Free Probability, Random Matrices, and Non-Commutative Rational Functions

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(joint work with S. Belinschi, J. W. Helton, T. Mai, and S. Yin)



Section 1

Introduction



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which "non-commuting variables"

- (random) matrices of size $N \times N$
- ullet operators on Hilbert spaces (corresponding to $N \to \infty)$



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- (maybe even: non-commutative analytic functions)



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- (maybe even: non-commutative analytic functions)

what means "distribution"

- algebraic/combinatorial distribution: collection of moments
- analytic distribution: probability measure

Definition

A non-commutative probability space (\mathcal{A}, φ) consists of

- \bullet a complex algebra ${\cal A}$ with unit $1_{\cal A}$ and
- a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying $\varphi(1_{\mathcal{A}}) = 1$ (expectation).

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Example

• $(L^{\infty}(\Omega, \mathbb{P}), \mathbb{E})$, where $(\Omega, \Sigma, \mathbb{P})$ is a classical probability space and \mathbb{E} the usual expectation that is given by $\mathbb{E}[X] = \int_{\Omega} X(\omega) d\mathbb{P}(\omega)$



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- $(M_N(\mathbb{C}), \operatorname{tr}_N)$, where tr_N is the normalized trace on $M_N(\mathbb{C})$



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• random matrices $((L^{\infty}(\Omega) \otimes M_n(\mathbb{C}), \mathbb{E} \otimes \operatorname{tr}_n)$



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We call (\mathcal{A}, φ) a C^* -probability space if

• \mathcal{A} is a unital C^* -algebra;

i.e., ${\mathcal A}$ consists of bounded operators on a Hilbert space ${\mathcal H}$

• φ is positive (i.e. $\varphi(x^*x) \ge 0$ for $x \in \mathcal{A}$) and hence a state on \mathcal{A} ;

i.e., φ is of the form $\varphi(x)=\langle\Omega,x\Omega\rangle$ for some unit vector $\Omega\in\mathcal{H}$

Non-commutative distributions

Definition ("combinatorial distribution")

Let (\mathcal{A}, φ) be a non-commutative probability space. Let $(x_i)_{i \in I}$ be a family of non-commutative random variables. We call the collection of all mixed moments

$$\{\varphi(x_{i_1}\cdots x_{i_k}) \mid k \in \mathbb{N}, i_1, \dots, i_k \in I\}$$

their (joint) distribution.



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Definition ("analytic distribution")

Let (\mathcal{A}, φ) be a C^* -probability space.

For any $x = x^* \in A$, the distribution of x can be identified with the unique Borel probability measure μ_x on the real line \mathbb{R} that satisfies

$$\varphi(x^k) = \int_{\mathbb{R}} t^k \, d\mu_x(t) \quad \text{for all } k \in \mathbb{N}_0.$$

Section 2

Random Matrices



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- $\bullet\,$ 1967: Marchenko and Pastur described $N\to\infty$ asymptotics of "Wishart matrices"
- 1972: Montgomery and Dyson discovered relation between zeros of the Riemann zeta function and eigenvalues of random matrices
- since 2000: random matrix theory developed into a central subject in mathematics, with many different connections





Oberwolfach workshop "Random Matrices" 2000



What are random matrices

- $\bullet\,$ random matrices are "typical" sequences of $N\times N$ matrices, with growing N
- more precisely: random matrices are sequences of $N \times N$ matrices whose entries are chosen randomly (according to a prescribed distribution)



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Fundamental observation

Many random matrices show for $N \to \infty$ almost surely a deterministic (and interesting) behavior



Definition

A Wigner random matrix $X_N = \frac{1}{\sqrt{N}} (x_{ij})_{i,j=1}^N$

- is symmetric: $X_N^* = X_N$, i.e. $x_{ij} = x_{ji}$ for all i, j
- entries $\{x_{ij} \mid 1 \le i \le j \le N\}$ are chosen according to independent coin tosses head = +1, tail = -1





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one-matrix case: limit of X_N

 \bullet the almost sure limit of μ_{X_N} is given by a probability measure μ



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multi-matrix case: limit of X_N, Y_N

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 $\lim_{N\to\infty} \operatorname{tr}_N(q(X_N,Y_N)) = \varphi(q(x,y)) \qquad \text{for all monomials } q$

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• this means then in particular, that

$$p(X_N, Y_N) \to p(x, y)$$
 for all polynomials p

Random matrices and operators

Fundamental observation of Voiculescu (1991)



limit of random matrices can often be described by "nice" and "interesting" operators on Hilbert spaces (which, in the case of several matrices, describe interesting von Neumann algebras)



One-matrix case: classical random matrix case





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Multi-matrix case: independent (non-commuting) Wigner matrices

• X_N , Y_N independent Wigner matrices with N = 3000

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Multi-matrix case: independent (non-commuting) Wigner matrices

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- $x = l_1 + l_1^*$, $y = l_2 + l_2^*$ two copies of one-sided shift in different directions (creation and annihilation operators on full Fock space; Cuntz algebra) $\varphi(a) = \langle \Omega, a\Omega \rangle$



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• $X_N, Y_N \rightarrow x, y$ in distribution (Voiculescu 1991)

- $p(X_N, Y_N) \rightarrow p(x, y)$ in distribution
- $\|p(X_N, Y_N)\| \to \|p(x, y)\|$ (Haagerup and Thorbjørnsen 2005)



What are those limit operators x, y good for?

Basic Theorem (Voiculescu 1991)

For many random matrix models X_N, Y_N (like for independent Wigner matrices) the limit operators x, y are

• free in the sense of Voiculescu's free probability theory



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Tools from free probability theory: for x and y free we have

- free convolution: the distribution of x + y can effectively be calculated in terms of the distribution of x and the distribution of y
- matrix-valued free convolution: the matrix-valued distribution of α₀ ⊗ 1 + α₁ ⊗ x + α₂ ⊗ y can be calculated in terms of the distribution of x and the distribution of y



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Problem: we want general polynomials, not just sums!

We are actually interested not just in the distribution of x + y, but much more general in the distribution of non-linear polynomials p(x, y)!

Section 3

Linearization



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non-linear problem \rightarrow operator-valued linear problem

 $p(x_1,\ldots,x_m) \to \hat{p} := \alpha_0 \otimes 1 + \alpha_1 \otimes x_1 + \cdots + \alpha_m \otimes x_m$



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Example

Is $p(x,y) = xy + yx + x^2$ invertible?

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Hence $p = xy + yx + x^2$ is invertible if and only if

$$\hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$

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is invertible. Actually: $p^{-1} = [\hat{p}^{-1}]_{1,1}$

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Example

Is
$$p(x,y) - \mathbf{z} = xy + yx + x^2 - \mathbf{z}$$
 (for $\mathbf{z} \in \mathbb{C}$) invertible? We have

$$\begin{pmatrix} xy + yx + x^2 - \mathbf{z} & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & y + \frac{x}{2} & x\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\mathbf{z} & x & y + \frac{x}{2}\\ x & 0 & -1\\ y + \frac{x}{2} & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0\\ y + \frac{x}{2} & 1 & 0\\ x & 0 & 1 \end{pmatrix}$$

Hence $p - \boldsymbol{z}$ is invertible if and only if

$$\begin{pmatrix} -\mathbf{z} & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix} - \begin{pmatrix} \mathbf{z} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{p} - \mathbf{\Lambda}(\mathbf{z})$$

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is invertible. Actually: $(p - z)^{-1} = [\hat{p} - \Lambda(z)]_{1,1}^{-1}$

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Theorem (Haagerup, Thorbjørnsen 2005 (+Schultz 2006); Anderson 2012)

Every polynomial $p(x_1, ..., x_m)$ has a (non-unique) linearization $\hat{p} = \alpha_0 \otimes 1 + \alpha_1 \otimes x_1 + \cdots + \alpha_m \otimes x_m$ such that

$$(z-p)^{-1} = [(\Lambda(z) - \hat{p})^{-1}]_{1,1}, \quad \text{where} \qquad \Lambda(z) = \begin{pmatrix} z & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$



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and hence also

$$G_p(z) = \varphi((z-p)^{-1}) = [\varphi \otimes 1(\Lambda(z) - \hat{p})^{-1}]_{1,1} = [G_{\hat{p}}(\Lambda(z))]_{1,1}.$$

• $G_p(z) = \varphi[(z-p)^{-1}]$ is the Cauchy transform of p• $G_{\hat{p}}(b) = \varphi \otimes \mathbb{1}[(b-\hat{p})^{-1}]$ is the operator-valued Cauchy transform of \hat{p}

Distribution of p(x, y) for x and y free

Calculation of distribution of p by linearization and operator-valued convolution (Belinschi, Mai, Speicher 2013)

• we want the distribution of $p(x,y) = xy + yx + x^2$

Distribution of $p(\boldsymbol{x},\boldsymbol{y})$ for \boldsymbol{x} and \boldsymbol{y} free

Calculation of distribution of p by linearization and operator-valued convolution (Belinschi, Mai, Speicher 2013)

- ${\ensuremath{\, \bullet }}$ we want the distribution of $p(x,y)=xy+yx+x^2$
- this is determined by the operator-valued distribution of its linearization

$$\hat{p}(x,y) = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$

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- but this is now an additive (operator-valued) problem

$$\hat{p}(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes y$$

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- for this we have analytic theory of operator-valued free convolution
- this allows to calculate the operator-valued Cauchy transform of $\hat{p},$ and thus the Cauchy transform of p

How to calculate operator-valued free convolution

Theorem (Belinschi, Mai, Speicher 2013)

Consider

$$\hat{p}(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes y =: X + Y$$

How to calculate operator-valued free convolution

Theorem (Belinschi, Mai, Speicher 2013)

Consider $\hat{p} = X + Y$ Then X and Y are free in the operator-valued sense and there exists a unique pair of (Fréchet-)holomorphic maps $\omega_1, \omega_2: \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$, such that

 $G_X(\omega_1(b)) = G_Y(\omega_2(b)) = G_{X+Y}(b), \quad b \in \mathbb{H}^+(\mathcal{B}).$

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 $G_X(\omega_1(b)) = G_Y(\omega_2(b)) = G_{X+Y}(b), \quad b \in \mathbb{H}^+(\mathcal{B}).$

Moreover, ω_1 and ω_2 can easily be calculated via the following fixed point iterations on $\mathbb{H}^+(\mathcal{B})$

$$w \mapsto h_Y(b + h_X(w)) + b$$
 for $\omega_1(b)$
 $w \mapsto h_X(b + h_Y(w)) + b$ for $\omega_2(b)$

where we put $h_X(b) := G_X(b)^{-1} - b$ and $h_Y(b) := G_Y(b)^{-1} - b$;

$$\mathcal{B} = M_3(\mathbb{C}), \qquad \mathbb{H}(\mathcal{B})^+ := \{ b \in \mathcal{B} \mid \Im b = \frac{b - b^*}{2i} > 0 \}$$

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- X_N , Y_N independent Wigner matrices with N = 3000
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Historical remark

Note that this linearization trick is also a well-known idea in many other mathematical communities, known under various names like

- Higman's trick (Higman "The units of group rings", 1940)
- recognizable power series (automata theory, Kleene 1956, Schützenberger 1961)
- linearization by enlargement (ring theory, Cohn 1985; Cohn and Reutenauer 1994, Malcolmson 1978)
- descriptor realization (control theory, Kalman 1963; Helton, McCullough, Vinnikov 2006)



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\sim Linearization even works for non-commutative rational functions!

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Section 4

Non-Commutative Rational Functions



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Non-commutative rational functions (Amitsur 1966, Cohn 1971)

• are given by rational expressions in non-commuting variables, like

$$r(x,y) := (4-x)^{-1} + (4-x)^{-1}y \left((4-x) - y(4-x)^{-1}y \right)^{-1} y (4-x)^{-1}$$

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- there is no standard form for a rational function
- in general nested inversions are needed
Non-commutative rational functions

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• can always be realized in terms of matrices of (linear) polynomials, like

$$r = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 + \frac{1}{4}x & \frac{1}{4}y \\ \frac{1}{4}y & -1 + \frac{1}{4}x \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

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• this gives also linearizations \hat{r} of any non-commutative rational function r, like

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• if x and y are free, this is an operator-valued free convolution problem and can be solved as before (Helton, Mai, Speicher 2016)

Distribution of random matrices and their limit for $r(x,y) = (4-x)^{-1} + (4-x)^{-1}y((4-x) - y(4-x)^{-1}y)^{-1}y(4-x)^{-1}$



- X_N, Y_N independent Wigner matrices for N = 3000
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But how do we know that we also have convergence for rational functions?

Rational functions of random matrices and their limit

Proposition (Sheng Yin 2017)

Consider selfadjoint random matrices X_N, Y_N which converge to selfadjoint operators x, y in the following strong sense: for any selfadjoint polyomial p we have

- $p(X_N, Y_N) \rightarrow p(x, y)$ in distribution
- $\lim_{N \to \infty} \|p(X_N, Y_N)\| = \|p(x, y)\|$

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Then this strong convergence remains also true for rational functions:Let r be a non-commutative rational function, such that r(x,y) is defined. Then we have almost surely that

- $r(X_N, Y_Y)$ is defined eventually for large N
- $r(X_N, Y_N) \rightarrow r(x, y)$ in distribution
- $\lim_{N \to \infty} ||r(X_N, Y_N)|| = ||r(x, y)||$

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Proof

- by recursion on complexity of formulas with respect to inversions
- main step: controlling taking inverse, by approximations by polynomials, uniformly in approximating matrices and limit operators



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$\begin{array}{rcl} \mbox{convergence for polynomials} & \rightsquigarrow \mbox{convergence for rational functions} \\ & \mbox{strong} & \Longrightarrow & \mbox{strong} \\ & \mbox{in distribution} & \not \Rightarrow & \mbox{in distribution} \end{array}$



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Section 5

Summary



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Roland Speicher (Saarland University) Free Probability and Random Matrices

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- Linearization is a powerful tool for reducing non-linear problems to operator-valued linear problems
- Free Probability can deal with operator-valued linear problems
- This gives an effective algorithm for calculating the asymptotic eigenvalue distribution of polynomials in random matrices
- This can actually be extended to non-commutative rational functions



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• extend to non-selfadjoint rational functions (Helton, Mai, Speicher)



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Image: A matrix and a matrix

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