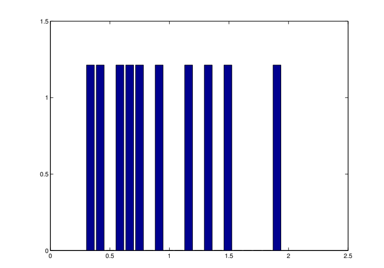
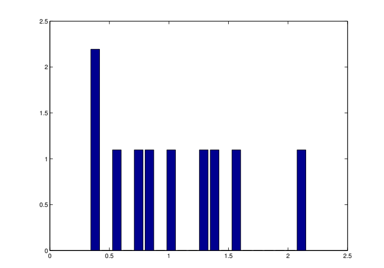
Example (eigenvalue distribution; $\lambda = 0.25$: $N = 10$, $M = 40$)





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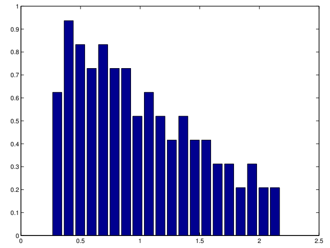
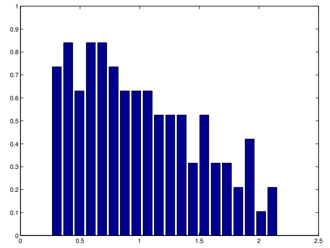
- $\{a_{ij} \mid 1 \leq i \leq N, 1 \leq j \leq M\}$ are independent and identically distributed, with a centered normal distribution of variance 1

Example (eigenvalue distribution; $\lambda = 0.25$: $N = 100$, $M = 400$)





Wishart Random Matrix (Wishart 1928)

Definition

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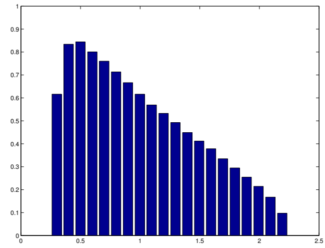
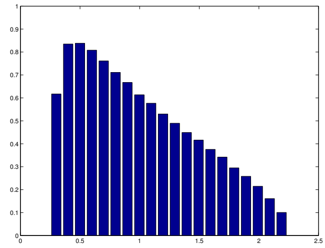
- A is an $N \times M$ matrix $A = \left(a_{ij}\right)_{\substack{i=1,\dots,N \\ j=1,\dots,M}}$
- $\{a_{ij} \mid 1 \le i \le N, 1 \le j \le M\}$ are independent and identically distributed, with a centered normal distribution of variance 1

Example (eigenvalue distribution; $\lambda = 0.25$: $N = 3000$, $M = 12000$)





Asymptotic Eigenvalue Distribution

Definition

For our random matrices X_N we are interested in the

eigenvalue distribution $$\mu_{X_N} := \frac{1}{N}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_N}),$$

where $\lambda_1, \ldots, \lambda_N$ are the (random) eigenvalues of X_N.

In our cases, μ_{X_N} converges, for $N \to \infty$, almost surely to a non-random probability distribution, the **limit eigenvalue distribution**.

The Cauchy Transform

Definition

For any probability measure μ on $\mathbb{R}$ we define its **Cauchy transform** by

$$G(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t)$$

$-G$ is also called **Stieltjes transform**.

This is an analytic function $G : \mathbb{C}^+ \to \mathbb{C}^-$ and we can recover μ from G by **Stieltjes inversion formula**.

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \Im G(t + i\varepsilon) dt$$

For our basic random matrix ensembles one can derive equations for the Cauchy transform of the limiting eigenvalue distribution, solve those equations and then get the density via Stieltjes inversion.

Example (Gaussian rm)

$$G(z)^2 + 1 = zG(z),$$

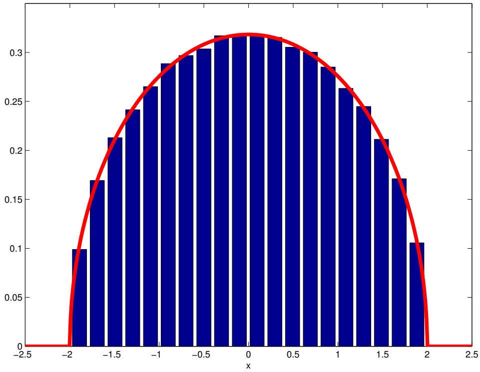
which can be solved as

$$G(z) = \frac{z - \sqrt{z^2 - 4}}{2},$$

thus

$$d\mu_s(t) = \frac{1}{2\pi}\sqrt{4 - t^2}dt$$

Wigners semicircle



For our basic random matrix ensembles one can derive equations for the Cauchy transform of the limiting eigenvalue distribution, solve those equations and then get the density via Stieltjes inversion.

Example (Wishart rm)

$$\frac{\lambda}{1-G(z)} + \frac{1}{G(z)} = z$$

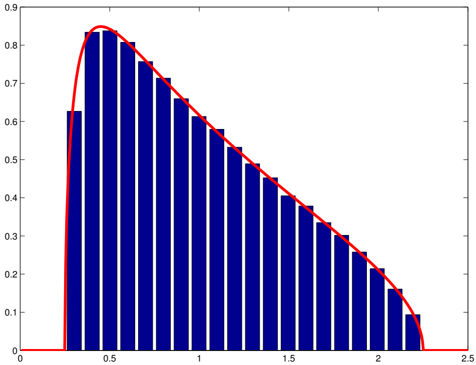
which can be solved as

$$G(z) = \frac{z+1-\lambda-\sqrt{(z-(1+\lambda))^2-4\lambda}}{2z}$$

and thus

$$d\mu(t) = \frac{1}{2\pi\lambda t}\sqrt{4\lambda-(t-(1+\lambda))^2}\,dt$$

Marchenko-Pastur distribution



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Polynomials in Several Independent Random Matrices

Problem

We are now interested in the limiting eigenvalue distribution of **general selfadjoint polynomials $p(X_1, \ldots, X_k)$** of **several independent** $N \times N$ random matrices $X_1, \ldots, X_k$

Typical phenomena:

- almost sure convergence to a deterministic limit eigenvalue distribution
- this limit distribution can be effectively calculated **only in very simple situations**

For simple situations one can derive equations for the Cauchy transform of the limiting eigenvalue distribution; those can usually not be solved explicitly; however, as fixed point equations they have a good analytic behaviour and can be solved numerically by iteration algorithms

Example (Gauss + Wishart)

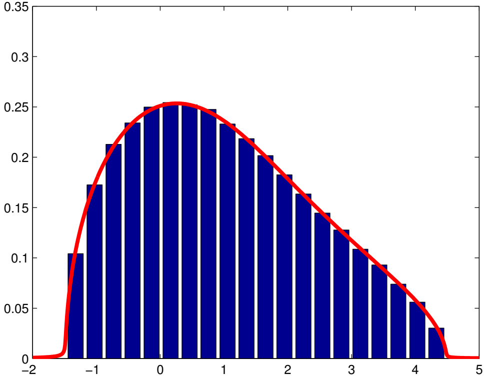
For

$$G(z) := G_{\text{Gauss+Wishart}}(z)$$

one finds the fixed point equation
(in **subordination form**)

$$G(z) = G_{\text{Wishart}}(z - G(z)),$$

which can be easily solved by iteration.



Existing Results for Calculations of the Limit Eigenvalue Distribution

- Marchenko, Pastur 1967: general Wishart matrices ADA^*
- Pastur 1972: deterministic + Wigner (deformed semicircle)
- Speicher, Nica 1998; Vasilchuk 2003: commutator or anti-commutator: $X_1X_2 \pm X_2X_1$
- more general models in wireless communications (Tulino, Verdu 2004; Couillet, Debbah, Silverstein 2011):

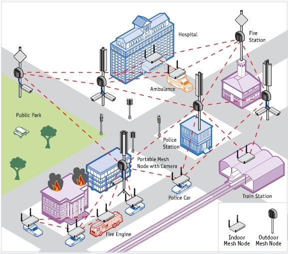
$$RADA^*R^*$$

or

$$\sum_i R_iA_iD_iA_i^*R_i^*$$

or

...



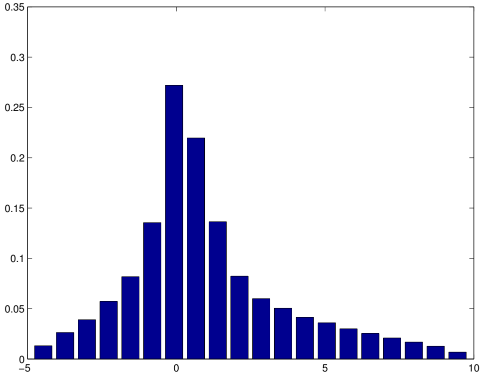
What Can We Say About More Complicated or Even General Selfadjoint Polynomials?

Example

Can we calculate the asymptotic eigenvalue distribution of

$$P(X,Y) = XY + YX + X^2$$

for independent Gaussian and Wishart random matrices X and Y, respectively?



Can we treat general polynomials $P(X_1, \ldots, X_k)$?

Asymptotic Freeness of Random Matrices

Basic Observation (Voiculescu 1991)

Large classes of independent random matrices (like Gaussian or Wishart matrices) become asymptoticially freely independent.

Definition (Voiculescu 1985)

Let $(\mathcal{A}, \varphi)$ be a **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$).

Example (Commutative Probability Space)

For a classical probability space (Ω, P) take

- $\mathcal{A} = L^\infty(\Omega, P)$
- $\varphi(x) = \int_\Omega x(\omega) dP(\omega)$ for $x \in \mathcal{A}$

Definition (Voiculescu 1985)

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Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever

- $a_i \in \mathcal{A}_{j(i)} \qquad j(i) \in I \quad \forall i$
- $j(1) \neq j(2) \neq \cdots \neq j(n)$
- $\varphi(a_i) = 0 \quad \forall i$

Random variables $x_1, \ldots, x_n \in \mathcal{A}$ are freely independent, if their generated unital subalgebras $\mathcal{A}_i :=$ algebra$(1, x_i)$ are so.

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What is Freeness?

Freeness between x and y is an infinite set of equations relating various moments in x and y:

$$\varphi\Big(p_1(x)q_1(y)p_2(x)q_2(y)\cdots\Big) = 0$$

Basic observation: free independence between x and y is actually a **rule for calculating mixed moments** in x and y from the moments of x and the moments of y:

$$\varphi\Big(x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots\Big) = \text{polynomial}\big(\varphi(x^i), \varphi(y^j)\big)$$

Example

If x and y are freely independent, then we have

$$\varphi(x^m y^n) = \varphi(x^m)\cdot\varphi(y^n)$$

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If x and y are freely independent, then we have

$$\varphi(x^m y^n) = \varphi(x^m) \cdot \varphi(y^n)$$

$$\varphi(x^{m_1} y^n x^{m_2}) = \varphi(x^{m_1+m_2}) \cdot \varphi(y^n)$$

but also

$$\varphi(xyxy) = \varphi(x^2) \cdot \varphi(y)^2 + \varphi(x)^2 \cdot \varphi(y^2) - \varphi(x)^2 \cdot \varphi(y)^2$$

Free independence is a rule for calculating mixed moments, analogous to the concept of independence for random variables.
Note: free independence is a different rule from classical independence; free independence occurs typically for **non-commuting random variables**, like operators on Hilbert spaces.

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This means of course that, for any polynomial p, the moments of $p(x, y)$ are determined in terms of the moments of x and the moments of y.

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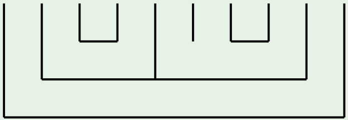
Combinatorial Structure of Freeness

Basic Observation (Speicher 1993)

The structure of the formulas for mixed moments is governed by the **lattice of non-crossing partitions**.

Example (Factorization of Non-Crossing Moments)

Let $x_1, \dots, x_5$ be free. Consider $x_1\, x_2\, x_3\, x_3\, x_2\, x_4\, x_5\, x_5\, x_2\, x_1$ Then



$$\varphi(x_1 x_2 x_3 x_3 x_2 x_4 x_5 x_5 x_2 x_1)$$
$$= \varphi(x_1 x_1) \cdot \varphi(x_2 x_2 x_2) \cdot \varphi(x_3 x_3) \cdot \varphi(x_4) \cdot \varphi(x_5 x_5)$$

Crossing moments are more complicated, but still have non-crossing structure: $\varphi(xyxy) = \varphi(x^2) \cdot \varphi(y)^2 + \varphi(x)^2 \cdot \varphi(y^2) - \varphi(x)^2 \cdot \varphi(y)^2$

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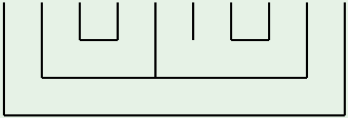

$$\varphi(x_1 x_2 x_3 x_3 x_2 x_4 x_5 x_5 x_2 x_1)$$
$$= \varphi(x_1 x_1) \cdot \varphi(x_2 x_2 x_2) \cdot \varphi(x_3 x_3) \cdot \varphi(x_4) \cdot \varphi(x_5 x_5)$$

- Many consequences of this are worked out in joint works with A. Nica
- Nica, Speicher: *Lectures on the Combinatorics of Free Probability*, 2006

Where Does Free Independence Show Up?

Free independence can be found in different situations; some of the main occurrences are:

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices

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Asymptotic Freeness of Random Matrices

Theorem (Voiculescu 1991)

Large classes of independent random matrices (like Gaussian or Wishart matrices) become asymptoticially freely independent, with respect to $\varphi = tr := \frac{1}{N} Tr$, if $N \to \infty$.

Example

This means, for example: if X_N and Y_N are independent $N \times N$ Wigner and Wishart matrices, respectively, then we have almost surely:

$$\lim_{N\to\infty} \mathrm{tr}(X_N Y_N X_N Y_N) = \lim_{N\to\infty} \mathrm{tr}(X_N^2) \cdot \lim_{N\to\infty} \mathrm{tr}(Y_N)^2 + \lim_{N\to\infty} \mathrm{tr}(X_N)^2 \cdot \lim_{N\to\infty} \mathrm{tr}(Y_N^2) - \lim_{N\to\infty} \mathrm{tr}(X_N)^2 \cdot \lim_{N\to\infty} \mathrm{tr}(Y_N)^2$$

Hence we have a rule for calculating asymptotically mixed moments of our matrices with respect to the normalized trace tr.

Asymptotic Freeness of Random Matrices

Theorem (Voiculescu 1991)

Large classes of independent random matrices (like Gaussian or Wishart matrices) become asymptoticially freely independent, with respect to $\varphi = tr := \frac{1}{N} Tr$, if $N \to \infty$.

Note that moments with respect to tr determine the eigenvalue distribution of a matrix.
For an $N \times N$ matrix $X = X^*$ with eigenvalues $\lambda_1, \ldots, \lambda_N$ its eigenvalue distribution

$$\mu_X := \frac{1}{N}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_N})$$

is determined by

$$\int_{\mathbb{R}} t^k d\mu_X(t) = \text{tr}(X^k) \qquad \text{for all } k = 0, 1, 2, \ldots$$

.

Consequence: Reduction of Random Matrix Problem to Problem of Polynomials in Freely Independent Variables

If the random matrices $X_N^{(1)}, \dots, X_N^{(k)}$ are asymptotically freely independent, then the distribution of a polynomial $p(X_N^{(1)}, \dots, X_N^{(k)})$ is asymptotically given by the distribution of $p(x_1, \dots, x_k)$, where

- $x_1, \dots, x_k$ are freely independent variables, and
- the distribution of x_i is the asymptotic distribution of $X_N^{(i)}$

Example

Consider independent Gaussian and Wishart random matrices X_N and Y_N, respectively. Then the asymptotic distribution of $X_N Y_N + Y_N X_N + X_N^2$ is given by the distribution of $xy + yx + x^2$, where x and y are free, and x has a semicircular distribution and y has a Marchenko-Pastur distribution.

Can We Actually Calculate Polynomials in Freely Independent Variables?

Free probability can deal effectively with simple polynomials in free variables:

- the sum of variables (Voiculescu 1986, R-transform)
- the product of variables (Voiculescu 1987, S-transform)
- the commutator of variables (Nica, Speicher 1998)

Basic Observation (Voiculescu, Biane, Götze, Chistyakov, Belinschi, Bercovici ...)

There are nice analytic descriptions in **subordination form**, e.g., for x and y free one has

$$G_{x+y}(z) = G_x\big(\omega(z)\big),$$

where $\omega : \mathbb{C}^+ \to \mathbb{C}^+$ is an analytic function which can be calculated effectively via fixpoint descriptions.

Can We Actually Calculate Polynomials in Freely Independent Variables?

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- the sum of variables (Voiculescu 1986, R-transform)
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- the commutator of variables (Nica, Speicher 1998)

However:

- **it is not clear how to deal with $xy + yx + x^2$**
- **there is actually no hope to calculate effectively general polynomials in freely independent variables with usual free probability theory!**

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Definition

Let $\mathcal{B} \subset \mathcal{A}$. A linear map $E : \mathcal{A} \to \mathcal{B}$ is a **conditional expectation** if

$$E[b] = b \qquad \forall b \in \mathcal{B}$$

and

$$E[b_1 a b_2] = b_1 E[a] b_2 \qquad \forall a \in \mathcal{A}, \quad \forall b_1, b_2 \in \mathcal{B}$$

An **operator-valued probability space** consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$

Example (Classical conditional expectation)

Let $\mathfrak{M}$ be a σ-algebra and $\mathfrak{N} \subset \mathfrak{M}$ be a sub-σ-algebra. Then

- $\mathcal{A} = L^\infty(\Omega, \mathfrak{M}, P)$
- $\mathcal{B} = L^\infty(\Omega, \mathfrak{N}, P)$
- $E[\cdot|\mathfrak{N}]$ is the classical conditional expectation from the bigger onto the smaller σ-algebra.

Definition of Operator-Valued Freeness

Definition (Voiculescu 1985)

Let $E : \mathcal{A} \to \mathcal{B}$ be an operator-valued probability space.
Subalgebras $\mathcal{A}_i$ $(i \in I)$, which contain $\mathcal{B}$, are **free over $\mathcal{B}$**, if
$E[a_1 \cdots a_n] = 0$ whenever

- $a_i \in \mathcal{A}_{j(i)}, \qquad j(i) \in I \quad \forall i$
- $j(1) \neq j(2) \neq \cdots \neq j(n)$
- $E[a_i] = 0 \quad \forall i$

Variables $x_1, \dots, x_n \in \mathcal{A}$ are free over $\mathcal{B}$, if the generated $\mathcal{B}$-subalgebras $\mathcal{A}_i :=$ algebra$(\mathcal{B}, x_i)$ are so.

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Can We Actually Calculate Polynomials in Operator-Valued Freely Independent Variables?

Again, in principle all operator-valued polynomials in freely independent variables are determined, but effectively we can again only deal with simple polynomials:

- **the sum of variables**
  Voiculescu 1995
  Belinschi, Mai, Speicher 2013
- **the product of variables**
  Voiculescu 1995; Dykema 2006
  Belinschi, Speicher, Treilhard, Vargas 2012

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Analytic Description of Operator-Valued Free Convolution

Definition

Consider an operator-valued probability space $E : \mathcal{A} \to \mathcal{B}$.
For a random variable $x \in \mathcal{A}$, we define the **operator-valued Cauchy transform**:

$$G(b) := E[(b - x)^{-1}] \qquad (b \in \mathcal{B}).$$

For $x = x^*$, this is well-defined and a nice analytic map on the operator-valued upper halfplane:

$$\mathbb{H}^+(\mathcal{B}) := \{b \in \mathcal{B} \mid \frac{b - b^*}{2i} > 0\}$$

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Subordination Formulation

Theorem (Belinschi, Mai, Speicher 2013)

Let x and y be selfadjoint operator-valued random variables free over $\mathcal{B}$. Then there exists a Fréchet analytic map $\omega\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ so that

$$G_{x+y}(b) = G_x(\omega(b)) \text{ for all } b \in \mathbb{H}^+(\mathcal{B}).$$

Moreover, if $b \in \mathbb{H}^+(\mathcal{B})$, then $\omega(b)$ is the unique fixed point of the map

$$f_b\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B}), \quad f_b(w) = h_y(h_x(w) + b) + b,$$

and

$$\omega(b) = \lim_{n\to\infty} f_b^{\circ n}(w) \qquad \text{for any } w \in \mathbb{H}^+(\mathcal{B}).$$

where

$$\mathbb{H}^+(\mathcal{B}) := \{b \in \mathcal{B} \mid \frac{b-b^*}{2i} > 0\}, \qquad h(b) := \frac{1}{G(b)} - b$$

The Linearization Philosophy:

In order to understand polynomials in non-commuting variables, it suffices to understand matrices of **linear** polynomials in those variables.

History (in operator algebras)

- Voiculescu 1987: motivation
- Haagerup, Thorbjørnsen 2005: largest eigenvalue
- Anderson 2012: the selfadjoint version
  ("Schur complement")

History (in other fields)

The same idea has been used in other fields under different names (like "descriptor system" in control theory), for example:

- Schützenberger 1961: automata theory
- Helton, McCullough, Vinnikov 2006: symmetric descriptor realization

Basic Observation

For each selfadjoint polynomial p there exists a selfadjoint matrix $\hat{p}$ of linear polynomials such that the Cauchy transform $G_p(z)$ of p with respect to φ is given as the (1,1)-entry of the operator-valued Cauchy transform of $\hat{p}$ with respect to $E = \mathrm{id} \otimes \varphi$:

$$G_{\hat{p}}(b) = \mathrm{id} \otimes \varphi\left[(b - \hat{p})^{-1}\right] = \begin{pmatrix} G_p(z) & * \\ * & * \end{pmatrix} \qquad \text{for} \quad b = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$$

Example

A selfadjoint linearization of

$$p = xy + yx + x^2 \qquad \text{is} \qquad \hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$

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The selfadjoint linearization $\hat{p}$ is now the sum of two selfadjoint operator-valued variables

$$\hat{p} = \hat{x} + \hat{y} = \begin{pmatrix} 0 & x & \frac{x}{2} \\ x & 0 & 0 \\ \frac{x}{2} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & -1 \\ y & -1 & 0 \end{pmatrix}$$

where

- we know the operator-valued distribution of $\hat{x}$ and the operator-valued distribution of $\hat{y}$
- and $\hat{x}$ and $\hat{y}$ are operator-valued freely independent!

So we can use operator-valued free convolution to calculate the operator-valued Cauchy transform of $\hat{x} + \hat{y}$.

$$G_{\hat{p}}(b) = G_{\hat{x}}(\omega(b))$$

and from this get the Caucy transform of $p(x, y)$.

Theorem (Belinschi, Mai, Speicher 2013)

1) The following algorithm allows the calculation of the distribution of any selfadjoint polynomial $p(x, y)$ in two non-commuting variables x and y, given the distribution of x and the distribution of y:

- *Linearize $p(x, y)$ to $\hat{p} = \hat{x} + \hat{y}$.*
- *Calculate $G_{\hat{x}}(b)$ out of $G_x(z)$ and $G_{\hat{y}}(b)$ out of $G_y(z)$*
- *Get $w_1(b)$ as the fixed point of the iteration*

$$w \mapsto G_{\hat{y}}(b + G_{\hat{x}}(w)^{-1} - w)^{-1} - (G_{\hat{x}}(w)^{-1} - w)$$

- *Calculate $G_{\hat{p}}(b) = G_{\hat{x}}(\omega_1(b))$ and recover $G_p(z)$ as one entry of $G_{\hat{p}}(b)$ for $b = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$*

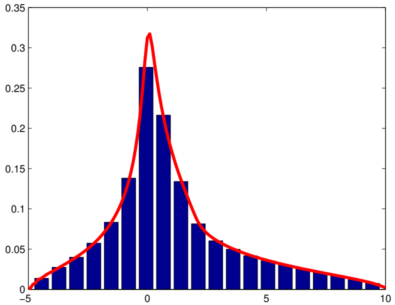
2) Iteration of step 3 of the above algorithm allows the calculation of the distribution of any selfadjoint polynomial $p(x_1, \ldots, x_k)$ in k non-commuting variables, given the distribution of each x_i.

Example

$P(X,Y) = XY + YX + X^2$

for independent X, Y; X is Gaussian and Y is Wishart

$$\hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$



$p(x,y) = xy + yx + x^2$

for free x, y; x is semicircular and y is Marchenko-Pastur

Summary and Outlook

- asymptotic eigenvalue distributions of polynomials in large classes of random matrices can be understood by looking at linear matrix-valued polynomials in free variables
- the latter can be calculated by using the analytic theory of operator-valued free convolution
- qualitative properties of polynomials in free variables can in principle be derived from qualitative properties of operator-valued free convolution
- approach can also be extended to larger classes of functions; for example, non-commutative rational functions
  (work in progress with Mai)
- generalization to non-selfadjoint polynomials (complex eigenvalues and Brown measure) are possible
  (joint work with Belinschi and Sniady)

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