Free Probability, Random Matrices, and Non-Commutative Rational Functions

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"Non-Commutative Distributions in Free Probability"

(joint work with S. Belinschi, J. W. Helton, and T. Mai)

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Section 1

Introduction

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- non-commutative rational functions
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what means "distribution"

- algebraic/combinatorial distribution: collection of moments
- analytic distribution: probability measure

Non-commutative probability spaces

Definition

A non-commutative probability space $(\mathcal{A}, \varphi)$ consists of

- a complex algebra $\mathcal{A}$ with unit $1_{\mathcal{A}}$ and
- a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying $\varphi(1_{\mathcal{A}}) = 1$ (expectation).

Elements $x \in \mathcal{A}$ are called non-commutative random variables.

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Example

- $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$, where $(\Omega, \Sigma, \mathbb{P})$ is a classical probability space and $\mathbb{E}$ the usual expectation that is given by $\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$

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- $(M_N(\mathbb{C}), \mathrm{tr}_N)$, where tr_N is the normalized trace on $M_N(\mathbb{C})$

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We call $(\mathcal{A}, \varphi)$ a C^*-probability space if

- $\mathcal{A}$ is a unital C^*-algebra;
  i.e., $\mathcal{A}$ consists of bounded operators on a Hilbert space $\mathcal{H}$
- φ is positive (i.e. $\varphi(x^*x) \geq 0$ for $x \in \mathcal{A}$) and hence a state on $\mathcal{A}$;
  i.e., φ is of the form $\varphi(x) = \langle \Omega, x\Omega \rangle$ for some unit vector $\Omega \in \mathcal{H}$

Non-commutative distributions

Definition ("combinatorial distribution")

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.
Let $(x_i)_{i \in I}$ be a family of non-commutative random variables. We call the collection of all mixed moments

$$\{\varphi(x_{i_1} \cdots x_{i_k}) \mid k \in \mathbb{N}, i_1, \ldots, i_k \in I\}$$

their (joint) distribution.

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Definition ("analytic distribution")

Let $(\mathcal{A}, \varphi)$ be a C^*-probability space.
For any $x = x^* \in \mathcal{A}$, the distribution of x can be identified with the unique Borel probability measure μ_x on the real line $\mathbb{R}$ that satisfies

$$\varphi(x^k) = \int_{\mathbb{R}} t^k \, d\mu_x(t) \qquad \text{for all } k \in \mathbb{N}_0.$$

Section 2

Random Matrices

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- 1972: Montgomery and Dyson discovered relation between zeros of the Riemann zeta function and eigenvalues of random matrices
- since 2000: random matrix theory developed into a central subject in mathematics, with many different connections

Oberwolfach workshop "Random Matrices" 2000

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Fundamental observation

Many random matrices show for $N \to \infty$ almost surely a *deterministic* (and interesting) behavior

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Wigner random matrices (Wigner 1955)

Definition

A **Wigner random matrix** $X_N = \frac{1}{\sqrt{N}}(x_{ij})_{i,j=1}^N$

- is symmetric: $X_N^* = X_N$, i.e. $x_{ij} = x_{ji}$ for all i, j
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$$\begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

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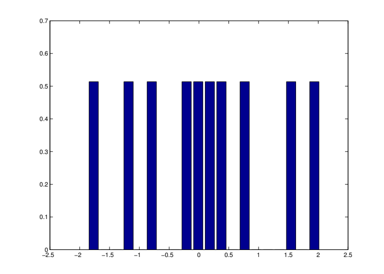
Example: eigenvalue distribution for $N = 10$



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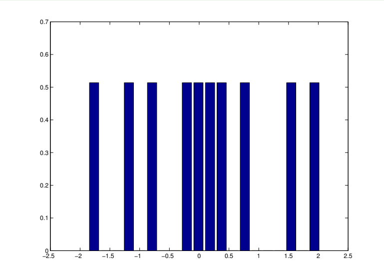

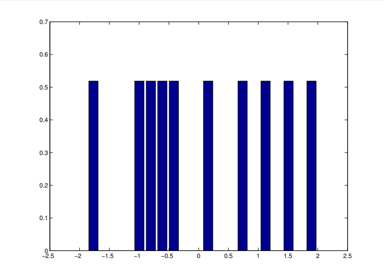

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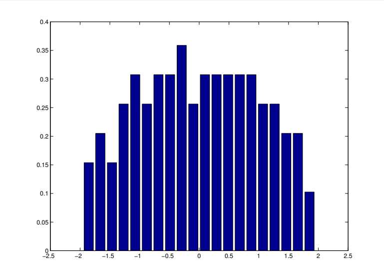
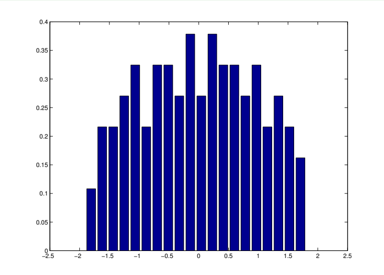
Example: eigenvalue distribution for $N = 100$





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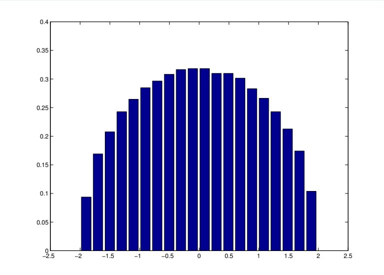
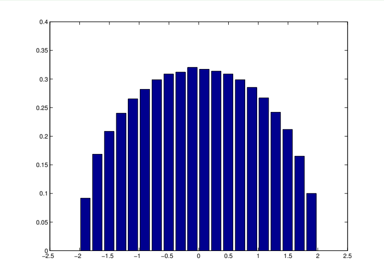
Example: eigenvalue distribution for $N = 3000$





How to describe the deterministic limit

one-matrix case: limit of X_N

- the almost sure limit of μ_{X_N} is given by a probability measure μ

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$$\lim_{N\to\infty} \mathrm{tr}_N(q(X_N, Y_N)) = \varphi(q(x,y)) \qquad \text{for all monomials } q$$

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- this means then in particular, that

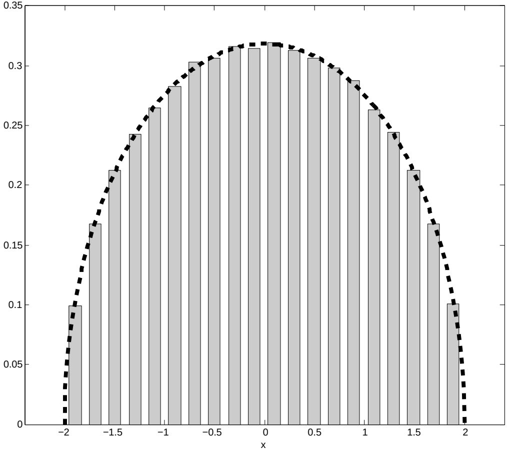
$$p(X_N, Y_N) \to p(x,y) \qquad \text{for **all** polynomials } p$$

Random matrices and operators

Fundamental observation of Voiculescu (1991)

limit of random matrices can often be described by "nice" and "interesting" operators on Hilbert spaces
(which, in the case of several matrices, describe interesting von Neumann algebras)

One-matrix case: classical random matrix case



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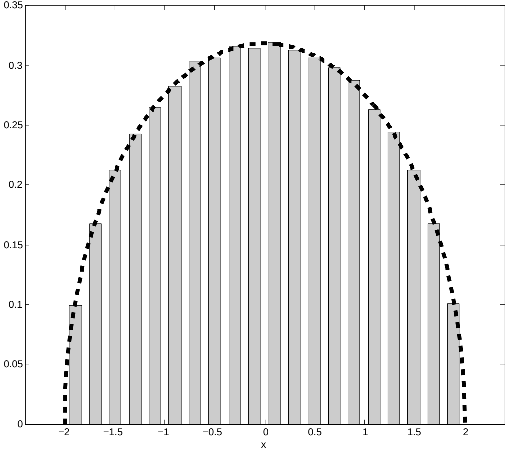

$x = l + l^*$,

l one-sided shift on $\bigoplus_{n\geq 0} \mathbb{C}e_n$

$le_n = e_{n+1}$
$l^* e_{n+1} = e_n, \; l^* e_0 = 0$

$\varphi(a) = \langle e_0, ae_0 \rangle$

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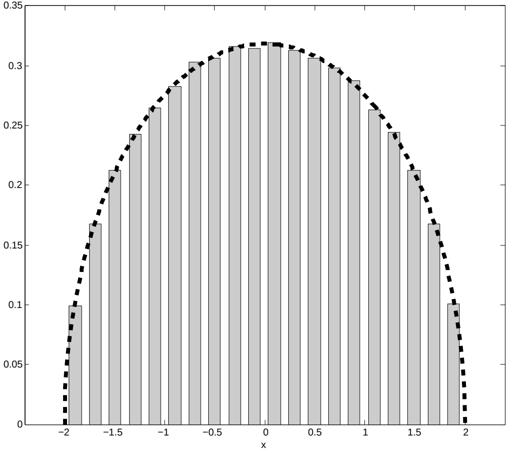

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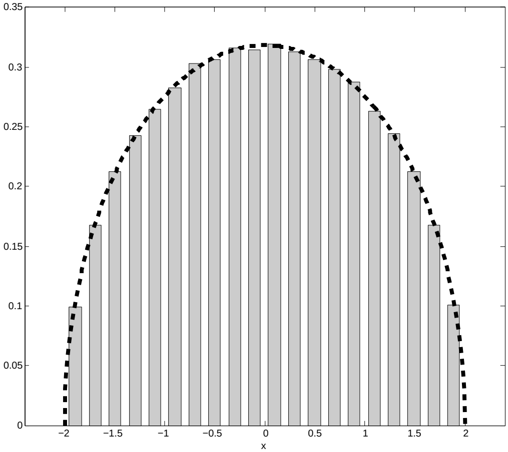

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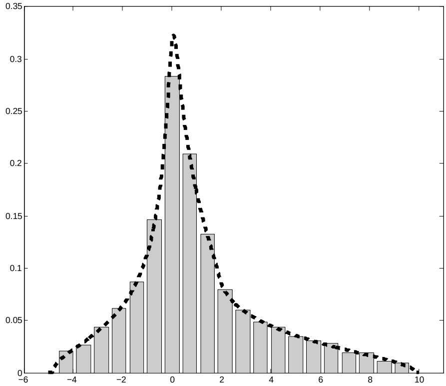
- $X_N \to x$ in distribution (Wigner 1955)
- $\|X_N\| \to \|x\| = 2$ (Füredi, Komlós 1981)

Multi-matrix case: non-commutative case; $p(x, y) = xy + yx + x^2$



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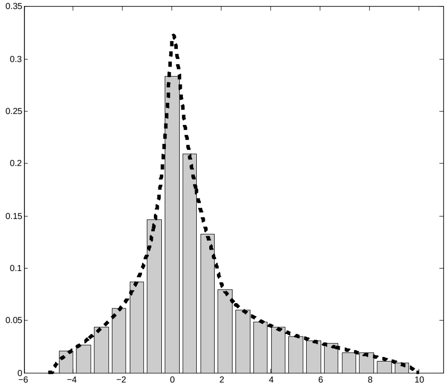

X_N, Y_N independent Wigner

$x = l_1 + l_1^*$, $y = l_2 + l_2^*$

two copies of one-sided shift in different directions (creation and annihilation operators on full Fock space; Cuntz algebra)

$\varphi(a) = \langle \Omega, a\Omega \rangle$

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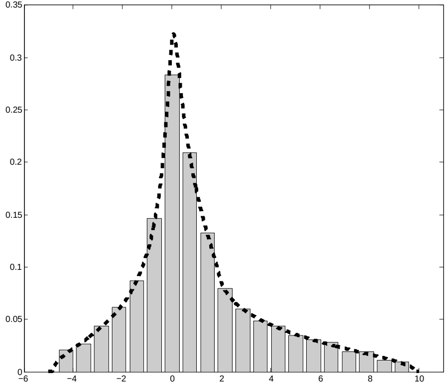

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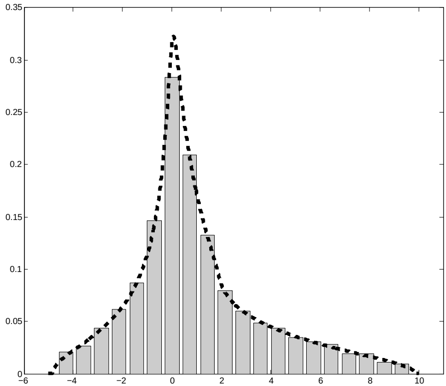

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- $\|p(X_N, Y_N)\| \to \|p(x,y)\|$ (Haagerup and Thorbjørnsen 2005)

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What are those limit operators x, y good for?

Basic Theorem (Voiculescu 1991)

For many random matrix models X_N, Y_N (like for independent Wigner matrices) the limit operators x, y are

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Tools from free probability theory: for x and y free we have

- **free convolution**: the distribution of $x + y$ can effectively be calculated in terms of the distribution of x and the distribution of y
- **matrix-valued free convolution**: the matrix-valued distribution of $\alpha_0 \otimes 1 + \alpha_1 \otimes x + \alpha_2 \otimes y$ can be calculated in terms of the distribution of x and the distribution of y

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Problem: what are such sums good for?

We are actually interested not just in the distribution of $x + y$, but much more general in the distribution of non-linear polynomials $p(x, y)$!

Section 3

Linearization

Idea of linearization

non-linear problem $\rightarrow$ operator-valued linear problem

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$$\begin{pmatrix} xy & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & x \\ y & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

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Hence xy is invertible if and only if

$$\begin{pmatrix} 0 & x \\ y & -1 \end{pmatrix}$$

is invertible

Idea of linearization

non-linear problem $\to$ operator-valued linear problem

$$p(x_1, \dots, x_m) \to \hat{p} := \alpha_0 \otimes 1 + \alpha_1 \otimes x_1 + \cdots \alpha_m \otimes x_m$$

Example

Is $p(x,y) = xy$ invertible? We have

$$\begin{pmatrix} xy & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & x \\ y & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$

Hence xy is invertible if and only if

$$\begin{pmatrix} 0 & x \\ y & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes y = \hat{p}$$

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Example

Is $p(x, y) = xy + yx + x^2$ invertible?

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Example

Is $p(x,y) = xy + yx + x^2$ invertible? We have

$$\begin{pmatrix} xy+yx+x^2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & y+\frac{x}{2} & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & x & y+\frac{x}{2} \\ x & 0 & -1 \\ y+\frac{x}{2} & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ y+\frac{x}{2} & 1 & 0 \\ x & 0 & 1 \end{pmatrix}$$

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Hence $p = xy + yx + x^2$ is invertible if and only if

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is invertible.

Idea of linearization

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is invertible.

Theorem (Haagerup, Thorbjørnsen 2005 (+Schultz 2006); Anderson 2012)

Every polynomial $p(x_1, \dots, x_m)$ *has a (non-unique) linearization* $\hat{p} = \alpha_0 \otimes 1 + \alpha_1 \otimes x_1 + \dots + \alpha_m \otimes x_m$ *such that*

$$(z - p)^{-1} = [(\Lambda(z) - \hat{p})^{-1}]_{1,1}, \quad \textit{where} \quad \Lambda(z) = \begin{pmatrix} z & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

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and hence also

$$G_p(z) = \varphi((z-p)^{-1}) = [\varphi \otimes 1(\Lambda(z) - \hat{p})^{-1}]_{1,1} = [G_{\hat{p}}(\Lambda(z))]_{1,1}.$$

- $G_p(z) = \varphi[(z-p)^{-1}]$ is the Cauchy transform of p
- $G_{\hat{p}}(b) = \varphi \otimes 1[(b-\hat{p})^{-1}]$ is the operator-valued Cauchy transform of $\hat{p}$

Distribution of $p(x, y)$ for x and y free

Calculation of distribution of p by linearization and operator-valued convolution (Belinschi, Mai, Speicher 2013)

- we want the distribution of $p(x, y) = xy + yx + x^2$

Distribution of $p(x,y)$ for x and y free

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- we want the distribution of $p(x,y) = xy + yx + x^2$
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$$\hat{p}(x,y) = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$

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$$\hat{p}(x, y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes y$$

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- for this we have analytic theory of operator-valued free convolution
- this allows to calculate the operator-valued Cauchy transform of $\hat{p}$, and thus the Cauchy transform of p

How to calculate operator-valued free convolution

Theorem (Belinschi, Mai, Speicher 2013)

Consider

$$\hat{p}(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes y =: X + Y$$

How to calculate operator-valued free convolution

Theorem (Belinschi, Mai, Speicher 2013)

Consider $\hat{p} = X + Y$ Then X and Y are free in the operator-valued sense and there exists a unique pair of (Fréchet-)holomorphic maps $\omega_1, \omega_2 : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$, such that

$$G_X(\omega_1(b)) = G_Y(\omega_2(b)) = G_{X+Y}(b), \quad b \in \mathbb{H}^+(\mathcal{B}).$$

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$$G_X(\omega_1(b)) = G_Y(\omega_2(b)) = G_{X+Y}(b), \quad b \in \mathbb{H}^+(\mathcal{B}).$$

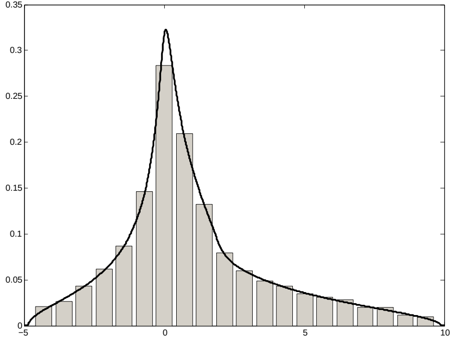
Moreover, ω_1 and ω_2 can easily be calculated via the following fixed point iterations on $\mathbb{H}^+(\mathcal{B})$

$$w \mapsto h_Y(b + h_X(w)) + b \qquad \text{for } \omega_1(b)$$
$$w \mapsto h_X(b + h_Y(w)) + b \qquad \text{for } \omega_2(b)$$

where we put $h_X(b) := G_X(b)^{-1} - b$ and $h_Y(b) := G_Y(b)^{-1} - b$;

$$\mathcal{B} = M_3(\mathbb{C}), \qquad \mathbb{H}(\mathcal{B}) := \{b \in \mathcal{B} \mid \Im b = \frac{b - b^*}{2i} > 0\}$$

Distribution of $p(x,y) = xy + yx + x^2$



$x = l_1 + l_1^*$, $y = l_2 + l_2^*$

two copies of one-sided shift in different directions (creation and annihilation operators on full Fock space; Cuntz algebra relations)

$\varphi(a) = \langle \Omega, a\Omega \rangle$

Historical remark

Note that this linearization trick is also a well-known idea in many other mathematical communities, known under various names like

- Higman's trick (Higman "The units of group rings", 1940)
- recognizable power series (automata theory, Kleene 1956, Schützenberger 1961)
- linearization by enlargement (ring theory, Cohn 1985; Cohn and Reutenauer 1994, Malcolmson 1978)
- descriptor realization (control theory, Kalman 1963; Helton, McCullough, Vinnikov 2006)

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$\curvearrowright$ **Linearization even works for non-commutative rational functions!**

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Section 4

Non-Commutative Rational Functions

Non-commutative rational functions

Non-commutative rational functions (Amitsur 1966, Cohn 1971)

- are given by rational expressions in non-commuting variables, like

$$r(x,y) := (4-x)^{-1} + (4-x)^{-1} y \big((4-x) - y(4-x)^{-1} y \big)^{-1} y (4-x)^{-1}$$

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- deciding whether $r = 0$ is not easy, for example

$$x_2^{-1} + x_2^{-1}(x_3^{-1}x_1^{-1} - x_2^{-1})^{-1}x_2^{-1} - (x_2 - x_3x_1)^{-1} = 0$$

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- there is no standard form for a rational function
- in general nested inversions are needed

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Non-commutative rational functions

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- $r(x,y) := (4-x)^{-1} + (4-x)^{-1} y \big((4-x) - y(4-x)^{-1} y\big)^{-1} y (4-x)^{-1}$
- can always be realized in terms of matrices of (linear) polynomials, like

$$r = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 + \frac{1}{4}x & \frac{1}{4}y \\ \frac{1}{4}y & -1 + \frac{1}{4}x \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

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- this gives also linearizations $\hat{r}$ of any non-commutative rational function r, like

$$\hat{r} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1+\frac{1}{4}x & \frac{1}{4}y \\ 0 & \frac{1}{4}y & -1+\frac{1}{4}x \end{pmatrix}$$

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- if x and y are free, this is an operator-valued free convolution problem and can be solved as before (Helton, Mai, Speicher 2016)

Rational functions of random matrices and their limit

Proposition (Sheng Yin 2016)

Consider selfadjoint random matrices X_N, Y_N which converge to selfadjoint operators x, y in the following strong sense: for any selfadjoint polyomial p we have

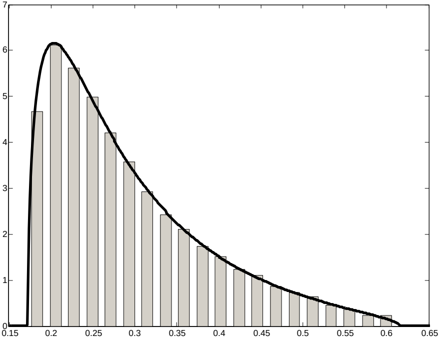
- $p(X_N, Y_N) \to p(x, y)$ in distribution
- $\lim_{N\to\infty} \|p(X_N, Y_N)\| = \|p(x, y)\|$

Then this strong convergence remains also true for rational functions: Let r be a non-commutative rational function, such that $r(x, y)$ is defined. Then we have almost surely that

- $r(X_N, Y_Y)$ is defined eventually for large N
- $r(X_N, Y_N) \to r(x, y)$ in distribution
- $\lim_{N\to\infty} \|r(X_N, Y_N)\| = \|r(x, y)\|$

Distribution of random matrices and their limit for

$$r(x,y) = (4-x)^{-1} + (4-x)^{-1}y\big((4-x) - y(4-x)^{-1}y\big)^{-1}y(4-x)^{-1}$$



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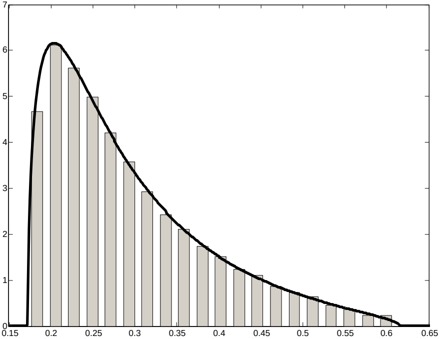

X_N, Y_N independent Wigner

$x = l_1 + l_1^*$, $y = l_2 + l_2^*$

two copies of one-sided shift in different directions (creation and annihilation operators on full Fock space; Cuntz algebra)

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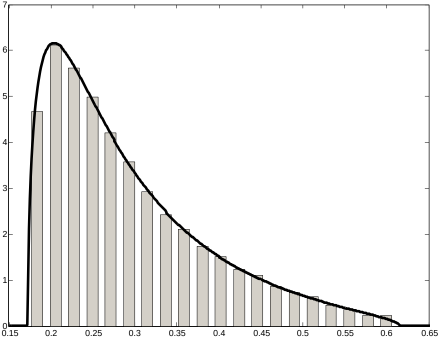

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- $r(X_N, Y_N) \to r(x,y)$ in distribution (Yin 2017)

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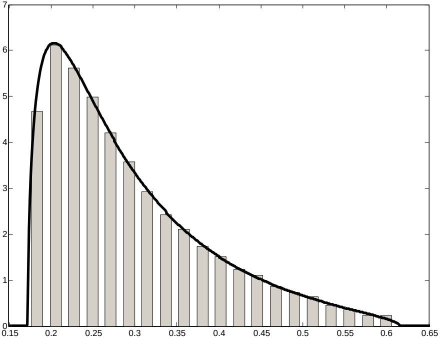

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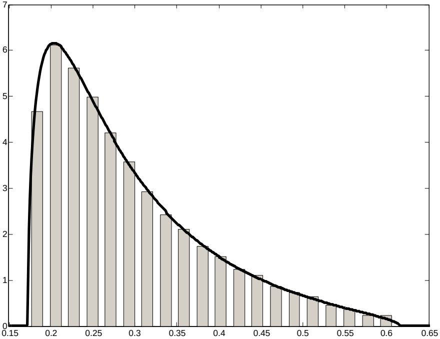

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- $\|r(X_N, Y_N)\| \to \|r(x,y)\|$ (Yin 2017)

Thank you!