Random Matrices and Their Limits

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"Non-Commutative Distributions in Free Probability"

Section 1

Random Matrices and Operators

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Deterministic limits of random matrices

Fundamental observation

Many random matrices show, via concentration, for $N \to \infty$ almost surely a *deterministic* behaviour.

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Interesting observation

Many random matrices show, via concentration, for $N \to \infty$ almost surely a deterministic *interesting* behaviour.

Interesting observation for the operator algebraic inclined

Many random matrices show, via concentration, for $N \to \infty$ almost surely a deterministic behaviour, which can be described by interesting *operators on Hilbert spaces* (or their generated C^*-algebras or von Neumann algebras)

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Random matrices and operators

Fundamental observation of Voiculescu (1991)

Limit of random matrices can often be described by "nice" and "interesting" operators on Hilbert spaces
(which, in the case of several matrices, describe interesting von Neumann algebras)

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Convergence of random matrices

$(X_1^{(N)}, \dots, X_m^{(N)})$	$\longrightarrow$	$(x_1, \dots, x_m)$
random matrices	almost surely	operators
$(M_N(\mathbb{C}), \mathrm{tr})$		$(\mathcal{A}, \tau), \mathcal{A} \subset B(\mathcal{H})$

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Convergence in distribution

We have for all polynomials p in m non-commuting variables

$$\lim_{N\to\infty} \mathrm{tr}[p(X_1^{(N)}, \dots, X_m^{(N)})] = \tau[p(x_1, \dots, x_m)]$$

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Strong convergence

- convergence in distribution
- and for all polynomials p

$$\lim_{N\to\infty} \|p(X_1^{(N)}, \dots, X_m^{(N)})\| = \|p(x_1, \dots, x_m)\|$$

Our most beloved example: independent GUE $\rightarrow$ free semicirculars

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One-matrix case: classical commuting case

Note: $X_N \rightarrow x$ means that all moments of X_N converge to corresponding moments of x, hence the distributions (in classical sense of probability measures on $\mathbb{R}$) converge

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$(X_N, Y_N) \rightarrow (x, y)$ means convergence of all moments, but this does not correspond to convergence of probability measures on $\mathbb{R}^2$

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... if we still want to see classical objects and pictures we can look on $p(X_N, Y_N) \to p(x, y)$ for (sufficiently many) functions p in X_N, Y_N ...

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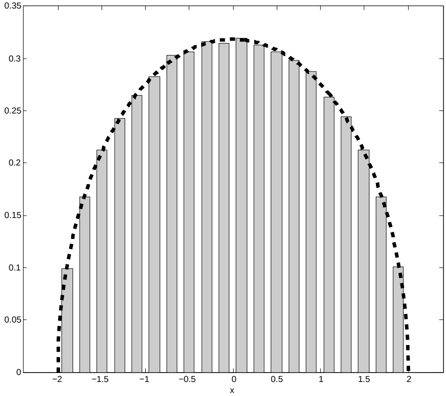
One-matrix case: classical random matrix case

X_N GUE random matrix



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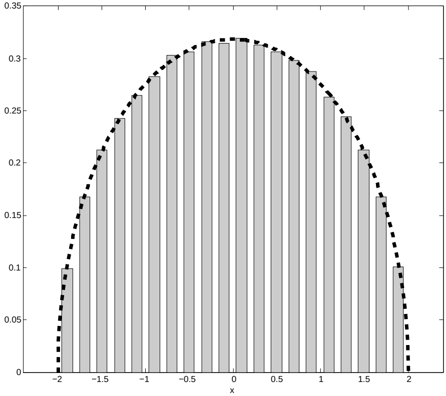

$$x = l + l^*,$$

l one-sided shift on $\bigoplus_{n\geq 0} \mathbb{C}e_n$

$$le_n = e_{n+1}$$
$$l^* e_{n+1} = e_n, \; l^* e_0 = 0$$

$$\tau(a) = \langle e_0, a e_0 \rangle$$

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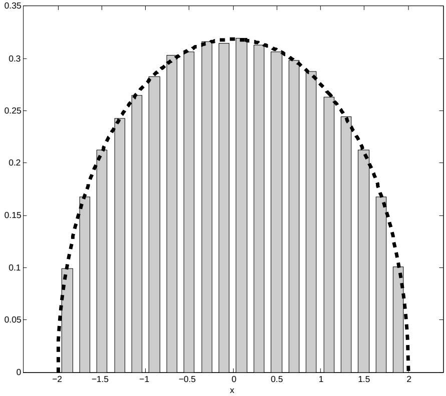

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- $X_N \to x$ in distribution (Wigner 1955)

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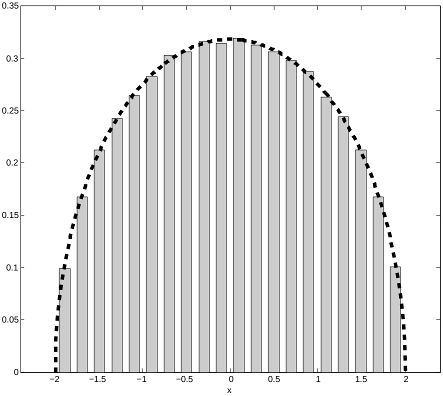

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- $X_N \to x$ in distribution (Wigner 1955)
- $\|X_N\| \to \|x\| = 2$ (Füredi, Komlós 1981)

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two copies of one-sided shift in different directions (creation and annihilation operators on full Fock space; Cuntz algebra)

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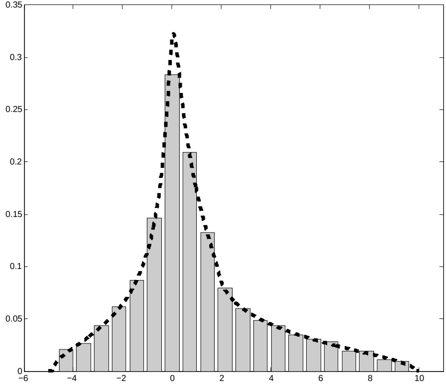

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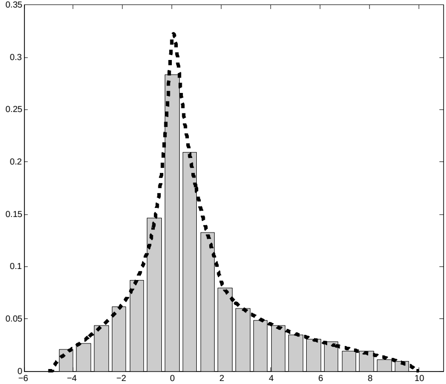

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- $\|p(X_N, Y_N)\| \to \|p(x,y)\|$ (Haagerup and Thorbjørnsen 2005)

Goal: Go over from Polynomials to More General Classes of Functions

For $m = 1$, one has

for all continuous f:

- $\lim_{N\to\infty} \operatorname{tr}[f(X^{(N)})] = \tau[f(x)]$
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Which classes of functions in non-commuting variables

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Which classes of functions in non-commuting variables

- continuous ???
- analytic $\rightsquigarrow$ free analysis
- rational !!!
- polynomials !!!

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Section 2

Non-Commutative Rational Functions

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Non-commutative rational functions

Non-commutative rational functions (Amitsur 1966, Cohn 1971)

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$$(4-y_1)^{-1} + (4-y_1)^{-1} y_2 \big((4-y_1) - y_2 (4-y_1)^{-1} y_2\big)^{-1} y_2 (4-y_1)^{-1}$$

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this is not obvious to decide, e.g., one has rational identities like

$$y_2^{-1}+y_2^{-1}(y_3^{-1}y_1^{-1}-y_2^{-1})^{-1}y_2^{-1}-(y_2-y_3y_1)^{-1}=0$$

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- after all, it works and gives a skew field

  $$\mathbb{C}\langle\!\langle y_1, \dots, y_m \rangle\!\rangle \qquad \text{“free field”}$$

A rational function $r(y_1, \dots, y_m)$ in non-commuting variables $y_1, \dots, y_m$

- can be realized more systematically by going over to matrices

$$r(y_1, \dots, y_m) = uQ^{-1}v$$

where, for some N,

  - u is $1 \times N$
  - Q is $N \times N$ and invertible in $M_N(\mathbb{C}\langle\!\langle y_1, \dots, y_m \rangle\!\rangle)$
  - v is $N \times 1$
  - and all entries are polynomials (can actually be chosen with degree ≤ 1)

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$$= \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 + \frac{1}{4} y_1 & \frac{1}{4} y_2 \\ \frac{1}{4} y_2 & -1 + \frac{1}{4} y_1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

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- this is essentially (in the case of polynomials) the "linearization trick" which we use in free probability, for example, to calculate distributions of polynomials in free variables

Historical remark

Note that this linearization trick is a well-known idea in many other mathematical communities, known under various names like

- Higman's trick (Higman "The units of group rings", 1940)
- recognizable power series (automata theory, Kleene 1956, Schützenberger 1961)
- linearization by enlargement (ring theory, Cohn 1985; Cohn and Reutenauer 1994, Malcolmson 1978)
- descriptor realization (control theory, Kalman 1963; Helton, McCullough, Vinnikov 2006)
- linearization trick (Haagerup, Thorbjørnsen 2005 (+Schultz 2006); Anderson 2012)

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... there are a couple of issues arising ...

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  - hence only work in stably finite algebras
  - this is the case in our situations, where we have a tracial state

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- In $\mathbb{C}\langle\!\langle y_1, \dots, y_m \rangle\!\rangle$ every $r(y_1, \dots, y_m) \neq 0$ is invertible. This is of course not true when we apply r to operators:

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  - even if $r(x_1,\dots,x_m)\neq 0$, it does not need to be invertible in general
- Possible solutions:
  - consider only r for which $r(x_1,\dots,x_m)$ is invertible as bounded operator
  - or allow also unbounded operators

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Rational functions of random matrices and their limit

Proposition (Sheng Yin 2017)

Consider random matrices $(X_1^{(N)}, \dots, X_m^{(N)})$ which converge to operators $(x_1, \dots, x_m)$ in the strong sense: for any polyomial $p \in \mathbb{C}\langle y_1, \dots, y_m\rangle$ we have

- $\lim_{N\to\infty} \mathrm{tr}[p(X_1^{(N)}, \dots, X_m^{(N)})] = \tau[p(x_1, \dots, x_m)]$
- $\lim_{N\to\infty} \|p(X_1^{(N)}, \dots, X_m^{(N)})\| = \|p(x_1, \dots, x_m)\|$

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Rational functions of random matrices and their limit

Proposition (Sheng Yin 2017)

Consider random matrices $(X_1^{(N)}, \dots, X_m^{(N)})$ which converge to operators $(x_1, \dots, x_m)$ in the strong sense: for any polyomial $p \in \mathbb{C}\langle y_1, \dots, y_m\rangle$ we have

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Distribution of random matrices and their limit for

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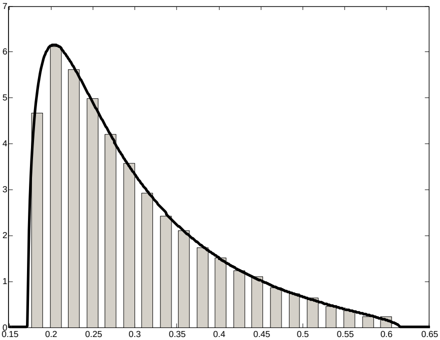

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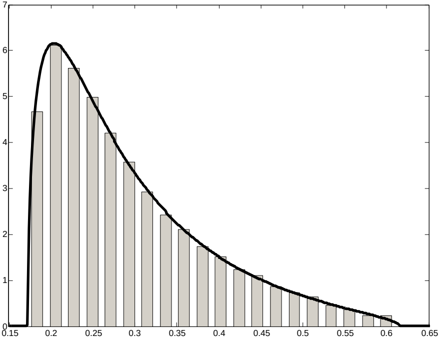

X_N, Y_N independent GUE

$x = l_1 + l_1^*,\ y = l_2 + l_2^*$

two copies of one-sided shift in different directions (creation and annihilation operators on full Fock space; Cuntz algebra)

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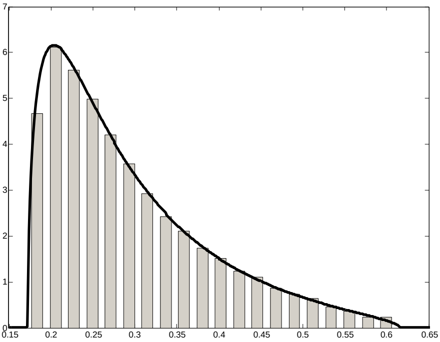

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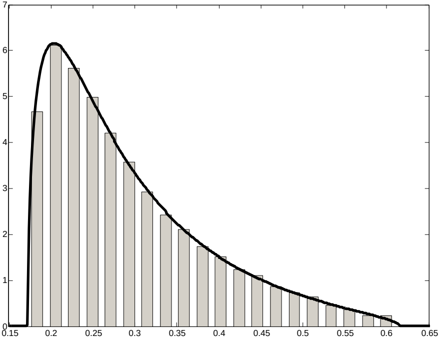

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Section 3

Unbounded Operators

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In the limit $(x_1, \dots, x_m) \subset (\mathcal{A}, \tau)$ we are in a II_1-situation

- unbounded operators $U(\mathcal{A})$ affiliated to vN algebra $\mathcal{A}$ form a $*$-algebra
- and $a \in U(\mathcal{A})$ is invertible if and only if a has no zero divisors

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we expect

$r(x_1, \ldots, x_m)$ for $r \in \mathbb{C}\langle\!\!\langle y_1, \ldots, y_m \rangle\!\!\rangle$ is well-defined and has, for $r \neq 0$, no zero divisors for

- $x_1, \ldots, x_m$ free semicirculars
- more general, limit operators of "nice" random matrix models

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we know (Charlesworth-Shlyakhtenko and Mai-Speicher-Weber)

$p(x_1, \dots, x_m)$ for $0 \neq p \in \mathbb{C}\langle y_1, \dots, y_m \rangle$ has no zero divisors for

- $x_1, \dots, x_m$ free semicirculars
- $x_1, \dots, x_m$ with maximal free entropy dimension $= m$

What do we expect of nice operators

$(X_1^{(N)}, \dots, X_m^{(N)})$	$\rightarrow$	$(x_1, \dots, x_m)$
nice random matrices	$\rightarrow$	nice operators

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What do we expect of nice operators

working definition of "nice"

operators $(x_1, \dots, x_m)$ are nice if $\delta(x_1, \dots, x_m) = m$.

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- we know
  - no polynomial relations
  - no "local" polynomial relations
- we don't know (but would like to)
  - no rational relations
  - no "local rational" relations

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no polynomial relations $\not\Rightarrow$ no rational relations

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There exist many embeddings

$$\mathbb{C}\langle y_1, \dots, y_m\rangle \subset \text{skew field } \mathbb{K}$$

The free field $\mathbb{C}\left(\!\!\left\langle y_1, \dots, y_m \right\rangle\!\!\right)$ is the *universal* field of fractions, every other $\mathbb{K}$ as above is a *localization* of $\mathbb{C}\left(\!\!\left\langle y_1, \dots, y_m \right\rangle\!\!\right)$

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Why no rational relations for nice operators?

- $0 = r \in \mathbb{C}\langle\!\langle y_1, \dots y_m\rangle\!\rangle \iff r(A_1, \dots, A_m) = 0$ for all matrices $A_1, \dots, A_m$ of all sizes

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- nice operators have many matrix tuples approximating them, thus should behave like a "generic" matrix tuple of arbitrary size

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Question

How much "regularity" of the distribution of $(x_1, \dots, x_m) \subset (\mathcal{A}, \tau)$ is necessary to have

$$\mathbb{C}\langle\!\langle x_1, \dots, x_m \rangle\!\rangle \quad \equiv \quad \text{free field}$$

where $\mathbb{C}\langle\!\langle x_1, \dots, x_m \rangle\!\rangle$ is the *division closure* of $\mathbb{C}\langle x_1, \dots, x_m \rangle$ in unbounded operators, i.e., the smallest algebra

- containing $\mathbb{C}\langle x_1, \dots, x_m \rangle$
- being closed under taking inverse, when possible (i.e, when no zero divisor)

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What do we know about the division closure in concrete cases

- if $u_1, \dots, u_m$ are free unitary elements, then (Linnell 1993)

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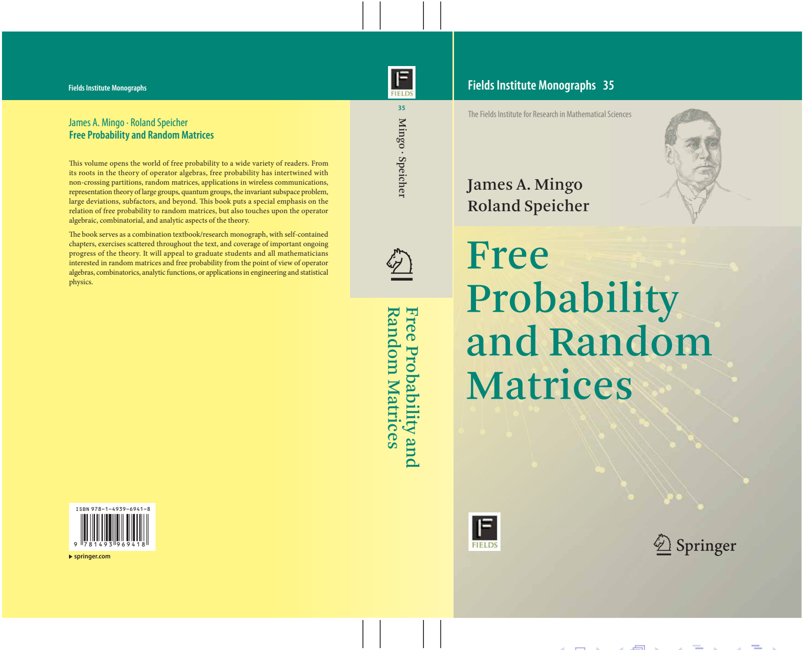
- what in the general nice case of maximal entropy dimension???



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