Second Order Freeness: Past and Future

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supported by ERC Advanced Grant
Non-Commutative Distributions in Free Probability

on joint work with Jamie

Section 1

Motivation

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Goal: to understand fluctuations of eigenvalues and traces

- around the year 2000, random matrix theory became main stream in mathematics
- main interest: local/global behaviour of eigenvalues

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- try to find some "free probability like" structure in those fluctuations
- main focus: combinatorial structure $\rightarrow$ non-crossing objects

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Section 2

History

History

Oberwolfach 2005

History

Banff 2004	Stochastic Eigenanalysis 2006

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History

Working hard on 2nd order freeness ....

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History

Oberwolfach 2015

Herstmonceux 2015

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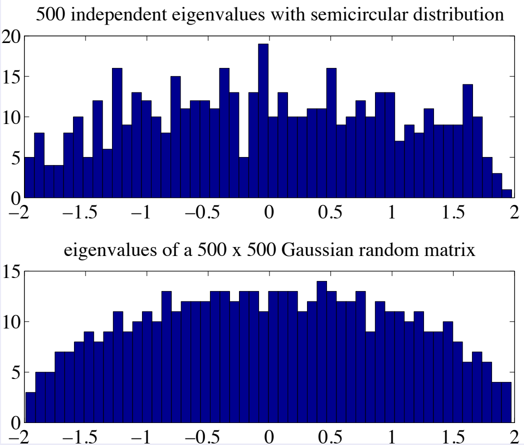
History

... still working hard on 2nd order freeness ....

Goal: Fluctuations of Eigenvalues and Traces

Fluctuations

We do not only look on the asymptotic eigenvalue distributions, but also how they fluctuate about the large N limit



Goal: Fluctuations of Eigenvalues and Traces

Second order limit distribution

An $N \times N$ random matrix ensemble $(A_N)_{N\in\mathbb{N}}$ has a **second order limit distribution** if for all $m, n \geq 1$ the limits

$$\alpha_n := \lim_{N\to\infty} E[\mathrm{tr}(A_N^n)]$$

and

$$\alpha_{m,n} := \lim_{N\to\infty} \mathrm{cov}\big(\mathrm{Tr}(A_N^m), \mathrm{Tr}(A_N^n)\big)$$

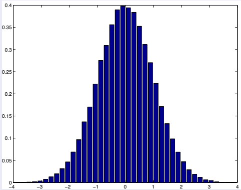
exist and if all higher classical cumulants of $\mathrm{Tr}(A_N^m)$ go to zero.
This means that the family

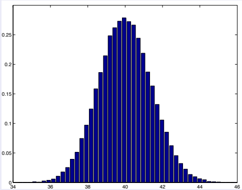
$$\big(\mathrm{Tr}(A_N^m) - E[\mathrm{Tr}(A_N^m)]\big)_{m\in\mathbb{N}}$$

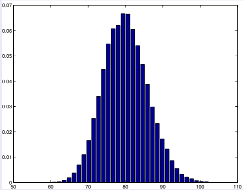
converges to a Gaussian family.

Goal: Fluctuations of Eigenvalues and Traces

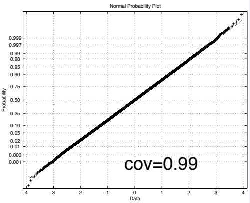
Example: Gaussian random matrix A ($N = 40$, trials=50.000)

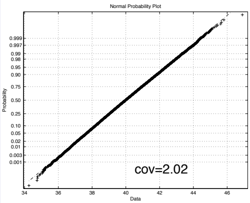


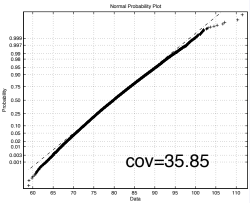




$\text{Var}(\text{Tr}(A)) = 1$ $\qquad$ $\text{Var}(\text{Tr}(A^2)) = 2$ $\qquad$ $\text{Var}(\text{Tr}(A^4)) = 36$

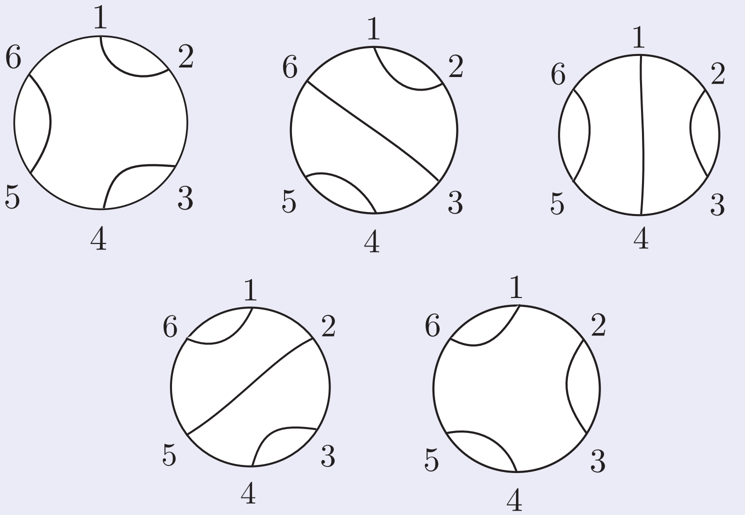






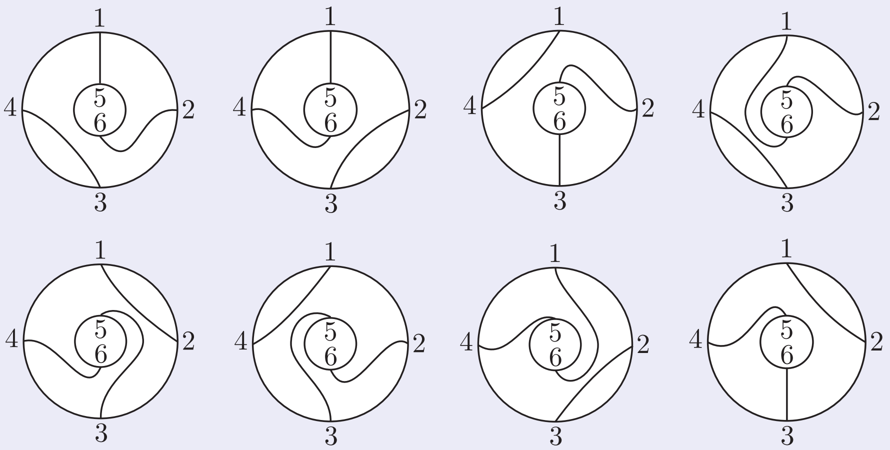
Goal: Fluctuations of Eigenvalues and Traces

Combinatorial understanding of expectation $\alpha_6 = 5$: one circle



Goal: Fluctuations of Eigenvalues and Traces

Combinatorial understanding of fluctuation $\alpha_{4,2} = 8$: two circles



Goal: Fluctuations of Eigenvalues and Traces

Isolation of the concept of “second order freeness“

Calculation of mixed fluctuations out of expectations and fluctuations of individual variables

$$
\begin{aligned}
\lim_{N\to\infty} & \operatorname{cov}\big(\mathrm{Tr}(A_N B_N),\mathrm{Tr}(A_N B_N)\big)\\
&= \lim_{N\to\infty}\Big\{E\big[\mathrm{tr}(A_N A_N)\big]\cdot E\big[\mathrm{tr}(B_N B_N)\big]\\
&\qquad - E\big[\mathrm{tr}(A_N A_N)\big]\cdot E\big[\mathrm{tr}(B_N)\big]\cdot E\big[\mathrm{tr}(B_N)\big]\\
&\qquad - E\big[\mathrm{tr}(A_N)\big]\cdot E\big[\mathrm{tr}(A_N)\big]\cdot E\big[\mathrm{tr}(B_N B_N)\big]\\
&\qquad + E\big[\mathrm{tr}(A_N)\big]\cdot E\big[\mathrm{tr}(A_N)\big]\cdot E\big[\mathrm{tr}(B_N)\big]\cdot E\big[\mathrm{tr}(B_N)\big]\\
&\qquad + \operatorname{cov}\big(\mathrm{Tr}(A_N),\mathrm{Tr}(A_N)\big)\cdot E\big[\mathrm{tr}(B_N)\big]\cdot E\big[\mathrm{tr}(B_N)\big]\\
&\qquad + E\big[\mathrm{tr}(A_N)\big]\cdot E\big[\mathrm{tr}(A_N)\big]\cdot \operatorname{cov}\big(\mathrm{Tr}(B_N),\mathrm{Tr}(B_N)\big)\Big\}
\end{aligned}
$$

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Goal: Fluctuations of Eigenvalues and Traces

Pushing the general theory forward

"partitioned permutations" and heavy cumulants machinery
joint work with Piotr Sniady and Benoit Collins
We have kind of satisfactory combinatorial theory of second order freeness

Goal: Fluctuations of Eigenvalues and Traces

Calculation of special examples

leads often to some independent nice structures (and papers with summer students!)

- combinatorics of Chebyshev polynomials (with T. Kusalik)
- non-commutative cycle lemma (with C. Armstrong, J. Wilson)

- cumulants with products as entries (with E. Tan)

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Generalizations and applications

- extension from complex to real/quaternionic case
  (Jamie and Mihai Popa; Emily Redelmeier)

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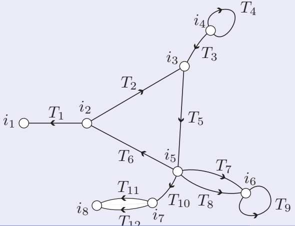
$$S_\tau = \sum_{i_1,i_2,\dots,i_7=1}^{N} t^{(1)}_{i_1i_2} t^{(2)}_{i_3i_2} t^{(3)}_{i_3i_4} t^{(4)}_{i_4i_4} t^{(5)}_{i_5i_3} t^{(6)}_{i_2i_5} t^{(7)}_{i_6i_5} t^{(8)}_{i_6i_5} t^{(9)}_{i_6i_6} t^{(10)}_{i_7i_5} t^{(11)}_{i_8i_7} t^{(12)}_{i_8i_7}.$$



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    - asymptotic freeness of (partial) transposes (Jamie with Mihai Popa)

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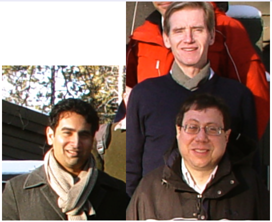
- application to eigenanalysis (with Raj Rao and Alan Edelman)



Section 3

The Future?

There are still many mysteries around second order freeness ...

What to do?

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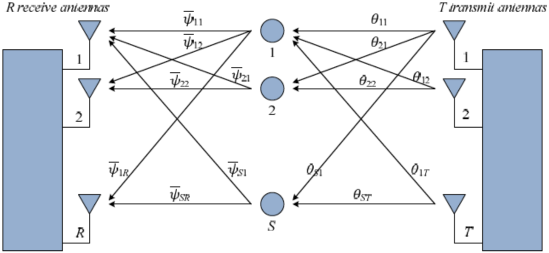
What to do?

- find operator realizations of second order freeness: cyclic Fock space
- better understanding of analytic properties of second order moment and cumulant generating functions
- calculate fluctuations of products of random matrices and apply this to wireless problems

  this relies on (forthcoming) paper of Jamie and Octavio Arizmendi around second order R-diagonal elements

Outage Capacity of Rayleigh Product Channels

(Zheng, Wei, Speicher, Müller, Hämäläinen, Corander)



$$y = Hx + n \qquad H = \Psi^* \Theta$$

$$\mathcal{I} = \log\det\big(I + \gamma HH^*)\big) = \sum_i \log(1 + \gamma\lambda_i(HH^*))$$

$$\sigma_{\mathcal{I}}^2 = \frac{1}{4\pi^2} \int_{\mathcal{C}_x, \mathcal{C}_y} \log(1+\gamma x)\log(1+\gamma y) \cdot \mathrm{cov}\big(G(x), G(y)\big) dxdy$$

In a Far, Far Future ....

Episode Infinity

THE NEW HOPE

directed by J. Mingo and R. Speicher

It is a period of darkness and commutativity.

Random Matrix spaceships have won their first victory against the Commutative Empire.

Can the new weapon,

FREE PROBABILITY AND RANDOM MATRICES

restore Non-Commutativity in the Mathematical Universe and, most importantly, WHEN ....

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Section 4

Conclusion

Second Order Freeness, Mathematics and Life in General is ...

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... sometimes a mass event ...

Second Order Freeness, Mathematics and Life in General is ...

... sometimes a mass event ...

... sometimes a late lonely afternoon at the Institute ...

... but nevertheless we are looking forward to what the next 60 years will bring ...

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Happy Birthday, Jamie!