QUANTUM GROUPS MADE EASY

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important organizing principle

SYMMETRY

Understanding symmetries of a problem might give:

- solving/better understanding of probem

- reduction to easier problem

Understanding symmetries/relations between roots of polynomial equations

explains

why fifth and higher degree polynomial equations are not solvable in general

Galois Theory $\sim$ 1830

concept of a group

Basic examples: matrix groups

Consider matrix groups, given by

matrices acting on vectors

$$A = (a_{ij})_{i,j=1}^{n} \quad \text{acting on} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1, \ldots, x_n)^t$$

Example: symmetry of (n-1)-sphere

The $(n-1)$-sphere

$$S^{n-1} := \{x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$$

is invariant under orthogonal transformations:

$$Ax \in S^{n-1} \quad \forall\, x \in S^{n-1} \quad \Longleftrightarrow \quad A \in M_n(\mathbb{R}) \quad \text{and} \quad A^t = A^{-1}$$

Let

$$Ax =: y = (y_1, \ldots, y_n)^t, \qquad \text{where} \quad y_k = \sum_{l=1}^{n} a_{kl} x_l$$

Then

$$\sum_{k=1}^{n} y_k^2 = \sum_{k=1}^{n} \sum_{l_1, l_2=1}^{n} a_{kl_1} a_{kl_2} x_{l_1} x_{l_2} = \sum_{l_1, l_2 = 1}^{n} \left(\sum_{k=1}^{n} a_{kl_1} a_{kl_2} \right) x_{l_1} x_{l_2} \stackrel{!}{=} \sum_{k=1}^{n} x_k^2,$$

thus we need

$$a_{kl_1} a_{kl_2} = \delta_{l_1 l_2}$$

Thus:

$$y \in S^{n-1} \quad \Longleftrightarrow \quad A^t A = 1,$$

i.e.,

$$Sym(S^{n-1}) = O_n \qquad \text{orthogonal group}$$

Example: symmetry of set of n distinct points

Set of n distinct points

$$P^n = \{p_1, \ldots, p_n\}$$

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Set of n distinct points

$$P^n = \{p_1, \dots, p_n\}$$

corresponds to functions $e_1, \dots, e_n$ on those points;

$$e_i : P^n \to \mathbb{R}; \quad e_i(p_j) = \delta_{ij} \qquad (i = 1, \dots, n)$$

latter can be algebraically described by

$$e_i^2 = e_i, \quad e_i e_j = 0 \, (i \neq j), \quad e_1 + \cdots + e_n = 1.$$

Thus

$$P^n \quad \widehat{=} \quad \{(e_1, \dots, e_n)^t \mid e_i^2 = e_i, \, e_i e_j = 0 \, (i \neq j), \, e_1 + \cdots + e_n = 1\}$$

Let

$$Ae =: f = (f_1, \ldots, f_n)^t, \qquad \text{where} \quad f_k = \sum_{l=1}^{n} a_{kl} e_l$$

Then

$$f_i = f_i^2$$

implies

$$\sum_j a_{ij} e_j \overset{!}{=} \left(\sum_j a_{ij} e_j \right)^2 = \sum_{j_1, j_2} a_{ij_1} a_{ij_2} \underbrace{e_{j_1} e_{j_2}}_{\delta_{j_1 j_2} e_{j_1}} = \sum_j a_{ij} a_{ij} e_j,$$

thus

$$a_{ij} = a_{ij}^2, \qquad \text{thus: } a_{ij} = 0 \text{ or } a_{ij} = 1$$

thus: $a_{ij} = 0$ or $a_{ij} = 1$

Similarly: one sees that

$$\sum_j a_{ij} = 1 = \sum_j a_{ji} \qquad \text{for all } i$$

thus: A has to be a permutation matrix

$$Sym(P^n) = S_n \qquad \text{permutation group}$$

General Frame

Note: in both cases we described our set as given by functions

- points: $e_i : P^n \to \mathbb{R}$

- sphere: can consider x_i as coordinate functions

$$x_i : S^{n-1} \to \mathbb{R};\ x \mapsto x_i$$

Our sets are given by functions subject to some constraint R

$$C = \{y = (y_1, \ldots, y_n)^t \mid y_1, \ldots, y_n \text{ functions satisfying } R\}$$

the corresponding symmetries are

$$Sym(C) = \{A = (a_{ij})_{i,j=1}^n \mid a_{ij} \text{ functions, } A \text{ preserves } R\}$$

Quantiziation

functions $\hat{=}$ commutative algebra	$\longrightarrow\longrightarrow\longrightarrow$	non-commutative algebra (operators on Hilbert spaces)

- physics: quantum mechanics needs operators for mathematical description (Heisenberg)

- mathematics: identify spaces with functions on those spaces, thus with commutative algebras and think of correponding classes of non-commutative algebras as functions on "non-commutative spaces"
(Gelfand, Grothendieck, Manin, Connes etc.)

Some important versions of non-commutative mathematics

space	**functions**	**non-commutative version**
topology	continuous	C^*-algebras
measure	measurable	von Neumann algebras
geometry	smooth differentials	non-commutative geometry

Quantum Symmetries

space	$\longrightarrow$	quantum space
$y = (y_1, \ldots, y_n)^t$		$y = (y_1, \ldots, y_n)^t$
$\{y_i\}$ commuting		$\{y_i\}$ non-commuting in general

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... but one can also consider quantum symmetries of a qspace

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Action of Quantum Symmetry

What means the action Ay for non-commutative entries: a_{ij} and y_k live on different spaces.

Put them on same space by taking tensor product!

$$Ay = (\tilde{y}_1, \dots, \tilde{y}_n)^t \qquad \text{with} \qquad \tilde{y}_i = \sum_{j=1}^{n} a_{ij} \otimes y_j$$

[This means that multiplication of our matrices (describing iterated action) should also be via tensor product:

$$A \odot A = (\tilde{a}_{ij}), \qquad \tilde{a}_{ij} = \sum_{k=1}^{n} a_{ik} \otimes a_{kj} \qquad]$$

Example: non-commutative (n-1)-sphere

$$QS^{n-1} := \{x = (x_1, \dots, x_n)^t \mid x_i = x_i^*, x_1^2 + \cdots + x_n^2 = 1\}$$

(universal C^*-algebra generated by those x_i)

Then we have the same classical symmetries as for commutative sphere:

$$Sym(QS^{n-1}) = O_n$$

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But there are more quantum symmetries:

$$QSym(QS^{n-1}) = \{A = (a_{ij}) \mid a_{ij} = a_{ij}^*, \sum_k a_{ik}a_{jk} = \delta_{ij}, \sum_k a_{ki}a_{kj} = \delta_{ij}\}$$

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The quantum orthogonal group $A_o(n) = C(O_n^+)$ is the universal unital C^*-algebra generated by a_{ij} $(i, j = 1, \ldots, n)$ subject to the relation

- $A = (a_{ij})_{i,j=1}^n$ is an orthogonal matrix

This means: for all i, j we have

$$\sum_{k=1}^{n} a_{ik}a_{jk} = \delta_{ij} \qquad \text{and} \qquad \sum_{k=1}^{n} a_{ki}a_{kj} = \delta_{ij}$$

Some remarks

- Quantum groups are no groups; they are generalizations of functions on groups

- O_n^+ does actually not exist, but only the non-commutative algebra $A_o(n) = "C(O_n^+)"$

- there is a kind of group like structure (Hopf algebra structure) on $C(O_n^+)$: if A_1 and A_2 satisfy relations, then so does

$$A_1 \odot A_2 = \Big(\sum_{k=1}^{n} a_{ik}^{(1)} \otimes a_{kj}^{(2)} \Big)_{i,j}$$

Some more remarks

- O_n^+ is "bigger" quantum group than O_n; quantum sphere has more symmetries than sphere

- Rigorously this means

$$C(O_n) \leftarrow A_o(n),$$

but often we prefer to write this as

$$O_n \subset O_n^+$$

Example: non-commutative space of n distinct points

$$QP^n = \{(e_1, \ldots, e_n)^t \mid e_i^2 = e_i,\, e_ie_j = 0\,(i \neq j),\, e_1 + \cdots + e_n = 1\}$$

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$$QP^n = P^n$$

But still: in non-commutative world there are more symmetries of P^n than in commutative world!

$$QSym(P^n) = \{A = (a_{ij}) \mid a_{ij}^* = a_{ij} = a_{ij}^2,\, \sum_j a_{ij} = 1 = \sum_j a_{ji}\, \forall i\}$$

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Quantum Permutation Group S_n^+ (Wang 1998)

The quantum permutation group $A_s(n) = C(S_n^+)$ is the universal unital C^*-algebra generated by a_{ij} $(i, j = 1, \ldots, n)$ subject to the relations

- $a_{ij}^2 = a_{ij} = a_{ij}^*$ for all $i, j = 1, \ldots, n$

- each row and column of $A = (a_{ij})_{i,j=1}^n$ is a partition of unity:

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad \sum_{i=1}^{n} a_{ij} = 1$$

If $n \geq 4$, then

$$S_n \subsetneq S_n^+$$

In general, we have

S_n	$\subset$	O_n	classical symmetry groups
$\cap$		$\cap$	
S_n^+	$\subset$	O_n^+	corresponding quantum symmetry groups

Those are examples of compact matrix quantum groups

Woronowiczs $\sim$ 1988

Note:

- S_n^+ is not a continuous deformation of S_n

- S_n^+ is a stronger symmetry than S_n

- S_n^+ plays the same role in the non-commutative world as S_n does in the commutative world

Question: Are there more such quantum strenghtenings of classical symmetries?

$$\begin{array}{ccccc} S_n & \subset & G & \subset & O_n \\ \cap & & & & \cap \\ S_n^+ & \subset & G^+ & \subset & O_n^+ \end{array} \quad \text{or} \quad S_n \subset G^* \subset O_n^+$$

- YES, but not clear how many and how to classify them
- there are "easy" ones, which have a "nice" structure (like S_n, O_n, S_n^+, O_n^+) and are easier to deal with

Classification of easy quantum groups

- [Banica, Speicher 2009]: There are exactly 6 easy classical groups G with $S_n \subset G \subset O_n$.

- [Banica, Speicher 2009, Weber 2011]: There are exactly 7 easy free groups G^+ with $S_n^+ \subset G^+ \subset O_n^+$.

- [Banica, Curran, Speicher 2010]: There are many more easy quantum groups G^* with $S_n \subset G^* \subset O_n^+$. We have only partial classification of them.

Definition and Proof

Depends on Woronowiczs's representation theory (Tannaka-Krein) for compact matrix quantum groups!

For

$$u = (u_{ij})_{i,j=1}^n \in M_n(\mathcal{A}) \qquad \text{compact matrix quantum group}$$

all irreducible representations can be found in tensor powers

$$u^{\otimes k} = (u_{i_1 j_1} \cdots u_{i k j_k})_{i_1,\dots,j_k=1}^n \in M_{n^k}(\mathcal{A})$$

of fundamental representation u

Information about them can be encoded in "intertwiners" between different powers

$$Int(k,l) := \{T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l} \mid Tu^{\otimes k} = u^{\otimes l} T\}.$$

Intertwiners $\longleftrightarrow$ Quantum Group

The collection of all intertwiner spaces

$$Int := \{Int(k,l) \mid k,l \in \mathbb{N}\}$$

forms a tensor category with duals and characterizes the corresponding quantum group.

$$S_n \subset G^* \subset O_n^+ \qquad \Leftrightarrow \qquad Int_{S_n} \supset Int_{G^*} \supset Int_{O_n^+}$$

What is Int_{S_n}?

Let $\pi \in \mathcal{P}(k,l)$ be a set partition of k upper and l lower points; define a corresponding linear operator

$$T_\pi : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$$

by

$$T_\pi e_{i_1} \otimes \cdots \otimes e_{i_k} = \sum_{\substack{\text{indices fit} \\ \text{with } \pi}} e_{j_1} \otimes \cdots \otimes e_{j_l}$$

Check:

$$T_\pi u^{\otimes k} = u^{\otimes l} T_\pi \qquad \forall\, \pi \in \mathcal{P}(k,l)$$

Those T_π generate actually all intertwiners:

$$Int_{S_n}(k,l) = \text{span}\{T_\pi \mid \pi \in \mathcal{P}(k,l)\}.$$

Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

$$
\begin{array}{ccccccc}
S_n^+ & \subset & O_n^+ & \qquad & Int_{S_n^+} & \supset & Int_{O_n^+} \\
\cup & & \cup & & \cap & & \cap \\
S_n & \subset & O_n & & Int_{S_n} & \supset & Int_{O_n}
\end{array}
$$

Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

Let $NC \subset \mathcal{P}$ be the subset of noncrossing partitions and $NC_2 \subset \mathcal{P}_2$ be (non-crossing) pair partitions.

$$\begin{array}{ccc}
\text{span}(T_\pi | \pi \in NC) = Int_{S_n^+} & \supset & Int_{O_n^+} = \text{span}(T_\pi | \pi \in NC_2) \\
\cap & & \cap \\
\text{span}(T_\pi | \pi \in \mathcal{P}) = Int_{S_n} & \supset & Int_{O_n} = \text{span}(T_\pi | \pi \in \mathcal{P}_2)
\end{array}$$

Easy Quantum Groups

(Banica, Speicher 2009)

A quantum group $S_n \subset G_n^* \subset O_n^+$ is called **easy** when its associated tensor category is of the form

$$
\begin{array}{c}
Int_{S_n} = \text{span}(T_\pi \mid \pi \in \mathcal{P}) \\
\cup \\
\mathbf{Int_{G_n^*}} \\
\cup \\
Int_{O_n^+} = \text{span}(T_\pi \mid \pi \in NC_2)
\end{array}
$$

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A quantum group $S_n \subset G_n^* \subset O_n^+$ is called **easy** when its associated tensor category is of the form

$$Int_{S_n} = \mathrm{span}(T_\pi \mid \pi \in \mathcal{P})$$

$$\cup$$

$$\mathbf{Int_{G_n^*}} = \mathbf{span}(\mathbf{T}_\pi \mid \pi \in \mathcal{P}_{\mathbf{G}^*}),$$

$$\cup$$

$$Int_{O_n^+} = \mathrm{span}(T_\pi \mid \pi \in NC_2)$$

for a certain collection of subsets $\mathcal{P}_{G^*} \subset \mathcal{P}$.

Classification Results [Banica, Speicher 2009; Weber 2011]:
There are

- 7 Categories of Noncrossing Partitions and
- 6 Categories of Partitions containing Basic Crossing:

$$
\begin{array}{ccccccc}
\left\{\begin{matrix}\text{singletons}\\ \text{pairings}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons}\\ \text{pairings}\\ \text{(even part)}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons}\\ \text{pairings}\\ \text{(resp. parity)}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all}\\ \text{pairings}\end{matrix}\right\} \\
\cap & & \cap & & & & \cap \\
\left\{\begin{matrix}\text{all}\\ \text{partitions}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all partitions}\\ \text{(even part)}\end{matrix}\right\} & \supset & & & \left\{\begin{matrix}\text{blocks of}\\ \text{even size}\end{matrix}\right\}
\end{array}
$$

Classification Results [Banica, Speicher 2009; Weber 2011]:
...and thus there are

- 7 free easy quantum groups $S_n^+ \subset G_n^+ \subset O_n^+$ and
- 6 classical easy groups $S_n \subset G_n \subset O_n$

$$\begin{array}{ccccccc} B_n^+ & \subset & B'^+_n & \subset & B_n^{\#+} & \subset & O_n^+ \\ \cup & & \cup & & & & \cup \\ S_n^+ & \subset & S'^+_n & & \subset & & H_n^+ \end{array}$$

The easy classical groups

- O_n and S_n

- $H_n = \mathbb{Z}_2 \wr S_n$: the hyperoctahedral group, consisting of monomial matrices with ± 1 nonzero entries.

- $B_n \simeq O_{n-1}$: the bistochastic group, consisting of orthogonal matrices having sum 1 in each row and each column.

- $S'_n = \mathbb{Z}_2 \times S_n$: permutation matrices multiplied by ± 1.

- $B'_n = \mathbb{Z}_2 \times B_n$: bistochastic matrices multiplied by ± 1.

de Finetti Theorem

(de Finetti 1931, Hewitt, Savage 1955)

The following are equivalent for an infinite sequence of classical, commuting random variables:

- the sequence is exchangeable (i.e., for each n the distribution of $x_1, \ldots, x_n$ is invariant under the action of S_n)

- the sequence is independent and identically distributed with respect to its tail σ-algebra

Non-commutative de Finetti Theorem

(Köstler, Speicher 2008)

The following are equivalent for an infinite sequence of non-commutative random variables:

- the sequence is quantum exchangeable (i.e., for each n the distribution of $x_1, \ldots, x_n$ is invariant under the action of S_n^+)

- the sequence is free (in the sense of Voiculescu's free probability theory) and identically distributed with respect to its tail algebra

classical world		**non-commutative world**
commuting variables		**maximal nc variables**
permutations rotations	$\longleftrightarrow$	quantum permutations quantum rotations

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classical world **commuting variables**		non-commutative world **maximal nc variables**
permutations rotations	$\longleftrightarrow$	quantum permutations quantum rotations
independence	$\longleftrightarrow$	free independence
derivative $(x^n)' = nx^{n-1}$	$\longleftrightarrow$	non-commutative derivative $D(x^n) = \sum_{k=0}^{n-1} x^k \otimes x^{n-k-1}$
analysis	$\longleftrightarrow$	free analysis
$\vdots$		$\vdots$