Invariance under Quantum Permutations and Free Probability

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Consider probability space $(\Omega, \mathfrak{A}, P)$. Denote expectation by φ,

$$\varphi(Y) = \int_\Omega Y(\omega) dP(\omega).$$

We say that random variables $X_1, X_2, \dots$ are **exchangeable** if their joint distribution is invariant under finite permutations, i.e. if

$$\varphi(X_{i(1)} \cdots X_{i(n)}) = \varphi(X_{\pi(i(1))} \cdots X_{\pi(i(n))})$$

for all $n \in \mathbb{N}$, all $i(1), \dots, i(n) \in \mathbb{N}$, and all permutations π

Examples:

$$\varphi(X_1^n) = \varphi(X_7^n)$$

$$\varphi(X_1^3 X_3^7 X_4) = \varphi(X_8^3 X_2^7 X_9)$$

Note that the X_i might all contain some common component; e.g., if all X_i are the same, then clearly

$$X, X, X, X, X \dots$$

is exchangeable.

Theorem of de Finetti says that an infinite sequence of exchangeable random variables is independent modulo its common part.

Formalize common part via **tail σ-algebra** of the sequence $X_1, X_2, \ldots$

$$\mathfrak{A}_{\text{tail}} := \bigcap_{i \in \mathbb{N}} \sigma(X_k \mid k \geq i)$$

Denote by E the **conditional expectation onto this tail σ-algebra**

$$E : L^\infty(\Omega, \mathfrak{A}, P) \to L^\infty(\Omega, \mathfrak{A}_{\text{tail}}, P)$$

de Finetti Theorem
(de Finetti 1931, Hewitt, Savage 1955)

The following are equivalent for an infinite sequence of random variables:

- the sequence is exchangeable
- the sequence is independent and identically distributed with respect to the conditional expectation E onto the tail σ-algebra of the sequence

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The following are equivalent for an infinite sequence of random variables:

- the sequence is exchangeable

- the sequence is independent and identically distributed with respect to the conditional expectation E onto the tail σ-algebra of the sequence

$$E[X_1^{m(1)} X_2^{m(2)} \cdots X_n^{m(n)}] = E[X_1^{m(1)}] \cdot E[X_2^{m(2)}] \cdots E[X_n^{m(n)}]$$

Non-commutative Random Variables

Replace random variables by operators on Hilbert spaces,

expectation by state on the algebra generated by those operators.

Setting: **non-commutative W^*-probability space $(\mathcal{A}, \varphi)$**, i.e.,

- $\mathcal{A}$ is von Neumann algebra (i.e., weakly closed subalgebra of bounded operators on Hilbert space)

- $\varphi : \mathcal{A} \to \mathbb{C}$ is faithful state on $\mathcal{A}$, i.e.,

$$\varphi(aa^*) \geq 0, \qquad \text{for all } a \in \mathcal{A}$$

$$\varphi(aa^*) = 0 \quad \text{if and only if } a = 0$$

Consider **non-commutative random variables** $x_1, x_2, \cdots \in \mathcal{A}$.

They are **exchangeable** if

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for all $n \in \mathbb{N}$, all $i(1), \ldots, i(n) \in \mathbb{N}$, and all permutations π.

Question: Does exchangeability imply anything like independence in this general non-commutative setting?

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Question: Does exchangeability imply anything like independence in this general non-commutative setting?

Answer: Not really. There are just too many possibilities out there in the non-commutative world, and exchangeability is a too weak condition!

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permutation group $\longrightarrow$ **quantum permutation group**

What are Quantum Permutations?

The permutation group S_k consists of automorphisms which preserve the structure of a set of k points.

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Permutation Group

k points $\quad\hat{=}\quad$ functions on k points

Permutation Group

k points $\quad\hat{=}\quad$ $e_1, \dots, e_k$ with

- $e_i^2 = e_i = e_i^*$
- $e_1 + \cdots + e_k = 1$

Permutation group (more precisely: $C(S_k)$) is the universal algebra, generated by **commuting** elements u_{ij} $(i, j = 1, \dots, k)$ such that $f_1, \dots, f_k$, given by

$$f_i := \sum_{j=1}^{k} u_{ij} e_j$$

satisfy the same relations as $e_1, \dots, e_k$.

Quantum Permutation Group

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Thus: quantum permutation group is the universal algebra, generated by **non-commuting elements** u_{ij} $(i, j = 1, \dots, k)$ such that $f_1, \dots, f_k$, given by

$$f_i := \sum_{j=1}^{n} u_{ij} \otimes e_j$$

satisfy the same relations as $e_1, \dots, e_k$.

Quantum Permutation Group (Wang 1998)

The quantum permutation group $A_s(k)$ is the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \ldots, k)$ subject to the relations

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i, j = 1, \ldots, k$

- each row and column of $u = (u_{ij})_{i,j=1}^k$ is a partition of unity:

$$\sum_{j=1}^{k} u_{ij} = 1 \qquad \sum_{i=1}^{k} u_{ij} = 1$$

$A_s(k)$ is a compact quantum group in the sense of Woronowicz.

Examples of $u = (u_{ij})_{i,j=1}^k$ satisfying these relations are:

- permutation matrices

- basic non-commutative example is of the form (for $k = 4$):

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & 1 \end{pmatrix}$$

  for (in general, non-commuting) projections p and q

Quantum Exchangeability

A sequence $x_1, x_2, \ldots$ in $(\mathcal{A}, \varphi)$ is quantum exchangeable if its distribution does not change under the action of quantum permutations, i.e., if

$$\varphi(x_{i(1)} \cdots x_{i(n)}) = \sum_{j(1),\ldots,j(n)=1}^{k} u_{i(1)j(1)} \cdots u_{i(n)j(n)} \varphi(x_{j(1)} \cdots x_{j(n)})$$

for all $u = (u_{ij})_{i,j=1}^{k}$ which satisfy the defining relations for $A_s(k)$.

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In particular: quantum exchangeable $\implies$ exchangeable

commuting variables are usually not quantum exchangeable

Question: What does quantum exchangeability for an infinite sequence $x_1, x_2, \ldots$ imply?

As before, constant sequences are trivially quantum exchangeable, thus we have to take out the common part of all the x_i.

Formally: Define the tail algebra of the sequence:

$$\mathcal{A}_{\mathsf{tail}} := \bigcap_{i \in \mathbb{N}} \mathsf{vN}(x_k \mid k \geq i),$$

then there exists conditional expectation

$$E : \mathsf{vN}(x_i \mid i \in \mathbb{N}) \to \mathcal{A}_{\mathsf{tail}}.$$

Question: Does quantum exchangeability imply an independence like property for this E?

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Thus

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In particular: if $E[x_2] = 0$, then $E[x_7 x_2 x_7 x_9] = 0$.

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However, for non-commuting variables there are many more expressions which cannot be treated like this.

basic example: $E[x_1 x_2 x_1 x_2] = ???$

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By quantum exchangeability we have

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$$= \sum_{j(1),\ldots,j(4)=1}^{k} u_{1j(1)} u_{2j(2)} u_{1j(3)} u_{2j(4)} E[x_{j(1)} x_{j(2)} x_{j(3)} x_{j(4)}]$$

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$$= \underbrace{\sum_{j(1)=j(3)\neq j(2)=j(4)} u_{1j(1)} u_{2j(2)} u_{1j(3)} u_{2j(4)}}_{\neq 1 \text{ for general } (u_{ij})} E[x_1 x_2 x_1 x_2]$$

Thus we have: if $E[x_1] = 0 = E[x_2]$, then $E[x_1 x_2 x_1 x_2] = 0$

This implies in general:

$$E[x_1 x_2 x_1 x_2] = E\Big[x_1 E[x_2] x_1\Big] \cdot E[x_2] + E[x_1] \cdot E\Big[x_2 E[x_1] x_2\Big]$$

$$- E[x_1]E[x_2]E[x_1]E[x_2]$$

In general, one shows in the same way that

$$E\Big[p_1(x_{i(1)})p_2(x_{i(2)})\cdots p_n(x_{i(n)})\Big] = 0$$

whenever

- $n \in \mathbb{N}$ and $p_1, \dots, p_n$ are polynomials in one variable

- $i(1) \neq i(2) \neq i(3) \neq \cdots \neq i(n)$

- $E[p_j(x_{i(j)})] = 0$ for all $j = 1, \dots, n$

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The x_i are free w.r.t E in the sense of Voiculescu's free probability theory.

Non-commutative de Finetti Theorem

(Köstler, Speicher 2008)

The following are equivalent for an infinite sequence of non-commutative random variables:

- the sequence is quantum exchangeable

- the sequence is free and identically distributed with respect to the conditional expectation E onto the tail-algebra of the sequence