

QUANTUM GROUPS MADE EASY

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important organizing principle

SYMMETRY

Understanding symmetries of a problem might give:

- solving/better understanding of problem
- reduction to easier problem

Understanding symmetries/relations between roots of polynomial equations

explains

why fifth and higher degree polynomial equations are not solvable in general

Galois Theory ~ 1830

concept of a group

Basic examples: matrix groups

Consider matrix groups, given by

matrices acting on vectors

$$A = (a_{ij})_{i,j=1}^n \quad \text{acting on} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1, \dots, x_n)^t$$

Example: symmetry of (n-1)-sphere

The $(n - 1)$ -sphere

$$S^{n-1} := \{x = (x_1, \dots, x_n)^t \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1\}$$

is invariant under orthogonal transformations:

$$Ax \in S^{n-1} \quad \forall x \in S^{n-1} \quad \iff \quad A \in M_n(\mathbb{R}) \quad \text{and} \quad A^t = A^{-1}$$

Let

$$Ax =: y = (y_1, \dots, y_n)^t, \quad \text{where} \quad y_k = \sum_{l=1}^n a_{kl} x_l$$

Then

$$\sum_{k=1}^n y_k^2 = \sum_{k=1}^n \sum_{l_1, l_2=1}^n a_{kl_1} a_{kl_2} x_{l_1} x_{l_2} = \sum_{l_1, l_2=1}^n \left(\sum_{k=1}^n a_{kl_1} a_{kl_2} \right) x_{l_1} x_{l_2} \stackrel{!}{=} \sum_{k=1}^n x_k^2,$$

thus we need

$$a_{kl_1} a_{kl_2} = \delta_{l_1 l_2}$$

Thus:

$$y \in S^{n-1} \iff A^t A = 1,$$

i.e.,

$$\text{Sym}(S^{n-1}) = O_n \quad \text{orthogonal group}$$

Example: symmetry of set of n distinct points

Set of n distinct points

$$P^n = \{p_1, \dots, p_n\}$$

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Set of n distinct points

$$P^n = \{p_1, \dots, p_n\}$$

corresponds to functions e_1, \dots, e_n on those points;

$$e_i : P^n \rightarrow \mathbb{R}; \quad e_i(p_j) = \delta_{ij} \quad (i = 1, \dots, n)$$

latter can be algebraically described by

$$e_i^2 = e_i, \quad e_i e_j = 0 \ (i \neq j), \quad e_1 + \dots + e_n = 1.$$

Thus

$$P^n \hat{=} \{(e_1, \dots, e_n)^t \mid e_i^2 = e_i, e_i e_j = 0 \ (i \neq j), e_1 + \dots + e_n = 1\}$$

Let

$$Ae =: f = (f_1, \dots, f_n)^t, \quad \text{where} \quad f_k = \sum_{l=1}^n a_{kl}e_l$$

Then

$$f_i = f_i^2$$

implies

$$\sum_j a_{ij}e_j \stackrel{!}{=} \left(\sum_j a_{ij}e_j \right)^2 = \sum_{j_1, j_2} a_{ij_1}a_{ij_2} \underbrace{e_{j_1}e_{j_2}}_{\delta_{j_1 j_2} e_{j_1}} = \sum_j a_{ij}a_{ij}e_j,$$

thus

$$a_{ij} = a_{ij}^2, \quad \text{thus: } a_{ij} = 0 \text{ or } a_{ij} = 1$$

thus: $a_{ij} = 0$ or $a_{ij} = 1$

Similarly: one sees that

$$\sum_j a_{ij} = 1 = \sum_j a_{ji} \quad \text{for all } i$$

thus: A has to be a permutation matrix

$$\text{Sym}(P^n) = S_n \quad \text{permutation group}$$

General Frame

Note: in both cases we described our set as given by functions

- points: $e_i : P^n \rightarrow \mathbb{R}$
- sphere: can consider x_i as coordinate functions

$$x_i : S^{n-1} \rightarrow \mathbb{R}; x \mapsto x_i$$

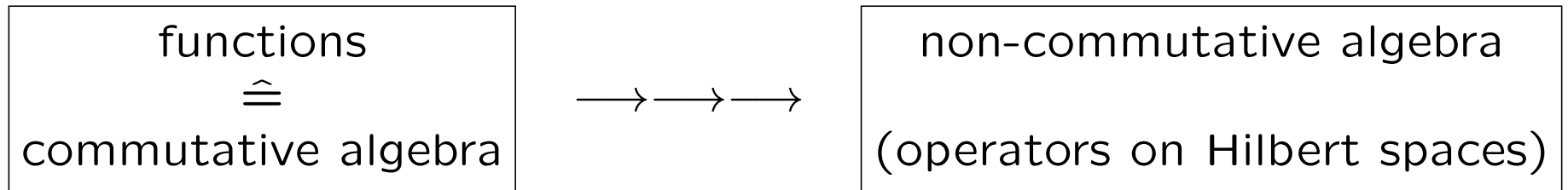
Our sets are given by functions subject to some constraint R

$$C = \{y = (y_1, \dots, y_n)^t \mid y_1, \dots, y_n \text{ functions satisfying } R\}$$

the corresponding symmetries are

$$\text{Sym}(C) = \{A = (a_{ij})_{i,j=1}^n \mid a_{ij} \text{ functions, } A \text{ preserves } R\}$$

Quantization

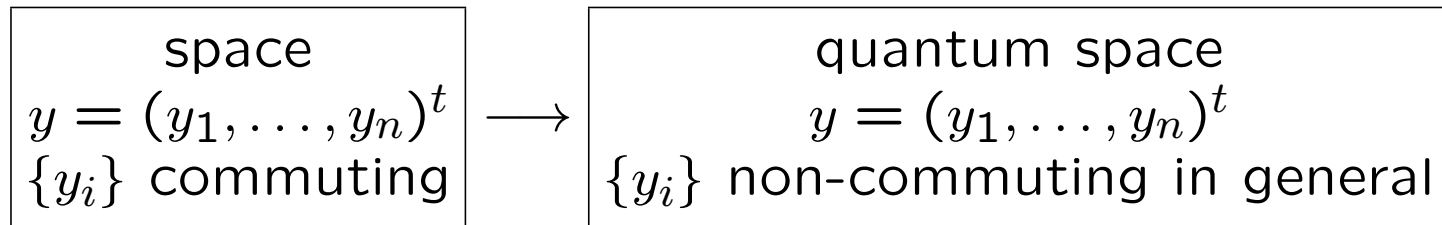


- physics: quantum mechanics needs operators for mathematical description (Heisenberg)
- mathematics: identify spaces with functions on those spaces, thus with commutative algebras and think of corresponding classes of non-commutative algebras as functions on "non-commutative spaces"
(Gelfand, Grothendieck, Manin, Connes etc.)

Some important versions of non-commutative mathematics

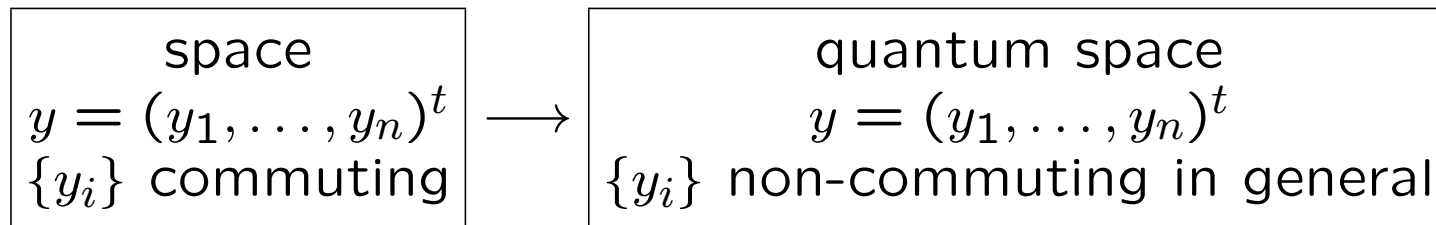
space	functions	non-commutative version
topology	continuous	C^* -algebras
measure	measurable	von Neumann algebras
geometry	smooth differentials	non-commutative geometry

Quantum Symmetries



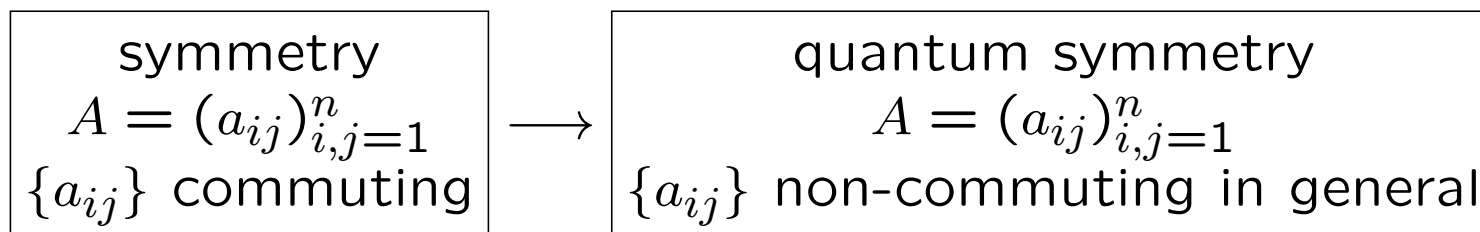
One can consider classical symmetry of a quantum space ...

Quantum Symmetries



One can consider classical symmetry of a quantum space ...

... but one can also consider quantum symmetries of a qspace



Action of Quantum Symmetry

What means the action Ay for non-commutative entries: a_{ij} and y_k live on different spaces.

Put them on same space by taking tensor product!

$$Ay = (\tilde{y}_1, \dots, \tilde{y}_n)^t \quad \text{with} \quad \tilde{y}_i = \sum_{j=1}^n a_{ij} \otimes y_j$$

[This means that multiplication of our matrices (describing iterated action) should also be via tensor product:

$$A \odot A = (\tilde{a}_{ij}), \quad \tilde{a}_{ij} = \sum_{k=1}^n a_{ik} \otimes a_{kj} \quad]$$

Example: non-commutative (n-1)-sphere

$$QS^{n-1} := \{x = (x_1, \dots, x_n)^t \mid x_i = x_i^*, x_1^2 + \dots + x_n^2 = 1\}$$

(universal C^* -algebra generated by those x_i)

Then we have the same classical symmetries as for commutative sphere:

$$\text{Sym}(QS^{n-1}) = O_n$$

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But there are more quantum symmetries:

$$Q\text{Sym}(QS^{n-1}) = \{A = (a_{ij}) \mid a_{ij} = a_{ij}^*, \sum_k a_{ik}a_{jk} = \delta_{ij}, \sum_k a_{ki}a_{kj} = \delta_{ij}\}$$

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The quantum orthogonal group $A_o(n) = C(O_n^+)$ is the universal unital C^* -algebra generated by a_{ij} ($i, j = 1, \dots, n$) subject to the relation

- $A = (a_{ij})_{i,j=1}^n$ is an orthogonal matrix

This means: for all i, j we have

$$\sum_{k=1}^n a_{ik}a_{jk} = \delta_{ij} \quad \text{and} \quad \sum_{k=1}^n a_{ki}a_{kj} = \delta_{ij}$$

Some remarks

- Quantum groups are no groups; they are generalizations of functions on groups
- O_n^+ does actually not exist, but only the non-commutative algebra $A_o(n) = "C(O_n^+)"$
- there is a kind of group like structure (Hopf algebra structure) on $C(O_n^+)$: if A_1 and A_2 satisfy relations, then so does

$$A_1 \odot A_2 = \left(\sum_{k=1}^n a_{ik}^{(1)} \otimes a_{kj}^{(2)} \right)_{i,j}$$

Some more remarks

- O_n^+ is "bigger" quantum group than O_n ; quantum sphere has more symmetries than sphere
- Rigorously this means

$$C(O_n) \leftarrow A_o(n),$$

but often we prefer to write this as

$$O_n \subset O_n^+$$

Example: non-commutative space of n distinct points

$$QP^n = \{(e_1, \dots, e_n)^t \mid e_i^2 = e_i, e_i e_j = 0 (i \neq j), e_1 + \dots + e_n = 1\}$$

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$$QP^n = P^n$$

But still: in non-commutative world there are more symmetries of P^n than in commutative world!

$$QSym(P^n) = \{A = (a_{ij}) \mid a_{ij}^* = a_{ij} = a_{ij}^2, \sum_j a_{ij} = 1 = \sum_j a_{ji} \forall i\}$$

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The quantum permutation group $A_s(n) = C(S_n^+)$ is the universal unital C^* -algebra generated by a_{ij} ($i, j = 1, \dots, n$) subject to the relations

- $a_{ij}^2 = a_{ij} = a_{ij}^*$ for all $i, j = 1, \dots, n$
- each row and column of $A = (a_{ij})_{i,j=1}^n$ is a partition of unity:

$$\sum_{j=1}^n a_{ij} = 1 \quad \sum_{i=1}^n a_{ij} = 1$$

If $n \geq 4$, then

$$S_n \subsetneq S_n^+$$

In general, we have

S_n	\subset	O_n	classical symmetry groups
\cap		\cap	
S_n^+	\subset	O_n^+	corresponding quantum symmetry groups

Those are examples of compact matrix quantum groups

Woronowicz \sim 1988

Note:

- S_n^+ is not a continuous deformation of S_n
- S_n^+ is a stronger symmetry than S_n
- S_n^+ plays the same role in the non-commutative world as S_n does in the commutative world

Question: Are there more such quantum strenghtenings of classical symmetries?

$$S_n \subset G \subset O_n$$

$$\cap \qquad \qquad \qquad \cap \qquad \text{or} \qquad S_n \subset G^* \subset O_n^+$$

$$S_n^+ \subset G^+ \subset O_n^+$$

- YES, but not clear how many and how to classify them
- there are "easy" ones, which have a "nice" structure (like S_n, O_n, S_n^+, O_n^+) and are easier to deal with

Classification of easy quantum groups

- [Banica, Speicher 2009]: There are exactly 6 easy classical groups G with $S_n \subset G \subset O_n$.
- [Banica, Speicher 2009, Weber 2011]: There are exactly 7 easy free groups G^+ with $S_n^+ \subset G^+ \subset O_n^+$.
- [Banica, Curran, Speicher 2010]: There are many more easy quantum groups G^* with $S_n \subset G^* \subset O_n^+$. We have only partial classification of them.

Definition and Proof

Depends on Woronowicz's representation theory (Tannaka-Krein) for compact matrix quantum groups!

For

$u = (u_{ij})_{i,j=1}^n \in M_n(\mathcal{A})$ compact matrix quantum group

all irreducible representations can be found in tensor powers

$$u^{\otimes k} = (u_{i_1 j_1} \cdots u_{i_k j_k})_{i_1, \dots, j_k=1}^n \in M_{n^k}(\mathcal{A})$$

of fundamental representation u

Information about them can be encoded in "intertwiners" between different powers

$$\text{Int}(k, l) := \{T : (\mathbb{C}^n)^{\otimes k} \rightarrow (\mathbb{C}^n)^{\otimes l} \mid Tu^{\otimes k} = u^{\otimes l}T\}.$$

Intertwiners \longleftrightarrow Quantum Group

The collection of all intertwiner spaces

$$Int := \{Int(k, l) \mid k, l \in \mathbb{N}\}$$

forms a tensor category with duals and characterizes the corresponding quantum group.

$$S_n \subset G^* \subset O_n^+ \quad \Leftrightarrow \quad Int_{S_n} \supset Int_{G^*} \supset Int_{O_n^+}$$

What is Int_{S_n} ?

Let $\pi \in \mathcal{P}(k, l)$ be a set partition of k upper and l lower points; define a corresponding linear operator

$$T_\pi : (\mathbb{C}^n)^{\otimes k} \rightarrow (\mathbb{C}^n)^{\otimes l}$$

by

$$T_\pi e_{i_1} \otimes \cdots \otimes e_{i_k} = \sum_{\substack{\text{indices fit} \\ \text{with } \pi}} e_{j_1} \otimes \cdots \otimes e_{j_l}$$

Check:

$$T_\pi u^{\otimes k} = u^{\otimes l} T_\pi \quad \forall \pi \in \mathcal{P}(k, l)$$

Those T_π generate actually all intertwiners:

$$Int_{S_n}(k, l) = \text{span}\{T_\pi \mid \pi \in \mathcal{P}(k, l)\}.$$

Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

S_n^+	\subset	O_n^+	$Int_{S_n^+}$	\supset	$Int_{O_n^+}$
U		U	\cap		\cap
S_n	\subset	O_n	Int_{S_n}	\supset	Int_{O_n}

Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

Let $NC \subset \mathcal{P}$ be the subset of noncrossing partitions and $NC_2 \subset \mathcal{P}_2$ be (non-crossing) pair partitions.

$$\text{span}(T_\pi | \pi \in NC) = \text{Int}_{S_n^+} \supset \text{Int}_{O_n^+} = \text{span}(T_\pi | \pi \in NC_2)$$

∩

∩

$$\text{span}(T_\pi | \pi \in \mathcal{P}) = \text{Int}_{S_n} \supset \text{Int}_{O_n} = \text{span}(T_\pi | \pi \in \mathcal{P}_2)$$

Easy Quantum Groups

(Banica, Speicher 2009)

A quantum group $S_n \subset G_n^* \subset O_n^+$ is called **easy** when its associated tensor category is of the form

$$\begin{array}{c}
 \text{Int}_{S_n} = \text{span}(T_\pi \mid \pi \in \mathcal{P}) \\
 \cup \\
 \text{Int}_{G_n^*} \\
 \cup \\
 \text{Int}_{O_n^+} = \text{span}(T_\pi \mid \pi \in NC_2)
 \end{array}$$

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A quantum group $S_n \subset G_n^* \subset O_n^+$ is called **easy** when its associated tensor category is of the form

$$\begin{aligned} \text{Int}_{S_n} &= \text{span}(T_\pi \mid \pi \in \mathcal{P}) \\ &\cup \\ \text{Int}_{G_n^*} &= \text{span}(\mathbf{T}_\pi \mid \pi \in \mathcal{P}_{G^*}), \\ &\cup \\ \text{Int}_{O_n^+} &= \text{span}(T_\pi \mid \pi \in NC_2) \end{aligned}$$

for a certain collection of subsets $\mathcal{P}_{G^*} \subset \mathcal{P}$.

Classification Results [Banica, Speicher 2009; Weber 2011]:

There are

- 7 Categories of Noncrossing Partitions and
- 6 Categories of Partitions containing Basic Crossing:

$$\begin{array}{ccccccc} \left\{ \begin{array}{l} \text{singletons} \\ \text{pairings} \end{array} \right\} & \supset & \left\{ \begin{array}{l} \text{singletons} \\ \text{pairings} \\ \text{(even part)} \end{array} \right\} & \supset & \left\{ \begin{array}{l} \text{singletons} \\ \text{pairings} \\ \text{(resp. parity)} \end{array} \right\} & \supset & \left\{ \begin{array}{l} \text{all} \\ \text{pairings} \end{array} \right\} \\ \cap & & \cap & & & & \cap \\ \left\{ \begin{array}{l} \text{all} \\ \text{partitions} \end{array} \right\} & \supset & \left\{ \begin{array}{l} \text{all partitions} \\ \text{(even part)} \end{array} \right\} & \supset & & & \left\{ \begin{array}{l} \text{blocks of} \\ \text{even size} \end{array} \right\} \end{array}$$

Classification Results [Banica, Speicher 2009; Weber 2011]:

...and thus there are

- 7 free easy quantum groups $S_n^+ \subset G_n^+ \subset O_n^+$ and
- 6 classical easy groups $S_n \subset G_n \subset O_n$

$$\begin{array}{ccccccc} B_n^+ & \subset & B'_n{}^+ & \subset & B_n^{\#}{}^+ & \subset & O_n^+ \\ \cup & & \cup & & & & \cup \\ S_n^+ & \subset & S'_n{}^+ & \subset & & \subset & H_n^+ \end{array}$$

The easy classical groups

- O_n and S_n
- $H_n = \mathbb{Z}_2 \wr S_n$: the hyperoctahedral group, consisting of monomial matrices with ± 1 nonzero entries.
- $B_n \simeq O_{n-1}$: the bistochastic group, consisting of orthogonal matrices having sum 1 in each row and each column.
- $S'_n = \mathbb{Z}_2 \times S_n$: permutation matrices multiplied by ± 1 .
- $B'_n = \mathbb{Z}_2 \times B_n$: bistochastic matrices multiplied by ± 1 .

de Finetti Theorem

(de Finetti 1931, Hewitt, Savage 1955)

The following are equivalent for an infinite sequence of classical, commuting random variables:

- the sequence is exchangeable (i.e., for each n the distribution of x_1, \dots, x_n is invariant under the action of S_n)
- the sequence is independent and identically distributed with respect to its tail σ -algebra

Non-commutative de Finetti Theorem

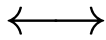
(Köstler, Speicher 2008)

The following are equivalent for an infinite sequence of non-commutative random variables:

- the sequence is quantum exchangeable (i.e., for each n the distribution of x_1, \dots, x_n is invariant under the action of S_n^+)
- the sequence is free (in the sense of Voiculescu's free probability theory) and identically distributed with respect to its tail algebra

classical world
commuting variables

permutations
rotations



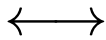
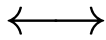
non-commutative world
maximal nc variables

quantum permutations
quantum rotations

classical world
commuting variables

permutations
rotations

independence



non-commutative world
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classical world
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non-commutative world
maximal nc variables

permutations
rotations

\longleftrightarrow

quantum permutations
quantum rotations

independence

\longleftrightarrow

free independence

derivative
 $(x^n)' = nx^{n-1}$

\longleftrightarrow

non-commutative derivative
 $D(x^n) = \sum_{k=0}^{n-1} x^k \otimes x^{n-k-1}$

analysis

\longleftrightarrow

free analysis

⋮

⋮