Polynomials in Asymptotically Free Random Matrices

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We are interested in the limiting eigenvalue distribution of an

$N \times N$ random matrix for $N \to \infty$.

Typical phenomena for basic random matrix ensembles:

- almost sure convergence to a deterministic limit eigenvalue distribution
- this limit distribution can be effectively calculated

The Cauchy (or Stieltjes) Transform

For any probability measure μ on $\mathbb{R}$ we define its Cauchy transform

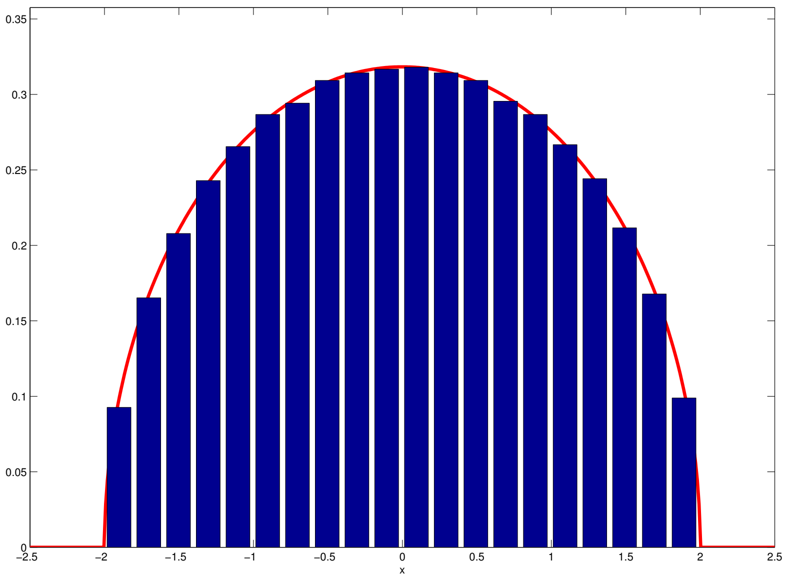
$$G(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t)$$

This is an analytic function $G : \mathbb{C}^+ \to \mathbb{C}^-$ and we can recover μ from G by **Stieltjes inversion formula**

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \Im G(t + i\varepsilon) dt$$

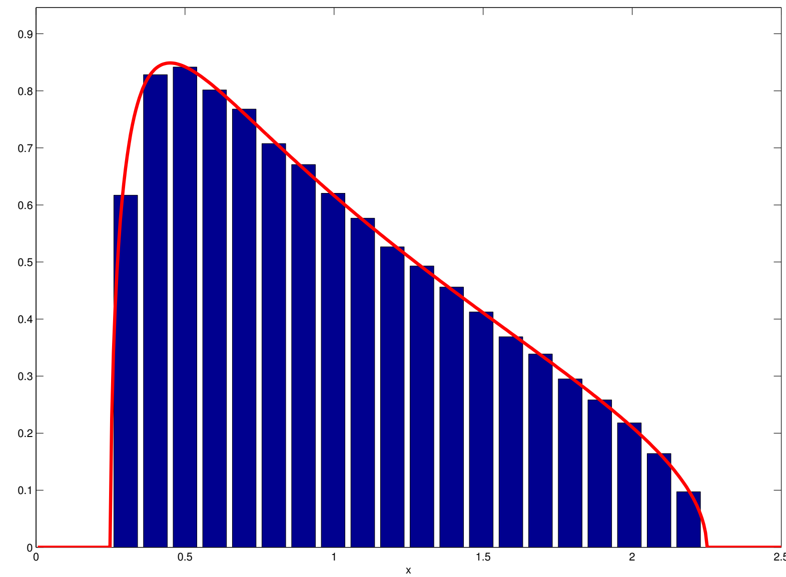
Wigner random matrix
and
Wigner's semicircle

$$G(z) = \frac{z - \sqrt{z^2 - 4}}{2}$$



Wishart random matrix
and
Marchenko-Pastur distribution

$$G(z) = \frac{z + 1 - \lambda - \sqrt{(z - (1 + \lambda))^2 - 4\lambda}}{2z}$$



We are now interested in the limiting eigenvalue distribution of

general (selfadjoint) polynomials $p(X_1, \dots, X_k)$

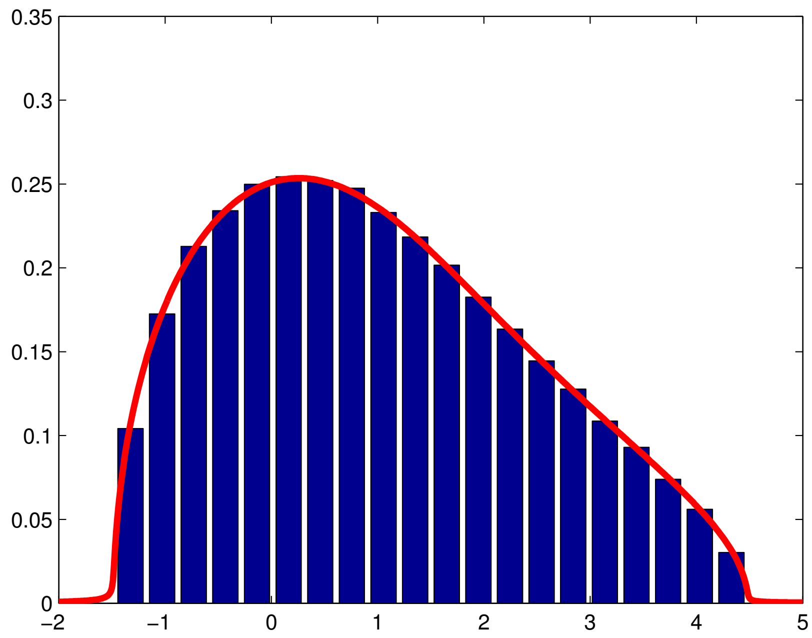
of **several** independent $N \times N$ random matrices $X_1, \dots, X_k$

Typical phenomena:

- almost sure convergence to a deterministic limit eigenvalue distribution
- this limit distribution can be effectively calculated **only in very simple situations**

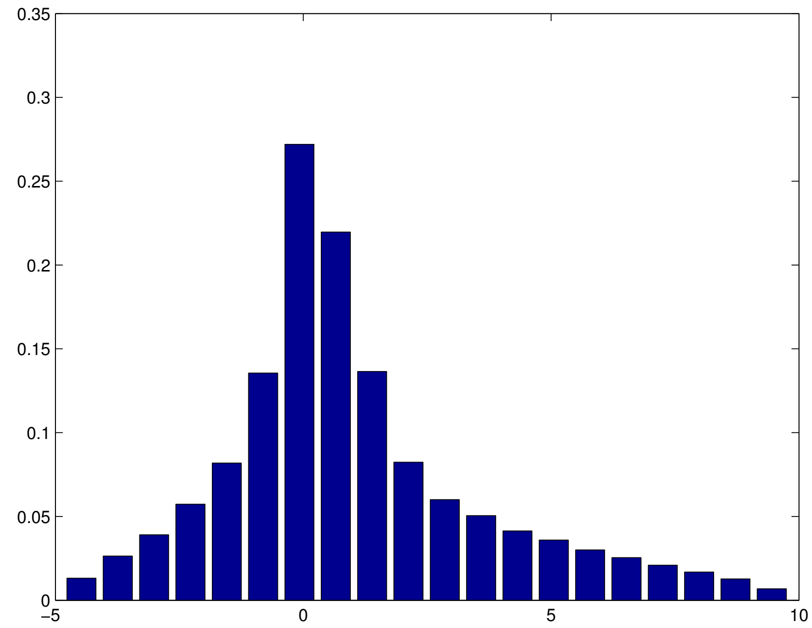
for X Wigner, Y Wishart

$$p(X,Y) = X + Y$$
$$G(z) = G_{\text{Wishart}}(z - G(z))$$



$$p(X,Y) = XY + YX + X^2$$

????



Existing Results for Calculations of the Limit Eigenvalue Distribution

- Marchenko, Pastur 1967: general Wishart matrices ADA^*
- Pastur 1972: deterministic + Wigner (deformed semicircle)
- Speicher, Nica 1998; Vasilchuk 2003: commutator or anti-commutator: $X_1X_2 \pm X_2X_1$
- more general models in wireless communications (Tulino, Verdu 2004; Couillet, Debbah, Silverstein 2011):

$$RADA^*R^* \qquad \text{or} \qquad \sum_i R_iA_iD_iA_i^*R_i^*$$

Asymptotic Freeness of Random Matrices

Basic result of Voiculescu (1991):
Large classes of independent random matrices (like Wigner or Wishart matrices) become asymptoticially freely independent.

Conclusion: Calculating the asymptotic eigenvalue distribution of polynomials in such matrices is the same as calculating the distribution of polynomials in free variables

We want to understand distribution of polynomials in free variables.

What we understand quite well is:

sums of free selfadjoint variables

So we should reduce:

$$\boxed{\text{arbitrary polynomial}} \longrightarrow \boxed{\textbf{sums} \text{ of } \textbf{selfadjoint} \text{ variables}}$$

This can be done on the expense of going over to operator-valued frame.

Let $\mathcal{B} \subset \mathcal{A}$. A linear map

$$E : \mathcal{A} \to \mathcal{B}$$

is a conditional expectation if

$$E[b] = b \qquad \forall b \in \mathcal{B}$$

and

$$E[b_1 a b_2] = b_1 E[a] b_2 \qquad \forall a \in \mathcal{A}, \quad \forall b_1, b_2 \in \mathcal{B}$$

An operator-valued probability space consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$

Consider an operator-valued probability space $E : \mathcal{A} \to \mathcal{B}$.

Random variables $x_i \in \mathcal{A}$ $(i \in I)$ are free with respect to E (or free with amalgamation over $\mathcal{B}$) if

$$E[a_1 \cdots a_n] = 0$$

whenever $a_i \in \mathcal{B}\langle x_{j(i)} \rangle$ are polynomials in some $x_{j(i)}$ with coefficients from $\mathcal{B}$ and

$$E[a_i] = 0 \quad \forall i \qquad \text{and} \qquad j(1) \neq j(2) \neq \cdots \neq j(n).$$

Consider an operator-valued probability space $E : \mathcal{A} \to \mathcal{B}$.

For a random variable $x \in \mathcal{A}$, we define the operator-valued Cauchy transform:

$$G(b) := E[(b - x)^{-1}] \qquad (b \in \mathcal{B}).$$

For $x = x^*$, this is well-defined and a nice analytic map on the operator-valued upper halfplane:

$$\mathbb{H}^+(B) := \{b \in B \mid (b - b^*)/(2i) > 0\}$$

Theorem (Belinschi, Mai, Speicher 2013): Let x and y be selfadjoint operator-valued random variables free over $\mathcal{B}$. Then there exists a Fréchet analytic map $\omega\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ so that

$$G_{x+y}(b) = G_x(\omega(b)) \text{ for all } b \in \mathbb{H}^+(\mathcal{B}).$$

Moreover, if $b \in \mathbb{H}^+(\mathcal{B})$, then $\omega(b)$ is the unique fixed point of the map

$$f_b\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B}), \qquad f_b(w) = h_y(h_x(w) + b) + b,$$

and

$$\omega(b) = \lim_{n\to\infty} f_b^{\circ n}(w) \qquad \text{for any } w \in \mathbb{H}^+(\mathcal{B}).$$

where

$$\mathbb{H}^+(\mathcal{B}) := \{b \in \mathcal{B} \mid (b - b^*)/(2i) > 0\}, \qquad h(b) := \frac{1}{G(b)} - b$$

The Linearization Philosophy:

In order to understand polynomials p in non-commuting variables, it suffices to understand matrices $\widehat{p}$ of **linear** polynomials in those variables.

- Voiculescu 1987: motivation
- Haagerup, Thorbjørnsen 2005: largest eigenvalue
- Anderson 2012: the selfadjoint version
  (based on Schur complement)

Theorem (Anderson 2012): One has

- for each p there exists a linearization $\widehat{p}$ (with an explicit algorithm for finding those)

- if p is selfadjoint, then this $\widehat{p}$ is also selfadjoint

Note: $\widehat{p}$ is the sum of operator-valued free variables!

Conclusion: Combination of linearization and operator-valued subordination allows to deal with case of selfadjoint polynomials.

Example: $p(x,y) = xy + yx + x^2$

p has linearization

$$\hat{p} = \begin{pmatrix} 0 & x & y+\frac{x}{2} \\ x & 0 & -1 \\ y+\frac{x}{2} & -1 & 0 \end{pmatrix}$$

and

$$G_{\hat{p}}(b) = \mathrm{id} \otimes \varphi((b-\hat{p})^{-1}) = \begin{pmatrix} \varphi((z-p)^{-1}) & \cdots \\ \cdots & \cdots \end{pmatrix} \qquad b = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

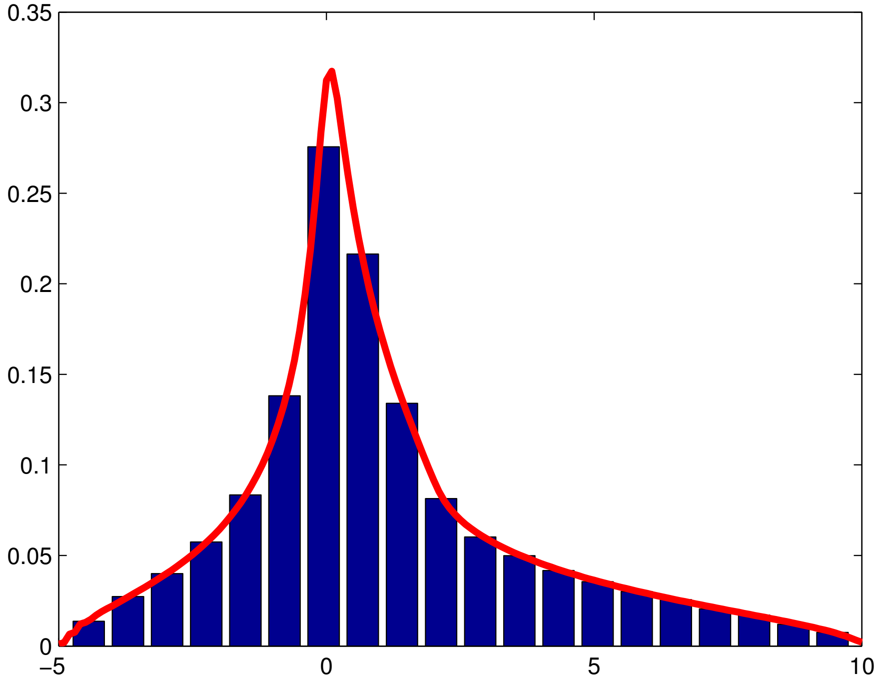
Example: $p(x,y) = xy + yx + x^2$

$$\hat{p} = \begin{pmatrix} 0 & x & \frac{x}{2} \\ x & 0 & -1 \\ \frac{x}{2} & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix} = \hat{X} + \hat{Y}$$

is now the sum of two selfadjoint variables which are free over $M_3(\mathbb{C})$, so we can use our subordination result

$$\begin{pmatrix} \varphi((z-p)^{-1}) & \cdots \\ \cdots & \cdots \end{pmatrix} = G_{\hat{p}}(b) = G_{\hat{X}+\hat{Y}}(b) = G_{\hat{X}}(\omega(b))$$

$P(X,Y) = XY + YX + X^2$
for independent X, Y; X is Wigner and Y is Wishart



$p(x,y) = xy + yx + x^2$
for free x, y; x is semicircular and y is Marchenko-Pastur

What about non-selfadjoint polynomials?

For a measure on $\mathbb{C}$ its Cauchy transform

$$G_\mu(\lambda) = \int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu(z)$$

is well-defined everywhere outside a set of $\mathbb{R}^2$-Lebesgue measure zero, however, it is analytic only outside the support of μ.

The measure μ can be extracted from its Cauchy transform by the formula (understood in distributional sense)

$$\mu = \frac{1}{\pi} \frac{\partial}{\partial \bar{\lambda}} G_\mu(\lambda),$$

Better approach by regularization:

$$G_{\epsilon,\mu}(\lambda) = \int_{\mathbb{C}} \frac{\bar{\lambda} - \bar{z}}{\epsilon^2 + |\lambda - z|^2} d\mu(z)$$

is well–defined for every $\lambda \in \mathbb{C}$. By sub-harmonicity arguments

$$\mu_\epsilon = \frac{1}{\pi} \frac{\partial}{\partial \bar{\lambda}} G_{\epsilon,\mu}(\lambda)$$

is a positive measure on the complex plane.

One has: $\lim_{\epsilon \to 0} \mu_\epsilon = \mu$ weak convergence

This can be copied for general (not necessarily normal) operators x in a tracial non-commutative probability space $(\mathcal{A}, \varphi)$.
Put

$$G_{\epsilon,x}(\lambda) := \varphi\Big((\lambda - x)^*\big((\lambda - x)(\lambda - x)^* + \epsilon^2\big)^{-1}\Big)$$

Then

$$\mu_{\epsilon,x} = \frac{1}{\pi}\frac{\partial}{\partial \bar{\lambda}} G_{\epsilon,\mu}(\lambda)$$

is a positive measure on the complex plane, which converges weakly for $\epsilon \to 0$,

$\mu_x := \lim_{\epsilon \to 0} \mu_{\epsilon,x}$ Brown measure of x

(L. Brown 1986; Haagerup, Larsen 2000)

Hermitization Method

(Janik, Nowak, Papp, Zahed 1997; Feinberg, Zee 1997)

For given x we need to calculate

$$G_{\epsilon,x}(\lambda) = \varphi\left((\lambda - x)^*\Big((\lambda - x)(\lambda - x)^* + \epsilon^2\Big)^{-1}\right)$$

Let

$$X = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in M_2(\mathcal{A}); \qquad \text{note: } X = X^*$$

Consider X in the $M_2(\mathbb{C})$-valued probability space with repect to $E = \mathrm{id} \otimes \varphi : M_2(\mathcal{A}) \to M_2(\mathbb{C})$ given by

$$E\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = \begin{pmatrix} \varphi(a_{11}) & \varphi(a_{12}) \\ \varphi(a_{21}) & \varphi(a_{22}) \end{pmatrix}.$$

For the argument

$$\Lambda_\epsilon = \begin{pmatrix} i\epsilon & \lambda \\ \bar{\lambda} & i\epsilon \end{pmatrix} \in M_2(\mathbb{C}) \qquad \text{and} \qquad X = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}$$

consider now the $M_2(\mathbb{C})$-valued Cauchy transform of X

$$G_X(\Lambda_\varepsilon) = E\Big[(\Lambda_\epsilon - X)^{-1}\Big] = \begin{pmatrix} g_{\epsilon,\lambda,11} & g_{\epsilon,\lambda,12} \\ g_{\epsilon,\lambda,21} & g_{\epsilon,\lambda,22} \end{pmatrix}.$$

One can easily check that

$$(\Lambda_\epsilon - X)^{-1} = \begin{pmatrix} -i\epsilon((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} & (\lambda - x)((\lambda - x)^*(\lambda - x) + \epsilon^2)^{-1} \\ (\lambda - x)^*((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} & -i\epsilon((\lambda - x)^*(\lambda - x) + \epsilon^2)^{-1} \end{pmatrix}$$

thus

$$g_{\epsilon,\lambda,12} = G_{\varepsilon,x}(\lambda).$$

So for a general polynomial we should

1. hermitize

2. linearise

3. subordinate

But: do (1) and (2) fit together???

We have now to linearize a polynomial in matrices!

Consider $p = xy$ with $x = x^*$, $y = y^*$.

For this we have to calculate the operator-valued Cauchy transform of

$$P = \begin{pmatrix} 0 & xy \\ yx & 0 \end{pmatrix}$$

Linearization means we should split this in sums of matrices in x and matrices in y.

Write

$$P = \begin{pmatrix} 0 & xy \\ yx & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = XYX$$

$P = XYX$ is now a selfadjoint polynomial in the selfadjoint variables

$$X = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}$$

XYX has linearization

$$\begin{pmatrix} 0 & 0 & X \\ 0 & Y & -1 \\ X & -1 & 0 \end{pmatrix}$$

thus

$$P = \begin{pmatrix} 0 & xy \\ yx & 0 \end{pmatrix}$$

has linearization

$$\begin{pmatrix} 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & -1 & 0 \\ 0 & 0 & y & 0 & 0 & -1 \\ x & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ x & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we can now calculate the operator-valued Cauchy transform of this via subordination.

Does eigenvalue distribution of polynomial in independent random matrices converge to Brown measure of corresponding polynomial in free variables?

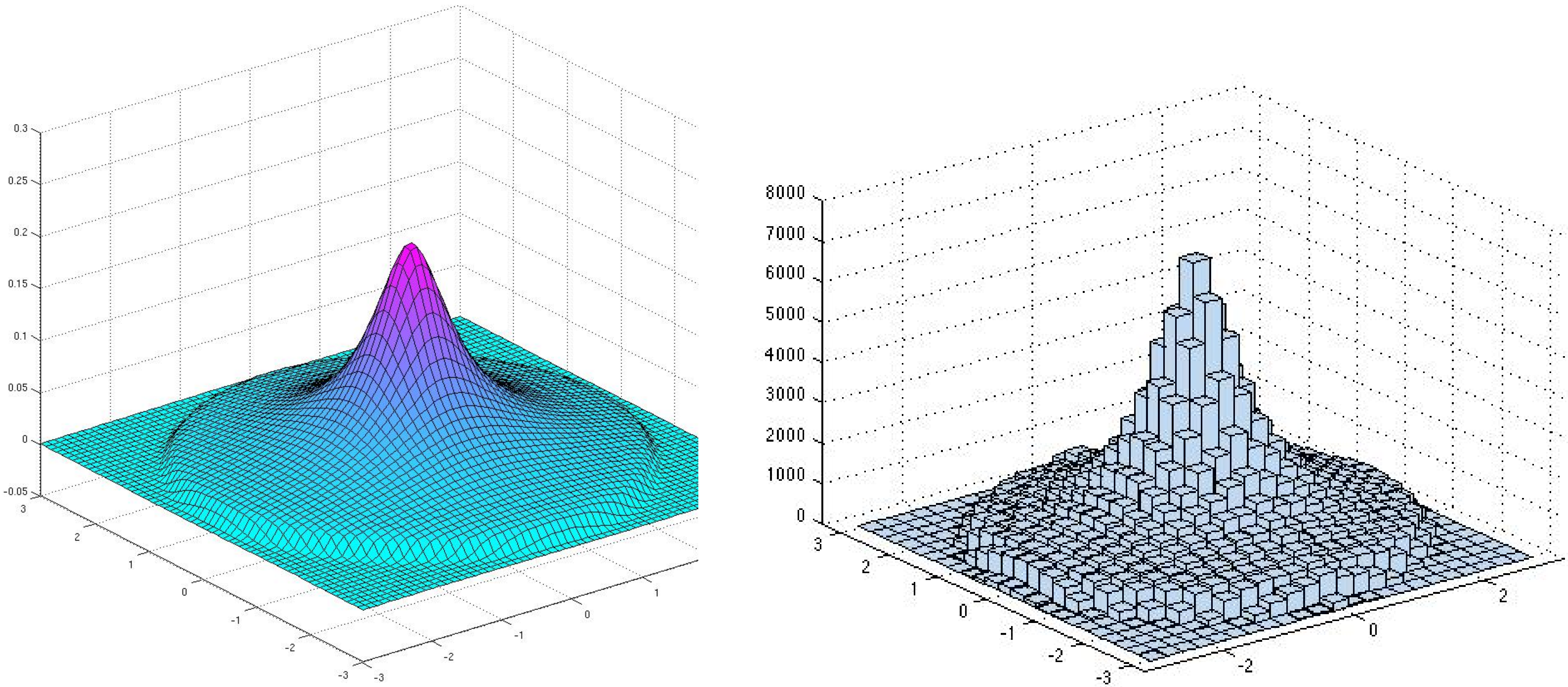
Conjecture: Consider m independent selfadjoint Gaussian (or, more general, Wigner) random matrices $X_N^{(1)}, \dots, X_N^{(m)}$ and put

$$A_N := p(X_N^{(1)}, \dots, X_N^{(m)}), \qquad x := p(s_1, \dots, s_m).$$

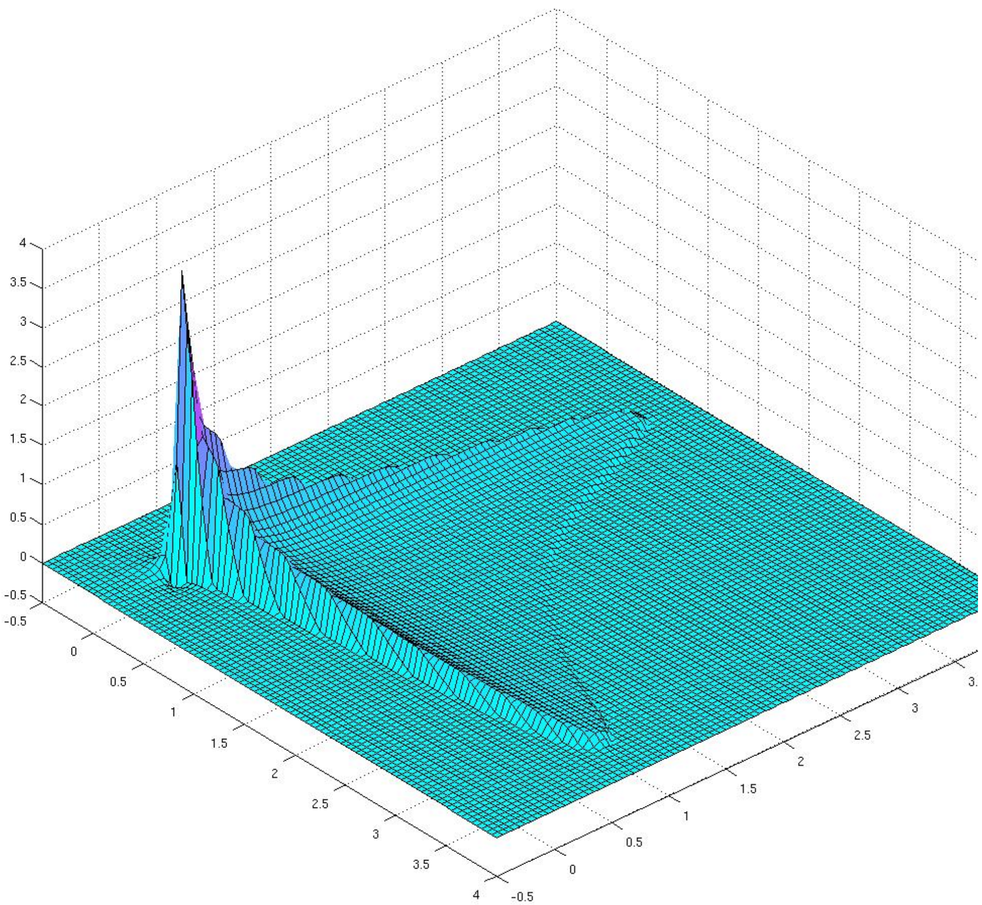
We conjecture that the eigenvalue distribution μ_{A_N} of the random matrices A_N converge to the Brown measure μ_x of the limit operator x.

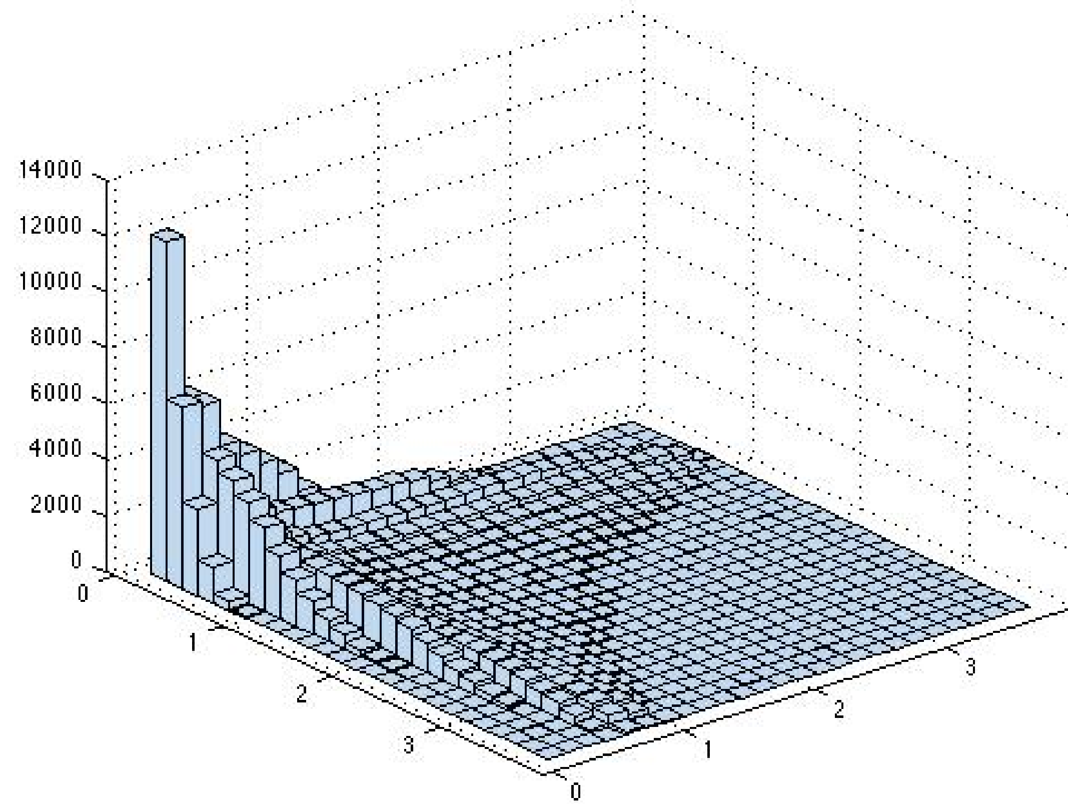
(compare: single ring theorem of Guionnet and Zeitouni 2011)

Brown measure of $xyz - 2yzx + zxy$ with x, y, z free semicircles



Brown measure of $x + iy$ with x, y free Poissons





Brown measure of $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$