

**Operator-Valued Semicircular Elements:  
at the Intersection of Combinatorics,  
Random Matrix Theory and Complex  
Analysis**

Roland Speicher  
Queen's University  
Kingston, Canada

joint work with Reza Rashidi Far, Tamer Oraby, Wlodek Bryc,  
Bill Helton

- Rashidi Far, Oraby, Bryc, Speicher:  
On slow-fading MIMO systems with non-separable correlation.  
IEEE Transactions on Information Theory 54 (2008)
- Helton, Rashidi Far, Speicher:  
Operator-valued Semicircular Elements: Solving A Quadratic  
Matrix Equation with Positivity Constraints.  
IMRN 2007

Consider **Gaussian**  $N \times N$ -random matrices

$$A_N = \left( a_{ij} \right)_{i,j=1}^N$$

i.e.,  $A_N$  is  $N \times N$ -matrix, where  $a_{ij}$  are random variables whose distribution is determined as follows:

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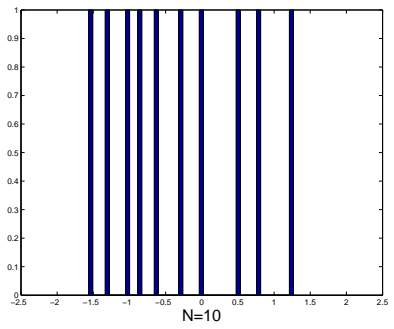
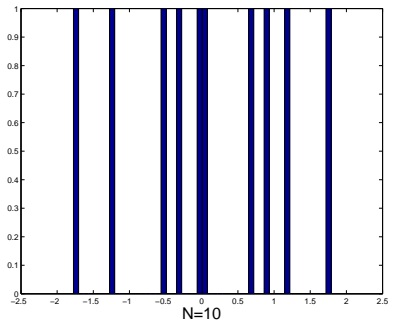
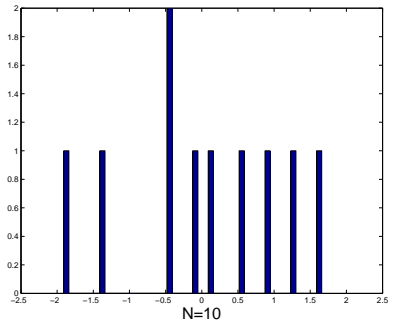
- $A_N$  is selfadjoint, i.e.,  $a_{ji} = \bar{a}_{ij}$

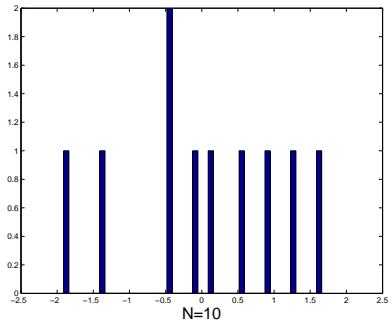
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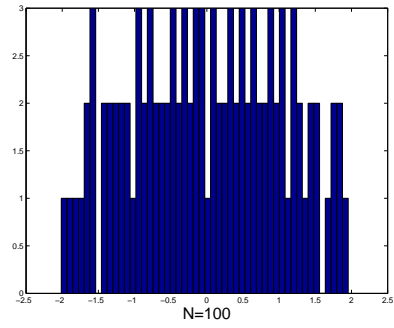
i.e.,  $A_N$  is  $N \times N$ -matrix, where  $a_{ij}$  are random variables whose distribution is determined as follows:

- $A_N$  is selfadjoint, i.e.,  $a_{ji} = \bar{a}_{ij}$
- otherwise, all entries are independent and identically distributed with centered normal distribution of variance  $1/N$

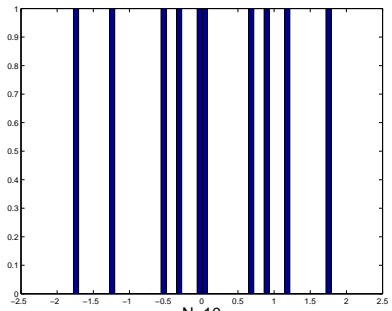




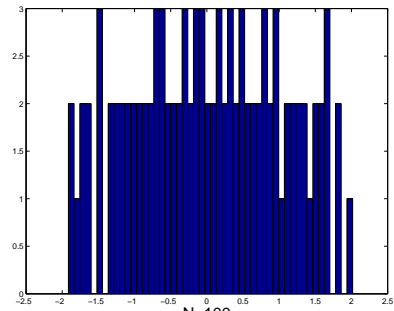
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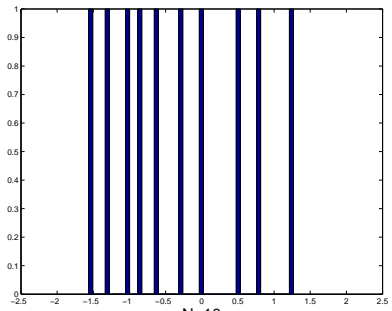
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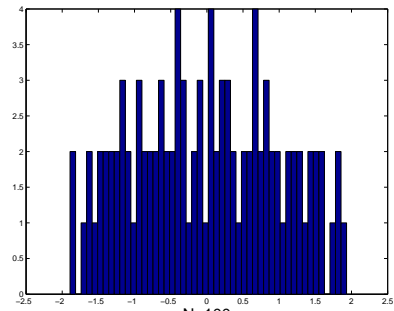
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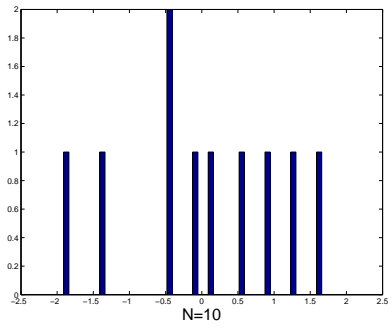
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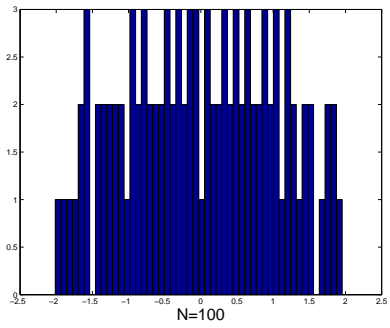
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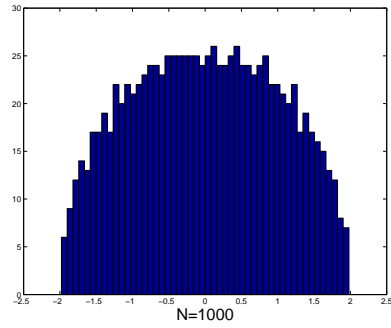
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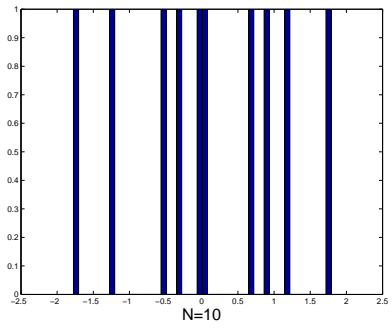
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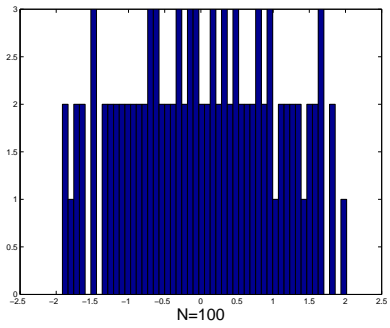
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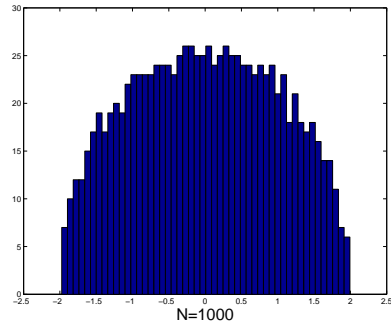
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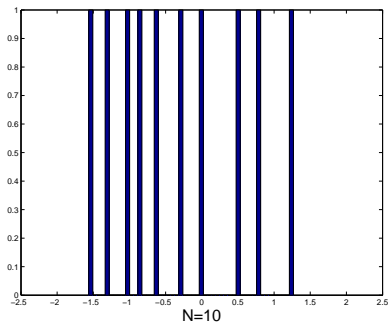
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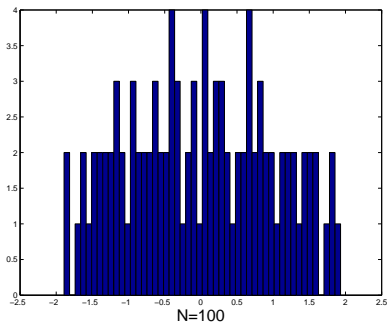
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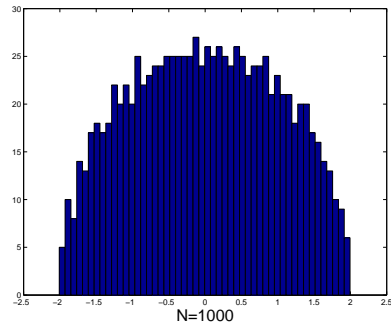
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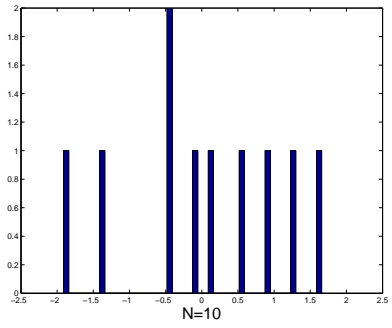


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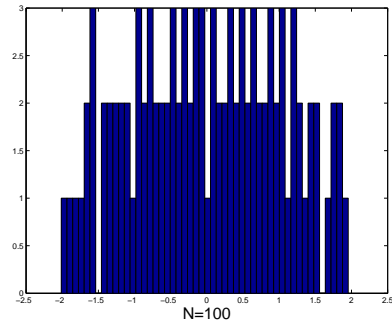


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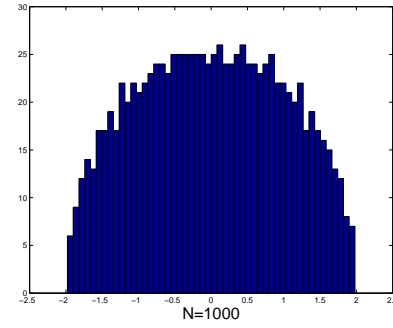




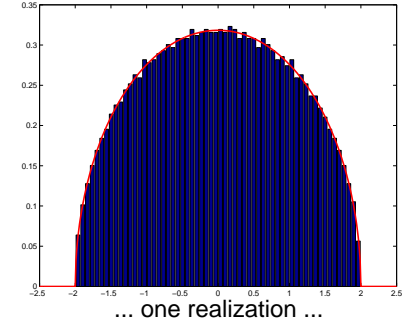
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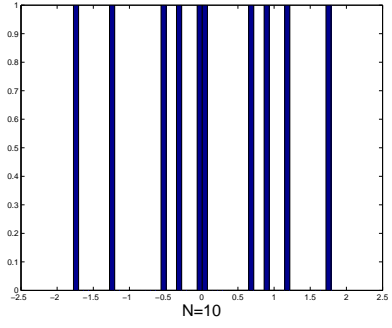
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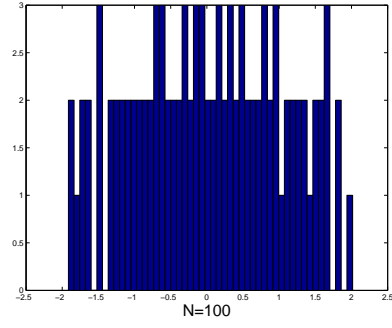
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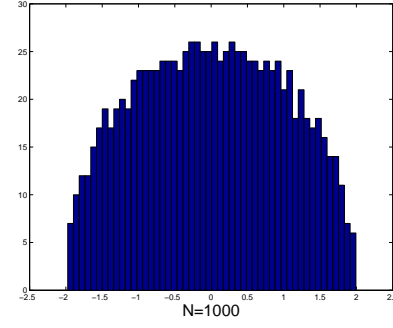
... one realization ...



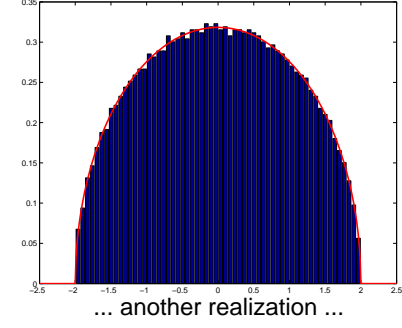
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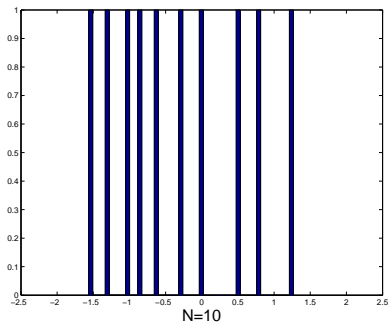
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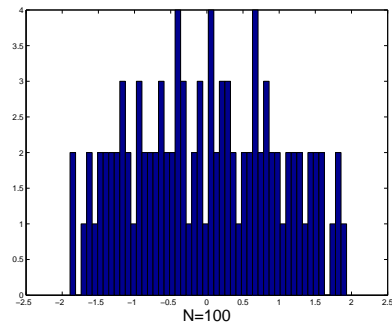
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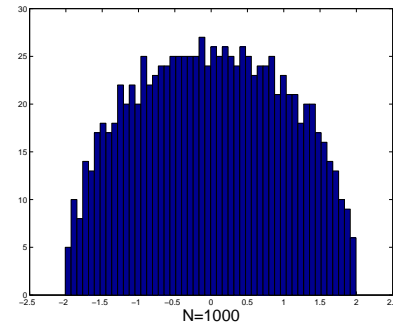
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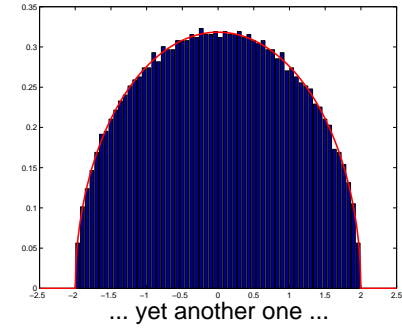
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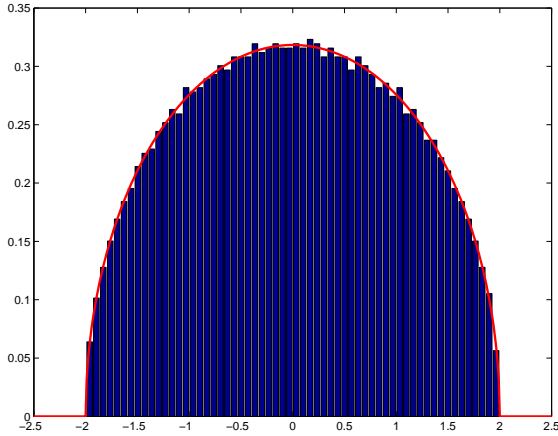
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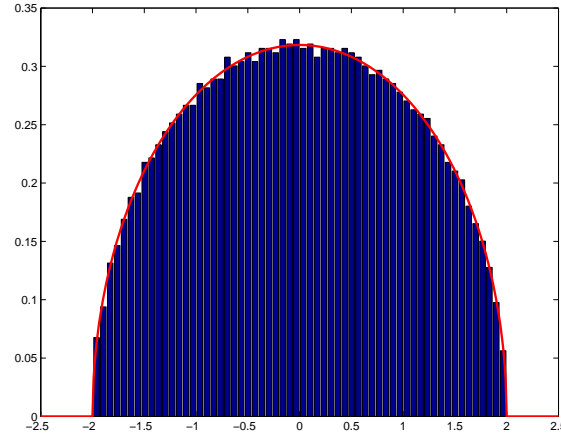
... yet another one ...

Convergence of **typical eigenvalue distribution** of Gaussian  $N \times N$  random matrices to **Wigner's semicircle**

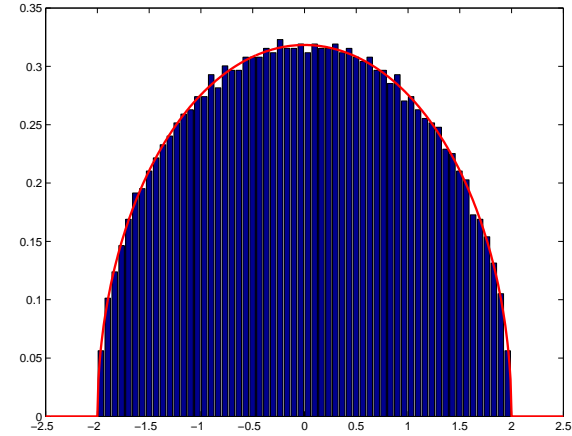
$N = 4000$



... one realization ...



... another realization ...



... yet another one ...

Consider **the empirical eigenvalue distribution of  $A_N$**

$$\mu_{A_N}(\omega) = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(\omega)}$$

$\lambda_i(\omega)$  are the  $N$  eigenvalues (counted with multiplicity) of  $A_N(\omega)$

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Then **Wigner's semicircle law** says that

$$\mu_{A_N} \implies \mu_W \quad \text{almost surely,}$$

i.e., for all continuous and bounded  $f$

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}} f(t) d\mu_{A_N}(t) = \int_{\mathbb{R}} f(t) d\mu_W(t) = \frac{1}{2\pi} \int_{-2}^2 f(t) \sqrt{4 - t^2} dt$$

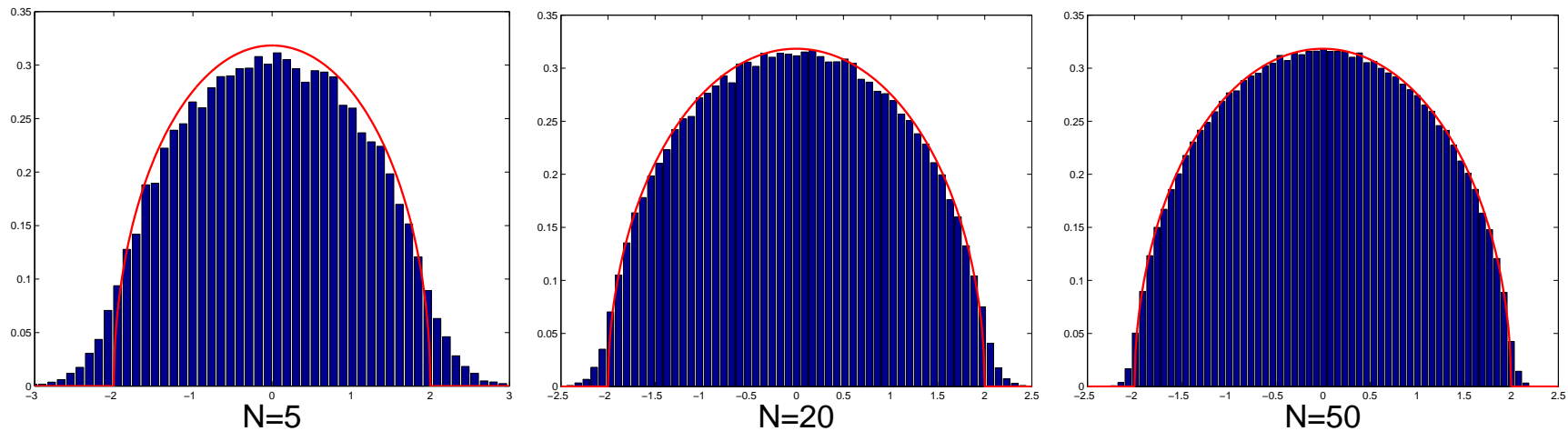
Show

$$\lim_{N \rightarrow \infty} \mu_{A_N}(f) = \mu_W(f) \quad \text{almost surely}$$

in two steps:

- $\lim_{N \rightarrow \infty} E[\mu_{A_N}(f)] = \mu_W(f)$
- $\sum_N \text{Var}[\mu_{A_N}(f)] < \infty$

Convergence of **averaged eigenvalue distribution** of Gaussian  $N \times N$  random matrices to **Wigner's semicircle**



number of realizations = 10000

For

$$\lim_{N \rightarrow \infty} E[\mu_{A_N}(f)] = \mu_W(f)$$

it suffices to treat **convergence of all averaged moments**, i.e.,

$$\lim_{N \rightarrow \infty} E\left[\int t^n d\mu_{A_N}(t)\right] = \int t^n d\mu_W(t) \quad \forall n \in \mathbb{N}$$

Note:

$$E\left[\int t^n d\mu_{A_N}(t)\right] = E\left[\frac{1}{N} \sum_{i=1}^N \lambda_i^n\right] = E[\text{tr}(A_N^n)]$$



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but

$$E[\text{tr}(A_N^n)] = \frac{1}{N} \sum_{i_1, \dots, i_n=1}^N E[a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}]$$

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Asymptotically, for  $N \rightarrow \infty$ , only **non-crossing pairings** survive:

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Define limiting **semicircle element**  $s$  by

$$\varphi(s^n) := \#NC_2(n).$$

( $s \in \mathcal{A}$ , where  $\mathcal{A}$  is some unital algebra,  $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ )

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Then we say that our Gaussian random matrices  $A_N$  converge in distribution to the semicircle element  $s$ ,

$$A_N \xrightarrow{\text{distr}} s$$

**What is distribution of  $s$ ?**

Claim:

$$\varphi(s^n) = \int t^n d\mu_W(t)$$

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more concretely:

$$\#NC_2(n) = \frac{1}{2\pi} \int_{-2}^{+2} t^n \sqrt{4 - t^2} dt$$

What is distribution of  $s$ ?

$$n = 2: \varphi(s^2) =$$

$$n = 4: \varphi(s^4) =$$

$$n = 6: \varphi(s^6) =$$



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□

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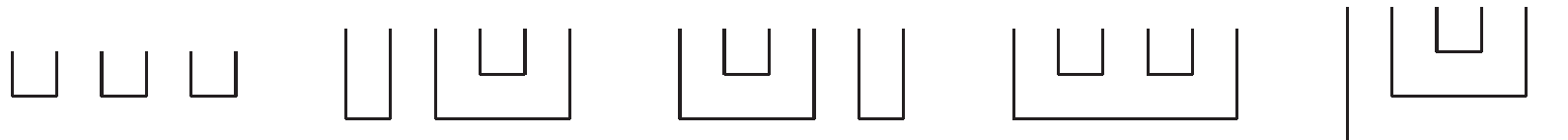
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$$n = 6: \varphi(s^6) = 5$$



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- $C_k = \frac{1}{k+1} \binom{2k}{k}$
- $C_k$  is determined by  $C_0 = C_1 = 1$  and the recurrence relation

$$C_k = \sum_{l=1}^k C_{l-1} C_{k-l}.$$

$$\varphi(s^{2k}) = \sum_{l=1}^k \varphi(s^{2l-2})\varphi(s^{2k-2l})$$

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Put

$$M(z) := \sum_{n=0}^{\infty} \varphi(s^n)z^n = 1 + \sum_{k=1}^{\infty} \varphi(s^{2k})z^{2k}$$



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$$\begin{aligned} M(z) &= 1 + z^2 \sum_{k=1}^{\infty} \sum_{l=1}^k \varphi(s^{2l-2})z^{2l-2} \varphi(s^{2k-2l})z^{2k-2l} \\ &= 1 + z^2 M(z) \cdot M(z) \end{aligned}$$

$$M(z) = 1 + z^2 M(z) \cdot M(z)$$

Instead of moment generating series  $M(z)$  consider **Cauchy transform**

$$G(z) := \varphi\left(\frac{1}{z-s}\right)$$

Note

$$G(z) = \sum_{n=0}^{\infty} \frac{\varphi(s^n)}{z^{n+1}} = \frac{1}{z} \sum_{n=0}^{\infty} \varphi(s^n) \left(\frac{1}{z}\right)^n = \frac{1}{z} M(1/z),$$

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thus

$$zG(z) = 1 + G(z)^2$$

For any probability measure  $\mu$  on  $\mathbb{R}$  corresponding Cauchy transform

$$G(z) := \int_{\mathbb{R}} \frac{1}{z - t} d\mu(t)$$

is analytic function on complex upper half plane  $\mathbb{C}^+$  and allows to recover  $\mu$  via **Stieltjes inversion formula**

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \operatorname{Im} G(t + i\varepsilon)$$

For semicircle  $s$ :

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implies

$$G(z) = \frac{z - \sqrt{z^2 - 4}}{2}$$

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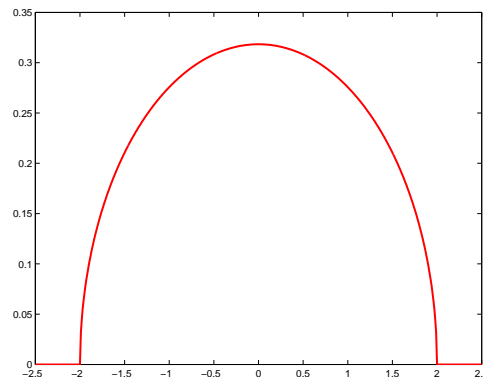
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and thus

$$d\mu(t) = \frac{1}{2\pi} \sqrt{4 - t^2} dt \quad \text{on } [-2, 2]$$



## Consider now more general random matrices

- Keep the entries independent, but change distribution of entries, globally or depending on position of entry

Arnold

Bai und Silverstein

Molchanov, Pastur, Khorunzhii (1996)

Khorunzhy, Khoruzhenko, Pastur (1996)

Shlyakhtenko (1996)

Guionnet (2002)

Anderson, Zeitouni (2006)



- Keep the distributions normal, but allow correlations between entries
  - for weak correlations one still gets semicircle
    - Chatterjee (2006)
    - Götze + Tikhomirov (2005)
    - Schenker und Schulz-Baldes (2006)
  - for stronger correlations other distributions occur
    - Boutet de Monvel, Khorunzhy, Vasilchuck (1996)
    - Girko (2001)
    - Hachem, Loubaton, Najim (2005)
    - Anderson, Zeitouni (2007)
    - Rashidi Far, Oraby, Bryc, Speicher (2007)

Consider **block matrix**

$$X_N = \begin{pmatrix} A_N & B_N & C_N \\ B_N & A_N & B_N \\ C_N & B_N & A_N \end{pmatrix},$$

where  $A_N, B_N, C_N$  are independent Gaussian  $N \times N$ -random matrices.

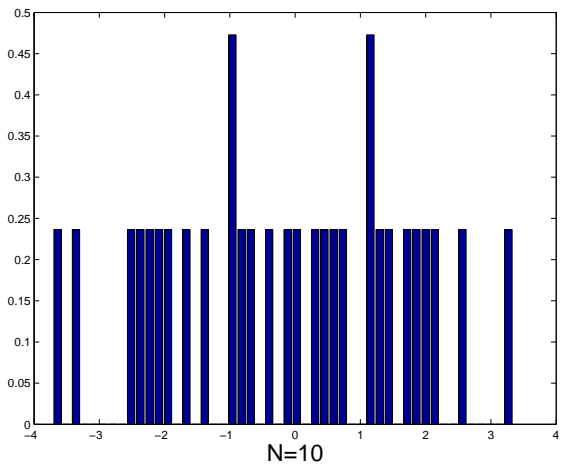
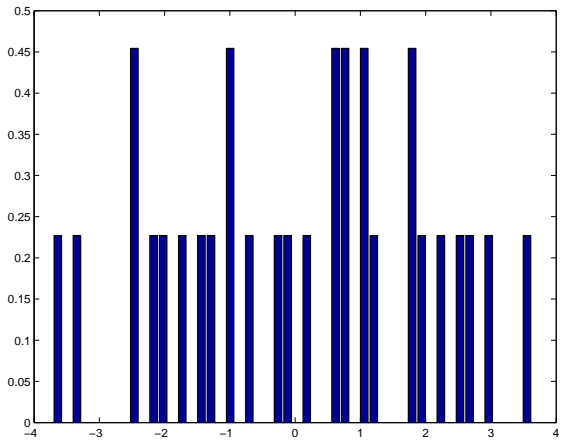
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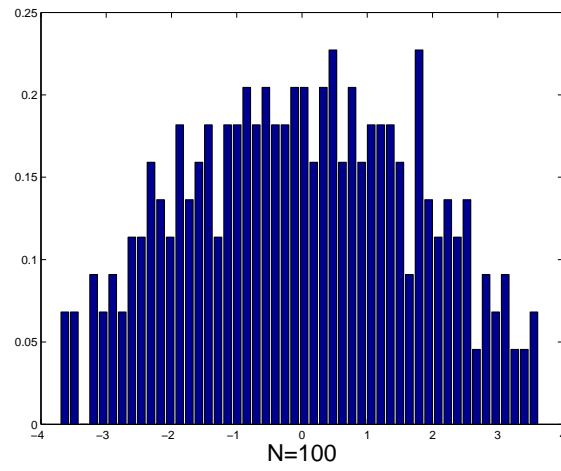
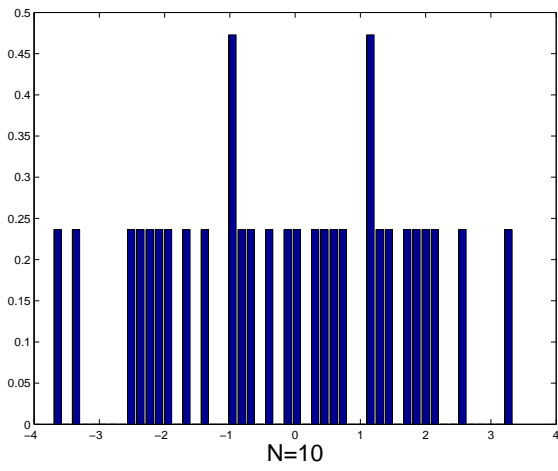
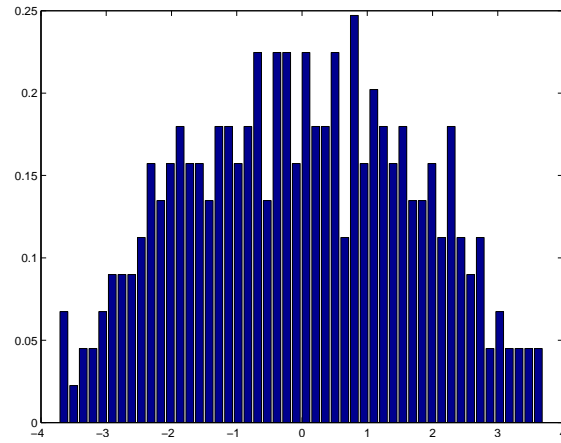
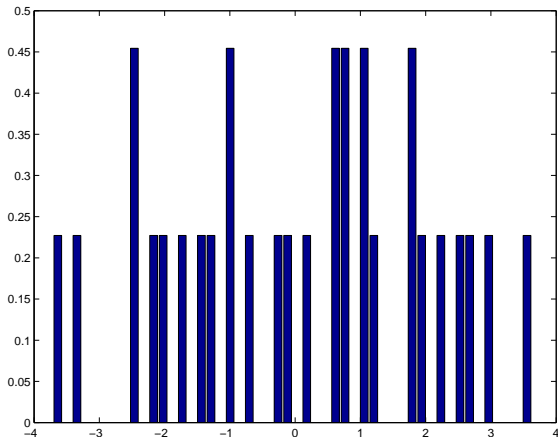
where  $A_N, B_N, C_N$  are independent Gaussian  $N \times N$ -random matrices.

**What is eigenvalue distribution of  $X_N$  for  $N \rightarrow \infty$ ?**

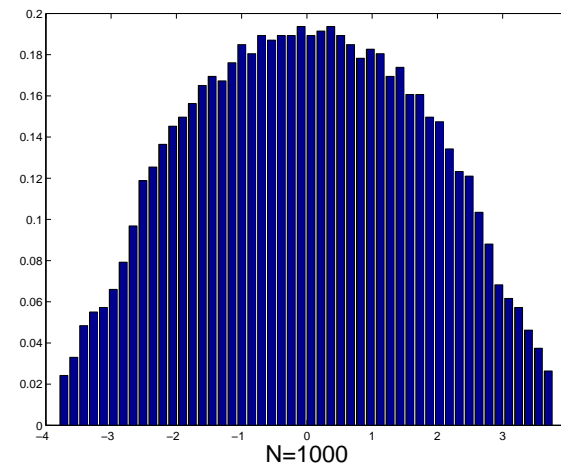
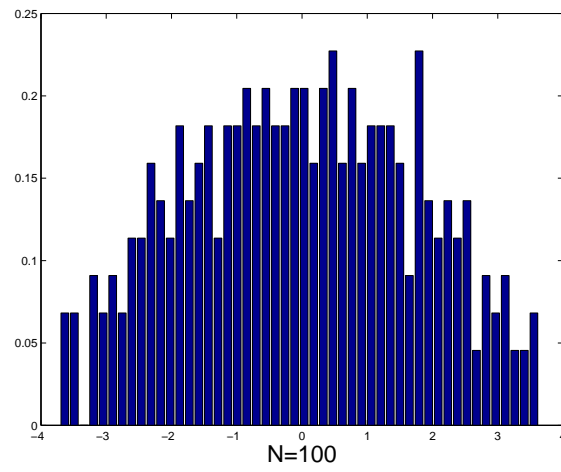
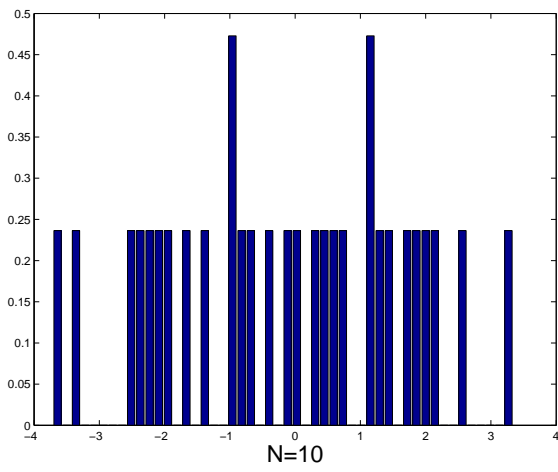
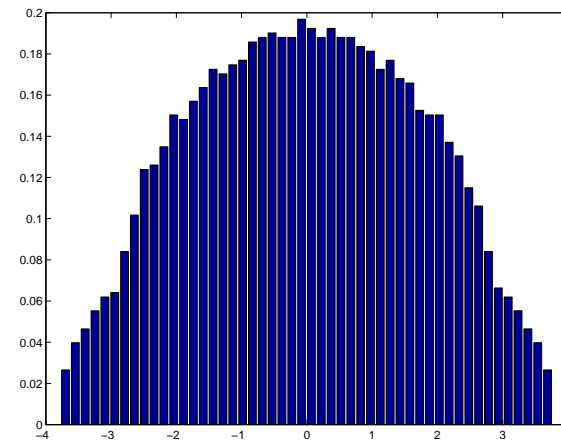
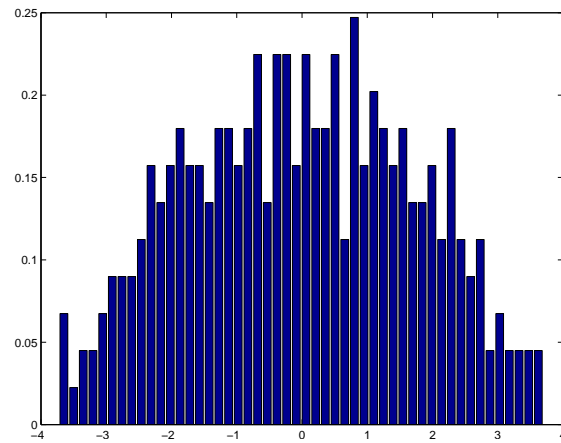
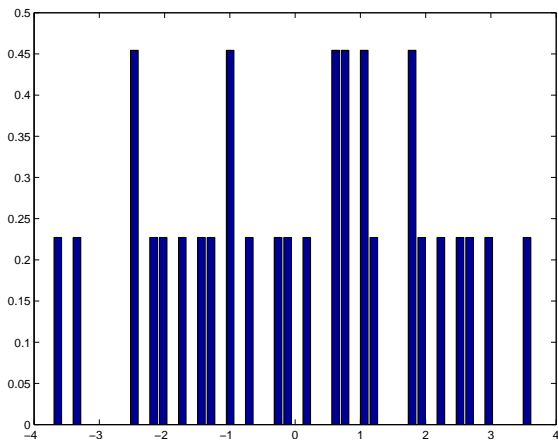
# eigenvalue distribution for one realisation



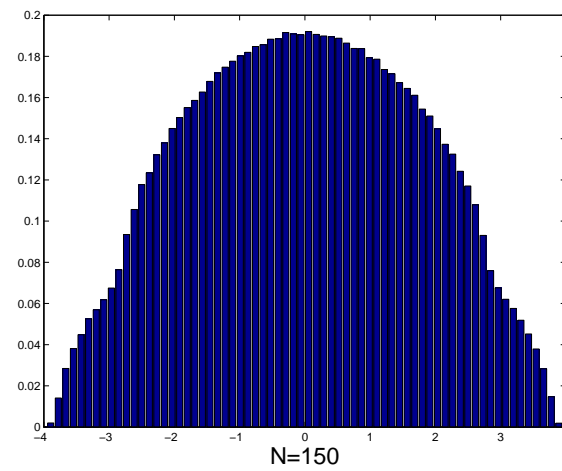
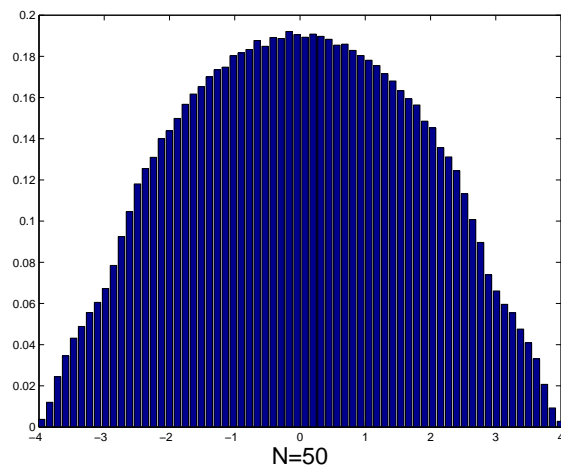
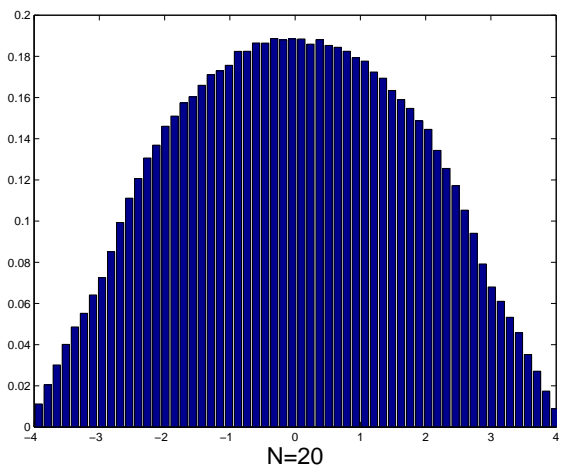
# eigenvalue distribution for one realisation



# eigenvalue distribution for one realisation



# averaged eigenvalue distributions



This limiting distribution is not a semicircle, and it cannot be described nicely within usual free probability theory.



This limiting distribution is not a semicircle, and it cannot be described nicely within usual free probability theory.

However, it fits well into the frame of

**operator-valued free probability theory!**

What is an **operator-valued probability space**?

scalars  $\longrightarrow$  operator-valued scalars  
 $\mathbb{C}$   $\mathcal{B}$

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$E : \mathcal{A} \rightarrow \mathcal{B}$

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$$E[b_1 a b_2] = b_1 E[a] b_2$$

moments  $\longrightarrow$  operator-valued moments

$\varphi(a^n)$

$E[ab_1 a b_2 a \cdots a b_{n-1} a]$

$\mathcal{B}$  can be a general algebra of bounded operators on a Hilbert space; in the following just consider

$$\mathcal{B} \cong \text{matrix algebra}$$

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Consider an operator-valued probability space

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- higher moments of  $s$  are given in terms of second moments by summing over non-crossing pairings



$$E[sbs] = \eta(b)$$

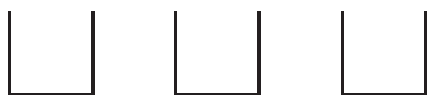
*sbs*



$$E[sbs] = \eta(b) \quad \begin{array}{c} sbs \\ \square \end{array}$$

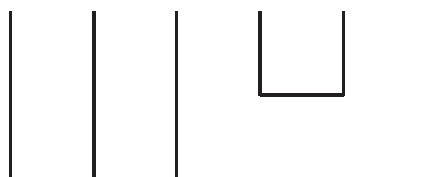
$$E[sb_1sb_2s \cdots sb_{n-1}s] = \sum_{\pi \in NC_2(n)} \left( \text{iterated application of } \eta \text{ according to } \pi \right)$$

$sb_1 sb_2 sb_3 sb_4 sb_5 s$



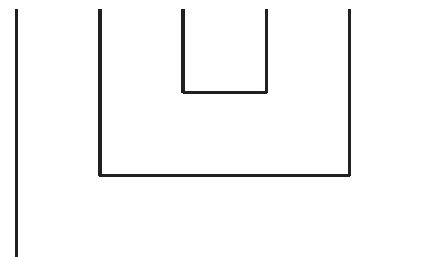
$$\eta(b_1) \cdot b_2 \cdot \eta(b_3) \cdot b_4 \cdot \eta(b_5)$$

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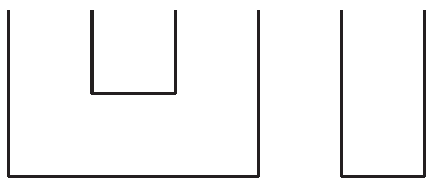
$$\eta(b_1) \cdot b_2 \cdot \eta(b_3 \cdot \eta(b_4) \cdot b_5)$$

$sb_1 sb_2 sb_3 sb_4 sb_5 s$



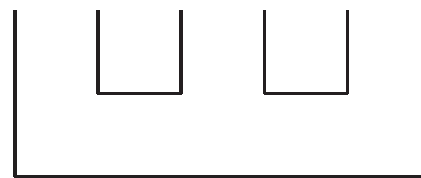
$$\eta(b_1 \cdot \eta(b_2 \cdot \eta(b_3) \cdot b_4) \cdot b_5)$$

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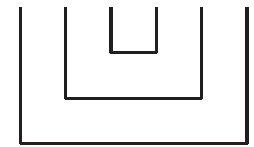
$$E[sb_1sb_2sb_3sb_4sb_5s] = \eta(b_1) \cdot b_2 \cdot \eta(b_3) \cdot b_4 \cdot \eta(b_5)$$



$$+ \eta(b_1) \cdot b_2 \cdot \eta(b_3 \cdot \eta(b_4) \cdot b_5)$$



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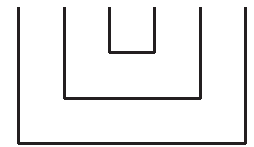
$$E[ssssss] = \eta(1) \cdot \eta(1) \cdot \eta(1)$$

$$+ \eta(1) \cdot \eta(\eta(1))$$

$$+ \eta(\eta(\eta(1)))$$

$$+ \eta(\eta(1)) \cdot \eta(1)$$

$$+ \eta(\eta(1) \cdot \eta(1))$$



We have the recurrence relation

$$E[s^{2k}] = \sum_{l=1}^k \eta(E[s^{2l-2}]) \cdot E[s^{2k-2l}].$$

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Put

$$M(z) := \sum_{n=0}^{\infty} E[s^n] z^n = 1 + \sum_{k=1}^{\infty} E[s^{2k}] z^{2k},$$

thus

$$M(z) = 1 + z\eta(M(z)) \cdot M(z)$$

Consider the **operator-valued Cauchy transform**

$$G(z) := E\left[\frac{1}{z - s}\right]$$

for  $z \in \mathbb{C}^+$ .

Note

$$G(z) = E\left[\frac{1}{z} \cdot \frac{1}{1 - sz^{-1}}\right] = \frac{1}{z}M(sz^{-1}),$$

thus

$$zG(z) = 1 + \eta(G(z)) \cdot G(z)$$



Thus: operator-valued Cauchy-transform of  $s$

$$G : \mathbb{C}^+ \rightarrow \mathcal{B}$$

satisfies

- $G$  analytic
- $G$  solution of

$$zG(z) = 1 + \eta(G(z)) \cdot G(z)$$

- $G(z) \sim \frac{1}{z}\mathbf{1}$  for  $z \rightarrow \infty$

## back to random matrices

special classes of random matrices are asymptotically described by operator-valued semicircular elements, e.g.

- band matrices (Shlyakhtenko 1996)
- block matrices (Rashidi Far, Oraby, Bryc, Speicher 2006)

Example:

$$X_N = \begin{pmatrix} A_N & B_N & C_N \\ B_N & A_N & B_N \\ C_N & B_N & A_N \end{pmatrix},$$

where  $A_N, B_N, C_N$  are independent Gaussian  $N \times N$  random matrices

For  $N \rightarrow \infty$ ,  $X_N$  converges to

$$s = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_2 \\ s_3 & s_2 & s_1 \end{pmatrix},$$

where  $s_1, s_2, s_3$  is free semicircular family.

$$s = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_2 \\ s_3 & s_2 & s_1 \end{pmatrix}, \quad s_1, s_2, s_3 \in (\tilde{\mathcal{A}}, \varphi)$$

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- $\eta : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$  given by  $\eta(D) = E[sDs]$

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Asymptotic eigenvalue distribution  $\mu$  of  $X_N$  is given by distribution of  $s$  with respect to  $\text{tr}_3 \otimes \varphi$ :

$$H(z) = \int \frac{1}{z-t} d\mu(t) = \text{tr}_3 \otimes \varphi\left(\frac{1}{z-s}\right)$$



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and  $G(z) = E \left[ \frac{1}{z-s} \right]$  is solution of

$$zG(z) = 1 + \eta(G(z)) \cdot G(z)$$

$$X = \begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix} :$$

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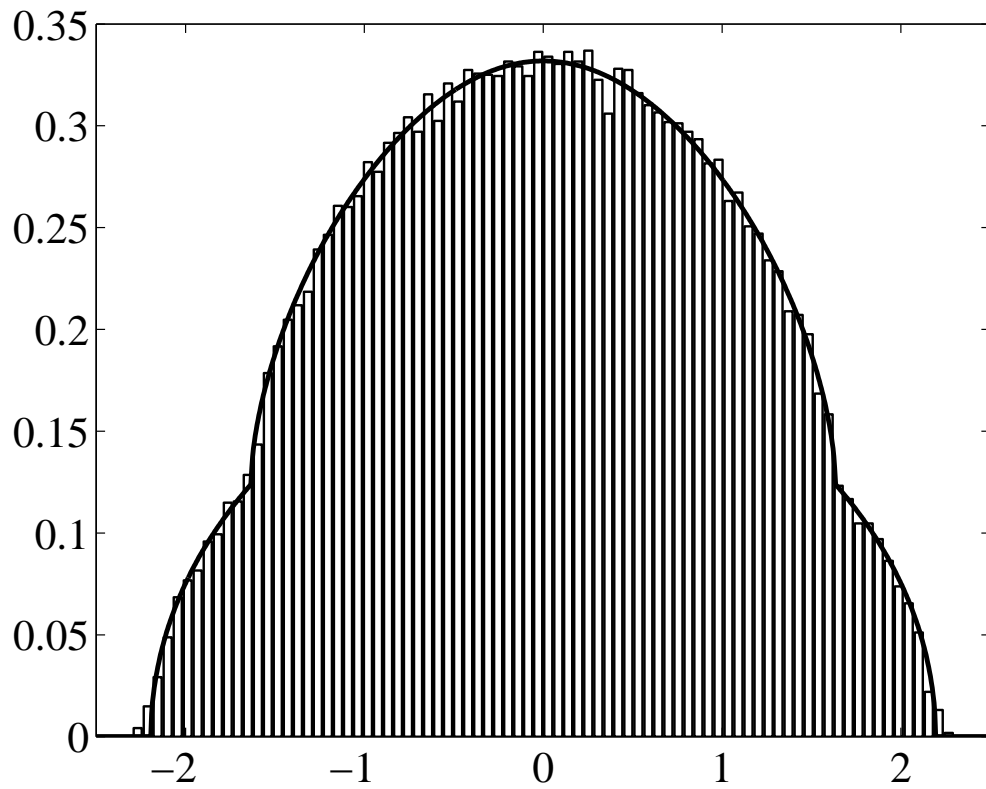
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$$zG(z) = 1 + \eta(G(z)) \cdot G(z)$$

$$H(z) = \text{tr}_3(G(z)) = \frac{1}{3}(2f(z) + g(z))$$

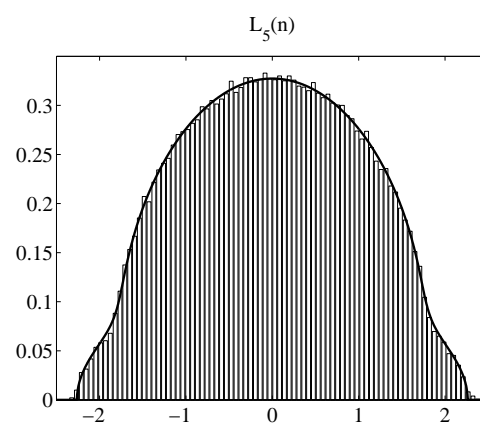
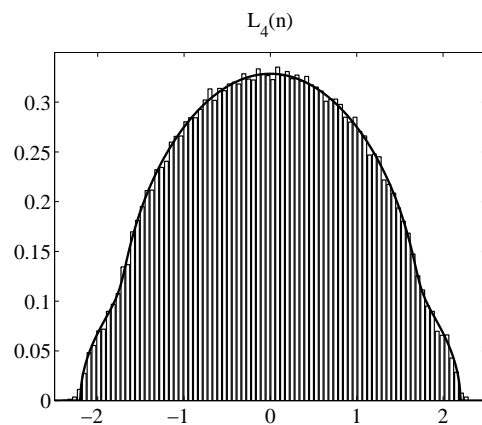
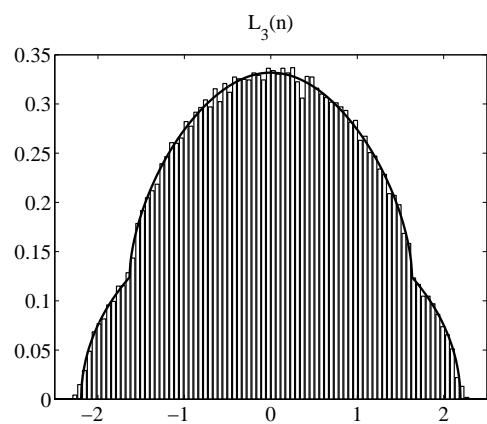
Comparison of this solution with simulations

$$L_3(n)$$





some more examples



$$\begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix}$$

$$\begin{pmatrix} A & B & C & D \\ B & A & B & C \\ C & B & A & B \\ D & C & B & A \end{pmatrix}$$

$$\begin{pmatrix} A & B & C & D & E \\ B & A & B & C & D \\ C & B & A & B & C \\ D & C & B & A & B \\ E & D & C & B & A \end{pmatrix},$$

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... however, we are now actually facing another problem:

How to deal with our special

**matrix-valued quadratic equations?**

**Problem:** We have to solve

$$zG(z) = 1 + \eta(G(z)) \cdot G(z) \quad (*)$$

for  $z$  close to real axis.

Recall: Stieltjes inversion formula

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \operatorname{Im} H(t + i\varepsilon)$$

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For fixed  $z$ ,

$$\begin{aligned} (*) &= \text{quadratic matrix equation} \\ &\hat{=} \text{system of quadratic equations} \end{aligned}$$

has many solutions!

**How do we find the right one?**

$$X = \begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix}, \quad G(z) = \begin{pmatrix} f(z) & 0 & h(z) \\ 0 & g(z) & 0 \\ h(z) & 0 & f(z) \end{pmatrix}$$

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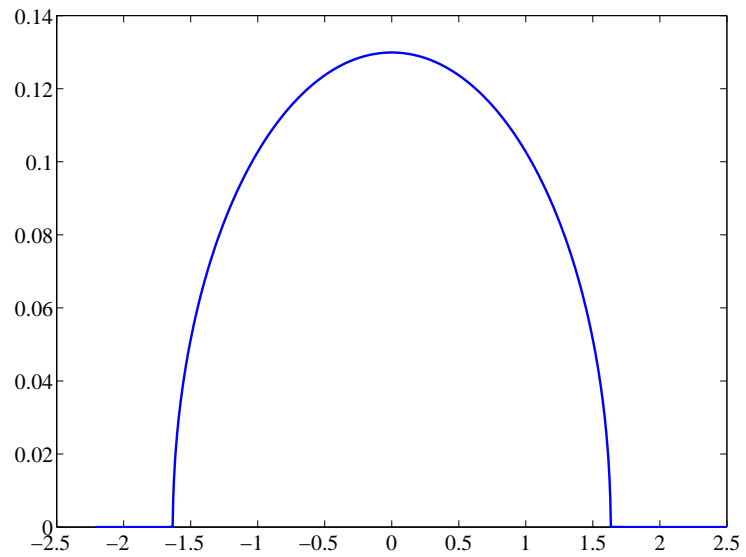
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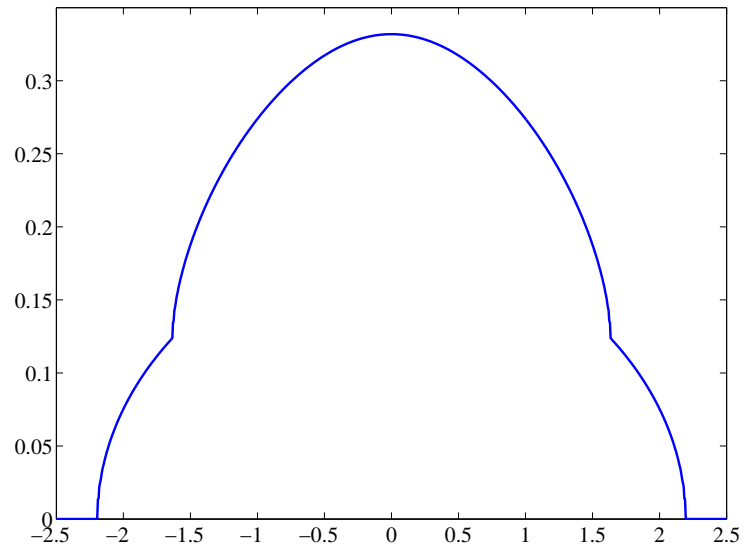
$$zf = 1 + \frac{g(f+h) + 2(f^2 + h^2)}{3}$$

$$zg = 1 + \frac{g(g + 2(f+h))}{3}$$

$$zh = \frac{4fh + g(f+h)}{3}.$$



solving the system  
by Newton's method  
with 'bad' initial values



correct solution



We are interested in a solution with a special **positivity property**:

For  $\text{Im } z > 0$  we have

$$\text{Im } G(z) = \frac{1}{2i} E\left[\frac{1}{z-s} - \frac{1}{\bar{z}-s}\right] \leq -\frac{\text{Im } z}{|z| + \|s\|^2} \mathbf{1}$$

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Thus: we look for

- a solution of  $zG(z) = 1 + \eta(G(z)) \cdot G(z)$
- with  $\text{Im } G(z) := \frac{G(z) - G(z)^*}{2i} < 0$

Set

$$G(z) := -iW(z)$$

Then

- $\operatorname{Re} W(z) := \frac{W(z) + W(z)^*}{2} > 0$
- $-izW(z) + \eta(W(z)) \cdot W(z) = 1$

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**Claim:** For each  $z$  with  $\operatorname{Im} z > 0$  there is exactly one solution with this property!

$$-izW(z) + \eta(W(z)) \cdot W(z) = 1 \quad \iff \quad \mathcal{F}_z(W) = W \quad (**)$$

for

$$\mathcal{F}_z(W) = \frac{1}{-iz1 + \eta(W)}$$

$$-izW(z) + \eta(W(z)) \cdot W(z) = 1 \quad \Longleftrightarrow \quad \mathcal{F}_z(W) = W \quad (**)$$

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Put

$$\mathcal{B}_+ := \{W \in \mathcal{B} \mid \operatorname{Re} W > 0\}$$

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**Theorem [Helton, Rashidi Far, Speicher 2007]:** For  $\operatorname{Im} z > 0$  there exists exactly one solution  $W \in \mathcal{B}_+$  to (\*\*); this  $W$  is the limit of iterates

$$W_n = \mathcal{F}_z^n(W_0)$$

for any  $W_0 \in \mathcal{B}_+$ .

Note:  $\mathcal{F}_z$  is **not** a contraction in operator norm.



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However, is contraction in some other, canonical metric!

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**Theorem [Earle, Hamilton 1968]:** Consider  $X$  complex Banach space,  $\emptyset \neq \mathcal{D}$  domain,  $h : \mathcal{D} \rightarrow \mathcal{D}$  bounded holomorphic function.

If  $h(\mathcal{D})$  lies strictly<sup>(\*)</sup> inside  $\mathcal{D}$ , then  $h$  is a **strict contraction in the Caratheodory metric  $\rho$** , thus has a unique fixed point in  $\mathcal{D}$ .

Furthermore,  $\exists m > 0 : \rho(x, y) \geq m\|x - y\|$  for all  $x, y \in \mathcal{D}$  and thus  $(h^n(x_0))_n$  converges in norm to fixed point, for all  $x_0 \in \mathcal{D}$

(\*)  $\exists \varepsilon > 0 \forall x \in \mathcal{D} : B_\varepsilon(h(x)) \subset \mathcal{D}$

in our case:

- $h = \mathcal{F}_z$  analytic
- $\mathcal{D} = R_b := \{W \in \mathcal{B}_+ \mid \|W\| < b\}$

Check that  $\mathcal{F}_z(R_b)$  lies strictly in  $R_b$ .

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Note:  $\mathcal{B}_+ = \bigcup_{b>0} R_b$

**Remark:** Same arguments work for

$$ZG(Z) = 1 + \eta(G(Z)) \cdot G(Z)$$

where  $Z \in \mathcal{B}$  with  $\text{Im } Z = \frac{Z - Z^*}{2i} > 0$  and invertible.

**Theorem [Helton, Rashidi Far, Speicher 2007]:** For fixed  $V \in \mathcal{B}_+$  there exists exactly one solution  $W \in \mathcal{B}_+$  to

$$VW + \eta(W)W = 1;$$

this  $W$  is the limit of iterates  $W_n = \mathcal{F}_V^n(W_0)$  for any  $W_0 \in \mathcal{A}_+$ , where

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Furthermore,

$$\|W\| \leq \|(\operatorname{Re} V)^{-1}\|, \quad \operatorname{Re} W \geq \frac{1}{m^2 \cdot \|(\operatorname{Re} V)^{-1}\|}$$

where

$$m := \|V\| + \|\eta\| \cdot \|(\operatorname{Re} V)^{-1}\|$$

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- operator-valued semicircular elements have an accessible operator-valued structure, in terms of combinatorics of non-crossing pairings
- corresponding equation for operator-valued Cauchy transform (**quadratic matrix equation**) has exactly one solution with the right positivity properties

## Literature on the combinatorial side of free probability

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