On Deformations and Independencies

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"Non-Commutative Distributions in Free Probability"

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Section 1

A long time ago in a galaxy far, far away ...

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The classical world (von Waldenfels, Hudson-Parthasarathy, Kümmerer, Accardi, etc)

- classical independence
- stochastic integration theories (bosonic, fermionic)
- notion of Gaussianity
- realization in bosonic and fermionic Fock spaces
- CCR and CAR algebras

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- free central limit theorem, notion of free Gaussianity ("semicircularity")
- realization on full Fock spaces
- Cuntz algebra

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The world of Marek

providing links between and challenges for the "ordinary" worlds

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Joint work with Marek

- 1991: ψ-independent and symmetrized white noises, QP VI
- 1991: An example of a generalized Brownian motion, CMP
- 1992: An example of a generalized Brownian motion II, QP VII
- 1994: Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces, Math. Ann.
- 1996: Convolution and limit theorems for conditionally free random variables (with Leinert), Pac. J. Math.
- 1996: Interpolations between bosonic and fermionic relations given by generalized Brownian motions, Math. Z.
- 1997: q-Gaussian processes: non-commutative and classical aspects (with Kümmerer), CMP
- 2011: The normal distribution is $\boxplus$-infinitely divisible (with Belinschi, Lehner), Adv. Math.

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Section 2

The q-interpolation between classical and free world

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- from operator theoretic point of view
  - classical independent Gauss variables are given by $x_i = a_i + a_i^*$ where a_i satisfy CCR relations

$$a_i a_j^* - a_j^* a_i = \delta_{ij} 1$$

  - freely independent semicircular variables are given $x_i = l_i + l_i^*$ where l_i satisfy Cuntz relations

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Interpolate between those by q-relations

$$a_i a_j^* - q a_j^* a_i = \delta_{ij}$$

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$$\varphi(x_{i(1)} \cdots x_{i(n)}) = \sum_{\pi \in \mathcal{P}(n)} \prod_{(p,q)\in\pi} \delta_{i(p)i(q)}$$

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Interpolate between those by the formula

$$\varphi(x_{i(1)} \cdots x_{i(n)}) = \sum_{\pi \in \mathcal{P}(n)} q^{crossings(\pi)} \prod_{(p,q) \in \pi} \delta_{i(p)i(q)}$$

All this is quite straightforward, the crucial point, however, is **positivity**.

- Are there operators satisfying the q-relations on a Hilbert space, such that a_i and a_i^* are adjoints?
- Is the linear functional φ defined by its moments via the q-Wick formula a state, i.e., positive?

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In the end all this leads to the question

Problem

Is the map

$$S_n \to \mathbb{C}, \quad \pi \mapsto q^{inv(\pi)}$$

on the permutation group S_n a positive definite function for each $n \in \mathbb{N}$.?
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Theorem (Bozejko, Speicher 1991)

Yes, indeed, for $-1 \leq q \leq 1$. This gives then the q-Fock space realization of the q-creation and annihilation operators and the existence of q-Gaussian random variables.

This was all very exciting and led to many new developments and questions ...

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  - given in terms of relations: q_{ij}-relations, relations determined by Yang-Baxter operators, etc ...
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- notions of independence
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- notions of convolutions
  - are there q-convolutions
  - ongoing quest of Marek for other (r-, Δ-,...) convolutions

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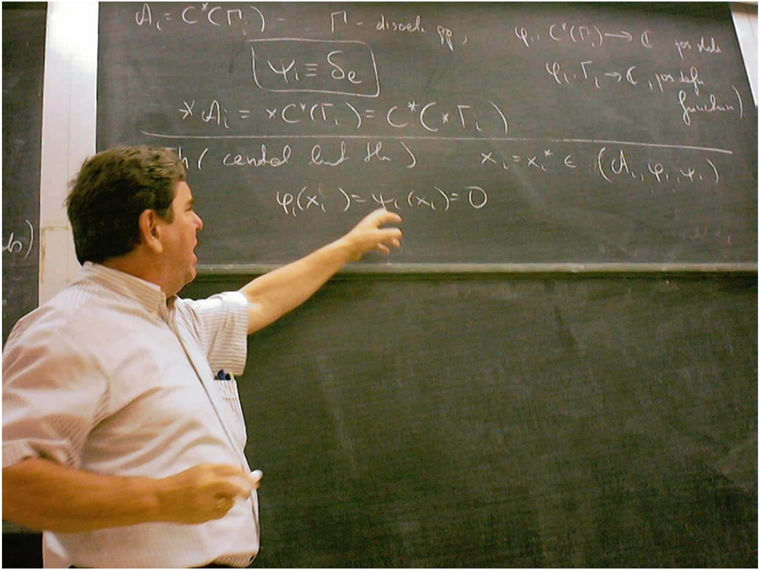
$\psi_i \equiv \delta_e$
$x_i = x_i^* \in (A_i, \varphi_i, \psi_i)$
$\varphi_i(x_i) = \psi_i(x_i) = 0$

Section 3

The intersection between the classical and the free world

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Question

Could it be that the classical Gauss distribution is infinitely divisible in the free sense?

"This fact might come as a surprise, since the classical Gaussian distribution has no special role in free probability theory. The first known explicit mentioning of that possibility to one of us was by Perez-Abreu at a meeting in Guanajuato in 2007. This conjecture had arisen out of joint work with Arizmendi. Later when the last three of the present authors met in Bielefeld in the fall of 2008 they were led to reconsider this question in the context of investigations about general Brownian motions."

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Theorem (Belinschi, Bozejko, Lehner, Speicher 2009)

Yes, indeed, the classical Gauss distribution is $\boxplus$*-infinitely divisible.*

Section 4

Mixtures of Classical and Free Independence

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History

- Green (1990): graph product of groups
- Młotkowski (2004): Λ-freeness
- Caspers + Fima (2014): graph product of operator algebras
- Wysoczanksi + Speicher (2016): ε-independence
- Speicher + Weber (2016): new corresponding quantum spheres and quantum groups

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Non-commutative probability spaces given by groups

$$G \quad \rightsquigarrow \quad (\mathbb{C}G, \tau) \quad \text{or} \quad (L(G), \tau)$$

nc $(W^*$-) probability space

where

$$\tau(\sum_g \alpha_g g) = \alpha_1 \quad \text{or} \quad \tau(x) = \langle x\delta_1, \delta_1 \rangle.$$

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freeness/independence $\hat{=}$ basic group constructions: free/direct product $\star_{i \in I} G_i, \quad \times_{i \in I} G_i$

$\uparrow$

only "universal" unital constructions which are uniform in $i \in I$

More General, Non-Homogeneous, Group Constructions

$$\text{direct product} = \text{free product} \Big/ \text{commutation relations between all } G_i$$

More General, Non-Homogeneous, Group Constructions

direct product = $\dfrac{\text{free product}}{\text{commutation relations between all } G_i}$

Generalize this to : partial commutation relations between groups

Fix symmetric matrix $(\varepsilon_{ij})_{i,j\in I} = (\Lambda_{ij})_{i,j\in I}$ with $0/1$ entries, with the idea that for $i \neq j$

$$\begin{array}{lcl} \varepsilon_{ij} = \varepsilon_{ji} = 1 & \hat{=} & [G_i, G_j] = 0 \\ \varepsilon_{ij} = \varepsilon_{ji} = 0 & \hat{=} & \text{no relation} \end{array}$$

[ε_{ii} is unspecified or sometimes it is convenient to put $\varepsilon_{ii} = 0$.]

Definition (Green 1990)

Let G_i $(i \in I)$ be groups. The **graph product** $\star_\varepsilon G_i$ (corresponding to graph with adjacency matrix ε) is

$$\star_{i\in I} G_i \Big/ \; G_i,\, G_j \; (i \neq j) \text{ commute whenever } \varepsilon_{ij} = 1$$

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Example

If $G_i = \mathbb{Z}$, then $\star_\varepsilon G_i$ has various names

- right angled Artin group
- free partially commutative group
- trace group
- etc

also: Cartier-Foata monoid, trace monoid, ...

Definition (Młotkowski 2004)

Subalgebras $(\mathcal{A}_i)_{i\in I}$ are **ε-independent** in $(\mathcal{A}, \varphi)$ if

- $[\mathcal{A}_i, \mathcal{A}_j] = 0$ for all $i \neq j$ with $\varepsilon_{ij} = 1$
- Consider $a_j \in \mathcal{A}_{i(j)}$ $(i = 1, \dots, n)$ with
  - $\varphi(a_j) = 0 \quad \forall j$
  - $i(j) \neq i(j+1)$ modulo commutation relations

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then there exists p with $k<p<l$ such that $i(k)\neq i(p)\neq i(l)$ and $\varepsilon_{i(k)i(p)}=0$

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There is a cumulant description (using free cumulants).

Relation to q-Gaussians

Theorem (Młotkowski)

Choose ε_{ij}, for $i > j$, independent and identically distributed, with $Prob(\varepsilon_{ij} = 1) = q$. Consider a sequence $x_1, x_2, \dots$ of ε-independent and identically distributed random variables with $\varphi(x_i) = 0$ and $\varphi(x_i^2) = 1$. Then

$$\frac{x_1 + \cdots + x_n}{\sqrt{n}}$$

converges, for $n \to \infty$, almost surely to a q-Gaussian distribution.

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Relation to q-Gaussians

Theorem

Choose the ε_{ij} as before. For each $n \in \mathbb{N}$, let $x_1^{(n)}, \dots, x_n^{(n)}$ be identically distributed and ε-independent. Assume furthermore that for each $k \in \mathbb{N}$ the limit

$$\lim_{n\to\infty} n\varphi((x_i^{(n)})^k) =: \alpha_k$$

exists. Then the random variable $x_1^{(n)} + \cdots + x_n^{(n)}$ converges in distribution, on average and also almost surely, for $n \to \infty$, to a variable x whose distribution is given by

$$\varphi(x^k) = \sum_{\pi \in \mathcal{P}(k)} q^{\text{crossing}(\pi)} \alpha_\pi,$$

where crossing(π) is the number of pairs of blocks of π which have a crossing, and α_π is the multiplicative extension according to the blocks of π.

Does there exist a nice q-convolution?

Convolution via cumulants

A formula like

$$\varphi(x^k) = \sum_{\pi \in \mathcal{P}(k)} q^{\text{crossing}(\pi)} \alpha_\pi,$$

can be used to define q-cumulants α_n; and q-convolution by additivity of the q-cumulants. The main problem is again positivity.

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Example

The definition of q-cumulants depends on the choice of *number of crossings* for a general partition. There have been various suggestions

- "left reduced number of crossings" (Nica)
  not positive
- "restricted number of crossings" (Anshelevich)
- the above suggests: take
  "number of pairs of blocks which cross"

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So maybe, a positive q-convolution exists in the end ....
which, of course, it should, because life and mathematics is nice.

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Happy Birthday, Marek!

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