Sharp Bounds for Sums Associated to Graphs of Matrices

Roland Speicher

Saarland University
Saarbrücken, Germany

joint work with James Mingo

Question

What is the asymptotic behaviour in N of

$$\sum_{\substack{j_1,\dots,j_{2m} \\ \text{some constraints on} \\ \text{equality of indices}}}^{N} t^{(1)}_{j_1 j_2} t^{(2)}_{j_3 j_4} \cdots t^{(m)}_{j_{2m-1} j_{2m}}$$

with given matrices

$$T_k = \left(t^{(k)}_{ij}\right)_{i,j=1}^{N}$$

Examples

$$\sum_{i,j=1}^{N} t_{ij}$$

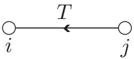
$$\sum_{i=1}^{N} t_{ii}$$

$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$

Examples and encoding of problem into graph

$$\sum_{i,j=1}^{N} t_{ij}$$



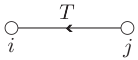
$$\sum_{i=1}^{N} t_{ii}$$

$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$

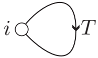
$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$

Examples and encoding of problem into graph

$$\sum_{i,j=1}^{N} t_{ij}$$



$$\sum_{i=1}^{N} t_{ii}$$

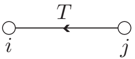


$$\sum_{i,j,k=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{ki}$$

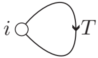
$$\sum_{i,j,k,l=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{jl}$$

Examples and encoding of problem into graph

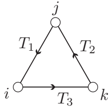
$$\sum_{i,j=1}^{N} t_{ij}$$



$$\sum_{i=1}^{N} t_{ii}$$



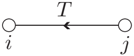
$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$



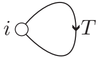
$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$

Examples and encoding of problem into graph

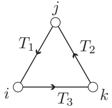
$$\sum_{i,j=1}^{N} t_{ij}$$



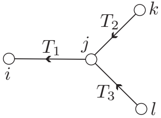
$$\sum_{i=1}^{N} t_{ii}$$



$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$



$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$

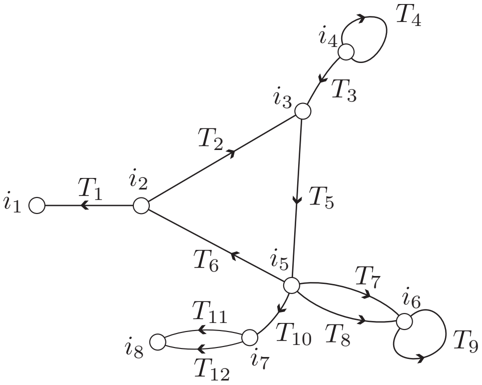


That's easy? Okay, so look on this:

$$\sum_{i_1,\ldots,i_8=1}^{N} t^{(1)}_{i_1 i_2} t^{(2)}_{i_3 i_2} t^{(3)}_{i_3 i_4} t^{(4)}_{i_4 i_4} t^{(5)}_{i_5 i_3} t^{(6)}_{i_2 i_5} t^{(7)}_{i_6 i_5} t^{(8)}_{i_6 i_5} t^{(9)}_{i_6 i_6} t^{(10)}_{i_7 i_5} t^{(11)}_{i_8 i_7} t^{(12)}_{i_8 i_7}$$

That's easy? Okay, so look on this:

$$\sum_{i_1,\ldots,i_8=1}^{N} t^{(1)}_{i_1 i_2} t^{(2)}_{i_3 i_2} t^{(3)}_{i_3 i_4} t^{(4)}_{i_4 i_4} t^{(5)}_{i_5 i_3} t^{(6)}_{i_2 i_5} t^{(7)}_{i_6 i_5} t^{(8)}_{i_6 i_5} t^{(9)}_{i_6 i_6} t^{(10)}_{i_7 i_5} t^{(11)}_{i_8 i_7} t^{(12)}_{i_8 i_7}$$



Problem

Graph sum of matrices

For a given directed graph G (multiple edges and loops allowed), with matrices attached to the edges, denote the corresponding sum by $S_G(N)$

$$S_G(N) := \sum_{i:V\to[N]} \prod_{e\in E} t^{(e)}_{i_{t(e)},i_{s(e)}}$$

Problem

Graph sum of matrices

For a given directed graph G (multiple edges and loops allowed), with matrices attached to the edges, denote the corresponding sum by $S_G(N)$

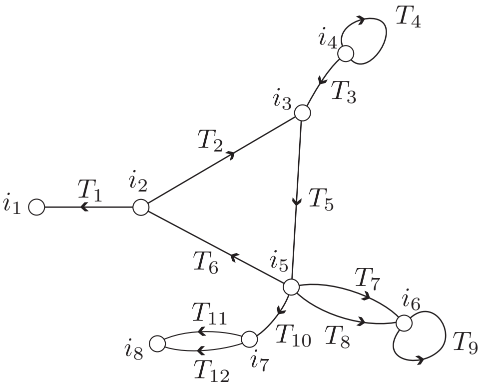
$$S_G(N) := \sum_{i:V\to[N]} \prod_{e\in E} t^{(e)}_{i_{t(e)},i_{s(e)}}$$

Question: asymptotics in N

Question: What is the optimal bound $r(G)$ in

$$|S_G(N)| \leq N^{\mathbf{r(G)}} \prod_{k=1}^{m} \|T_k\|$$

What is optimal asymptotics in N?



$$\left|\sum_{1}^{N} t^{(1)}_{i_1 i_2} t^{(2)}_{i_3 i_2} t^{(3)}_{i_3 i_4} t^{(4)}_{i_4 i_4} t^{(5)}_{i_5 i_3} t^{(6)}_{i_2 i_5} t^{(7)}_{i_6 i_5} t^{(8)}_{i_6 i_5} t^{(9)}_{i_6 i_6} t^{(10)}_{i_7 i_5} t^{(11)}_{i_8 i_7} t^{(12)}_{i_8 i_7}\right| \le N^{\mathbf{r}(\mathbf{G})} \cdot \prod_{i=1}^{12} \|T_i\|$$

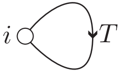
Motivation: Why do we care about such sums?

Such sums appear and have to be asymptotically controlled in calculations of moments of random matrices (in particular, for products of such matrices)

- Yin + Krishnaiah, 1983
- Bai (+ Silverstein), 1999 (2006)
- Mingo + Speicher, 2012 JFA
  (asymptotic freeness of Wigner and deterministic matrices)
- Male, 2012 $\rightarrow$ "traffics"

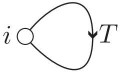
Example

$$\sum_{i=1}^{N} t_{ii}$$



Example

$$\sum_{i=1}^{N} t_{ii}$$

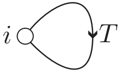


Then

$$|\sum_{i=1}^{N} t_{ii}| \leq \sum_{i=1}^{N} \|T\| = N\|T\|$$

Example

$$\sum_{i=1}^{N} t_{ii}$$



Then

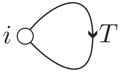
$$|\sum_{i=1}^{N} t_{ii}| \leq \sum_{i=1}^{N} \|T\| = N\|T\|$$

or

$$|\sum_{i=1}^{N} t_{ii}| = |\mathsf{Tr}(T)| \leq N\|T\|$$

Example

$$\sum_{i=1}^{N} t_{ii}$$



$$\mathbf{r(G) = 1}$$

Then

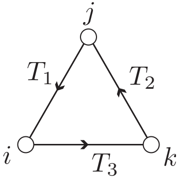
$$|\sum_{i=1}^{N} t_{ii}| \leq \sum_{i=1}^{N} \|T\| = N\|T\|$$

or

$$|\sum_{i=1}^{N} t_{ii}| = |\mathsf{Tr}(T)| \leq N\|T\|$$

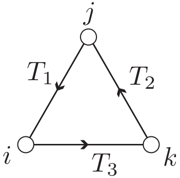
Example

$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$



Example

$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$

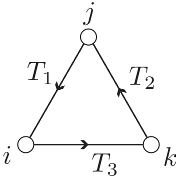


$$\mathbf{r(G) = 3}???$$

trivial estimate: $$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)} \leq \sum_{i,j,k=1}^{N} \|T_1 T_2 T_3\| = N^3 \|T_1\| \|T_2\| \|T_3\|$$

Example

$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$



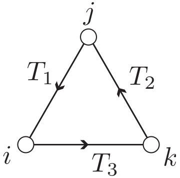
$$\mathbf{r(G) = 3???}$$

trivial estimate: $$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)} \le \sum_{i,j,k=1}^{N} \|T_1 T_2 T_3\| = N^3 \|T_1\| \|T_2\| \|T_3\|$$

$$|\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}| = |\mathsf{Tr}(T_1 T_2 T_3)|$$

Example

$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$



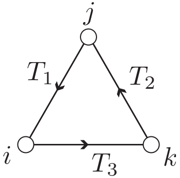
$\mathbf{r(G) = 3}$???

trivial estimate: $$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)} \leq \sum_{i,j,k=1}^{N} \|T_1 T_2 T_3\| = N^3 \|T_1\| \|T_2\| \|T_3\|$$

$$\left| \sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)} \right| = |\mathrm{Tr}(T_1 T_2 T_3)| \leq N \|T_1 T_2 T_3\| \leq N \|T_1\| \|T_2\| \|T_3\|$$

Example

$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$



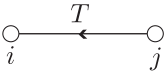
$\mathbf{r(G) = 1}$

trivial estimate: $$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)} \leq \sum_{i,j,k=1}^{N} \|T_1 T_2 T_3\| = N^3 \|T_1\| \|T_2\| \|T_3\|$$

$$\left| \sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)} \right| = |\mathrm{Tr}(T_1 T_2 T_3)| \leq N \|T_1 T_2 T_3\| \leq N \|T_1\| \|T_2\| \|T_3\|$$

Example:

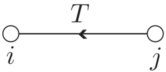
$$\sum_{i,j=1}^{N} t_{ij}$$



trivial estimate: $$|\sum_{i,j=1}^{N} t_{ij}| \leq \sum_{i,j=1}^{N} \|T\| = N^2 \|T\|$$

Example:

$$\sum_{i,j=1}^{N} t_{ij}$$

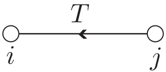


$\mathbf{r(G) = 2}???$

trivial estimate: $\quad |\sum_{i,j=1}^{N} t_{ij}| \leq \sum_{i,j=1}^{N} \|T\| = N^2 \|T\|$

Example:

$$\sum_{i,j=1}^{N} t_{ij}$$



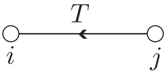
$$\mathbf{r(G) = 2}???$$

trivial estimate: $$|\sum_{i,j=1}^{N} t_{ij}| \leq \sum_{i,j=1}^{N} \|T\| = N^2 \|T\|$$

But we can also here do better ...

Example:

$$\sum_{i,j=1}^{N} t_{ij}$$



$$\mathbf{r(G) = 2???}$$

trivial estimate: $$|\sum_{i,j=1}^{N} t_{ij}| \leq \sum_{i,j=1}^{N} \|T\| = N^2 \|T\|$$

But we can also here do better ... with using the vectors

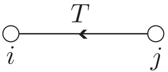
$$e_i := (0, \dots, \underset{i}{1}, \dots, 0), \qquad e := (1, 1, \dots, 1)$$

we can write

$$|\sum_{i,j=1}^{N} t_{ij}| = |\sum_{i,j=1}^{N} \langle e_i, Te_j \rangle|$$

Example:

$$\sum_{i,j=1}^{N} t_{ij}$$



$$\mathbf{r(G) = 2???}$$

trivial estimate: $$|\sum_{i,j=1}^{N} t_{ij}| \leq \sum_{i,j=1}^{N} \|T\| = N^2 \|T\|$$

But we can also here do better ... with using the vectors

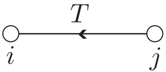
$$e_i := (0, \ldots, \underset{i}{1}, \ldots, 0), \qquad e := (1, 1, \ldots, 1)$$

we can write

$$|\sum_{i,j=1}^{N} t_{ij}| = |\sum_{i,j=1}^{N} \langle e_i, Te_j \rangle| = |\langle e, Te \rangle|$$

Example:

$$\sum_{i,j=1}^{N} t_{ij}$$



$$\mathbf{r(G) = 2???}$$

trivial estimate: $$|\sum_{i,j=1}^{N} t_{ij}| \leq \sum_{i,j=1}^{N} \|T\| = N^2 \|T\|$$

But we can also here do better ... with using the vectors

$$e_i := (0, \dots, \underset{i}{1}, \dots, 0), \qquad e := (1, 1, \dots, 1)$$

we can write

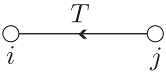
$$|\sum_{i,j=1}^{N} t_{ij}| = |\sum_{i,j=1}^{N} \langle e_i, Te_j \rangle| = |\langle e, Te \rangle| \leq \|e\|^2 \|T\| = N \|T\|$$

since

$$\|e\| = \sqrt{N}$$

Example:

$$\sum_{i,j=1}^{N} t_{ij}$$



$\mathbf{r(G) = 1}$

trivial estimate: $$|\sum_{i,j=1}^{N} t_{ij}| \leq \sum_{i,j=1}^{N} \|T\| = N^2\|T\|$$

But we can also here do better ... with using the vectors

$$e_i := (0, \dots, \underset{i}{1}, \dots, 0), \qquad e := (1, 1, \dots, 1)$$

we can write

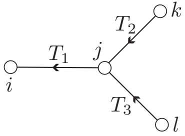
$$|\sum_{i,j=1}^{N} t_{ij}| = |\sum_{i,j=1}^{N} \langle e_i, Te_j\rangle| = |\langle e, Te\rangle| \leq \|e\|^2\|T\| = N\|T\|$$

since

$$\|e\| = \sqrt{N}$$

Example:

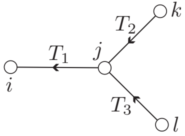
$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$



$\mathbf{r(G) = 4}???$

Example:

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$

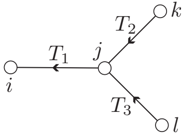


$\mathbf{r(G)} = \mathbf{4}???$

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)} = \sum_{i,j,k,l} \langle e_i, T_1 e_j \rangle \underbrace{\langle e_j, T_2 e_k \rangle \langle e_j, T_3 e_l \rangle}$$

Example:

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$

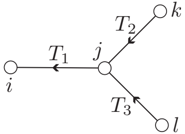


$\mathbf{r(G)} = \mathbf{4}???$

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)} = \sum_{i,j,k,l} \langle e_i, T_1 e_j \rangle \underbrace{\langle e_j, T_2 e_k \rangle \langle e_j, T_3 e_l \rangle}_{\langle e_j \otimes e_j, T_2 \otimes T_3 e_k \otimes e_l \rangle}$$

Example:

$$\sum_{i,j,k,l=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{jl}$$



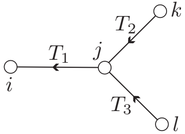
$$\mathbf{r(G) = 4???}$$

$$\sum_{i,j,k,l=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{jl} = \sum_{i,j,k,l} \langle e_i, T_1 e_j \rangle \underbrace{\langle e_j, T_2 e_k \rangle \langle e_j, T_3 e_l \rangle}_{\langle e_j \otimes e_j, T_2 \otimes T_3 e_k \otimes e_l \rangle}$$

$$= \langle e, T_1 \underbrace{\left(\sum_j |e_j\rangle\langle e_j \otimes e_j| \right)}_{V : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N} T_2 \otimes T_3, e \otimes e \rangle$$

Example:

$$\sum_{i,j,k,l=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{jl}$$



$\mathbf{r(G) = 4???}$

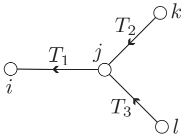
$$\sum_{i,j,k,l=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{jl} = \sum_{i,j,k,l} \langle e_i, T_1 e_j \rangle \underbrace{\langle e_j, T_2 e_k \rangle \langle e_j, T_3 e_l \rangle}_{\langle e_j \otimes e_j, T_2 \otimes T_3 e_k \otimes e_l \rangle}$$

$$= \langle e, T_1 \underbrace{\left(\sum_j |e_j\rangle \langle e_j \otimes e_j| \right)}_{V : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N} T_2 \otimes T_3, e \otimes e \rangle$$

$$= \langle e, T_1 V (T_2 \otimes T_3)\, e \otimes e \rangle$$

Example:

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$

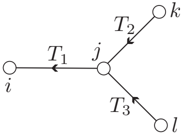


$\mathbf{r(G)} = \mathbf{4}???$

$$|\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}| = |\langle e, T_1 V (T_2 \otimes T_3)\, e \otimes e \rangle|$$

Example:

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$



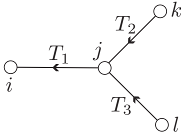
$\mathbf{r(G) = 4???}$

$$|\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}| = |\langle e, T_1 V (T_2 \otimes T_3)\, e \otimes e\rangle|$$

$$\leq \|e\| \cdot \underbrace{\|e \otimes e\|}_{\|e\|\cdot\|e\|} \cdot \|T_1\| \cdot \underbrace{\|V\|} \cdot \|T_2 \otimes T_3\|$$

Example:

$$\sum_{i,j,k,l=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{jl}$$



$\mathbf{r(G) = 4???}$

$$\Big|\sum_{i,j,k,l=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{jl}\Big| = |\langle e, T_1 V (T_2 \otimes T_3)\, e \otimes e\rangle|$$

$$\leq \|e\| \cdot \underbrace{\|e \otimes e\|}_{\|e\|\cdot\|e\|} \cdot \|T_1\| \cdot \underbrace{\|V\|}_{=1} \cdot \|T_2 \otimes T_3\|$$

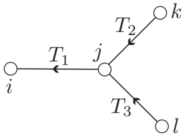
since

$$V : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N, \qquad e_i \otimes e_j \mapsto \begin{cases} e_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

is a partial isometry, and thus $\|V\| = 1$.

Example:

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$



$\mathbf{r(G) = 3/2}$

$$\left| \sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)} \right| = |\langle e, T_1 V (T_2 \otimes T_3)\, e \otimes e \rangle|$$

$$\leq \|e\| \cdot \underbrace{\|e \otimes e\|}_{\|e\| \cdot \|e\|} \cdot \|T_1\| \cdot \underbrace{\|V\|}_{=1} \cdot \|T_2 \otimes T_3\|$$

$$= N^{\mathbf{3/2}} \cdot \|T_1\| \cdot \|T_2\| \cdot \|T_3\|$$

since

$$V : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N, \qquad e_i \otimes e_j \mapsto \begin{cases} e_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

is a partial isometry, and thus $\|V\| = 1$.

General structure

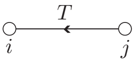
In these examples, our sum $S(N)$ is given as inner product

$$S(N) \qquad \hat{=} \qquad \langle \text{output}, \textit{stuff} \text{ input} \rangle$$

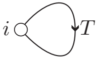
where

- each input and each output vertex contributes factor $N^{1/2}$
- internal vertices do not contribute, summation over them corresponds to matrix multiplication or, more general, partial isometries

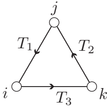
General structure:



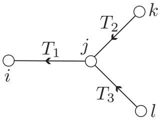
$r(G) = 1$



$r(G) = 1$



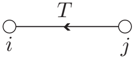
$r(G) = 1$



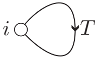
$r(G) = 3/2$

General structure: input-output graph

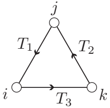
In the examples we could read our graph as a flow diagram from some input vertices to some output vertices, each contributing $1/2$ to $r(G)$



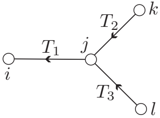
$$r(G) = 1$$



$$r(G) = 1$$



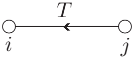
$$r(G) = 1$$



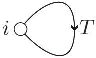
$$r(G) = 3/2$$

General structure: input-output graph

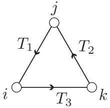
In the examples we could read our graph as a flow diagram from some input vertices to some output vertices, each contributing $1/2$ to $r(G)$



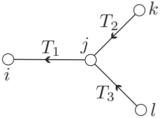
j input, i output $\quad r(G) = 1$



$r(G) = 1$



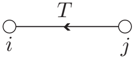
$r(G) = 1$



$r(G) = 3/2$

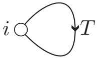
General structure: input-output graph

In the examples we could read our graph as a flow diagram from some input vertices to some output vertices, each contributing $1/2$ to $r(G)$



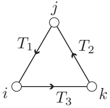
j input, i output

$r(G) = 1$

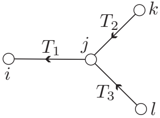


i both input and output

$r(G) = 1$



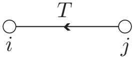
$r(G) = 1$



$r(G) = 3/2$

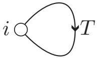
General structure: input-output graph

In the examples we could read our graph as a flow diagram from some input vertices to some output vertices, each contributing $1/2$ to $r(G)$



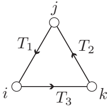
j input, i output

$r(G) = 1$



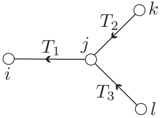
i both input and output

$r(G) = 1$



i both input and output

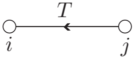
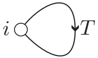
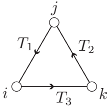
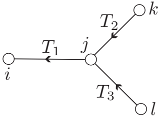
$r(G) = 1$



$r(G) = 3/2$

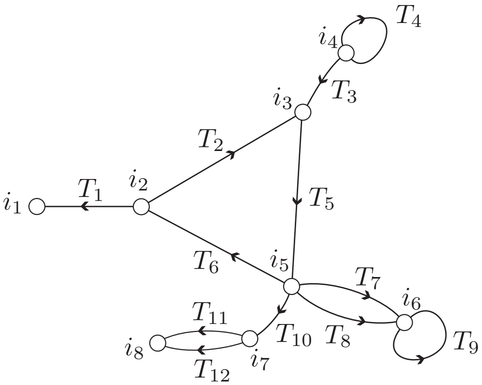
General structure: input-output graph

In the examples we could read our graph as a flow diagram from some input vertices to some output vertices, each contributing $1/2$ to $r(G)$

	j input, i output	$r(G) = 1$
	i both input and output	$r(G) = 1$
	i both input and output	$r(G) = 1$
	k and l input, i output	$r(G) = 3/2$

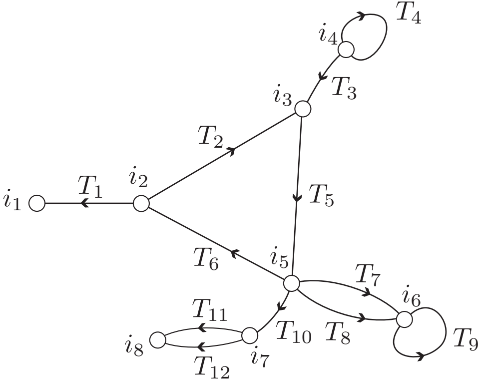
General structure: input-output graph

But how about more complicated graphs? This does not look like an input-output graph.



General structure: input-output graph

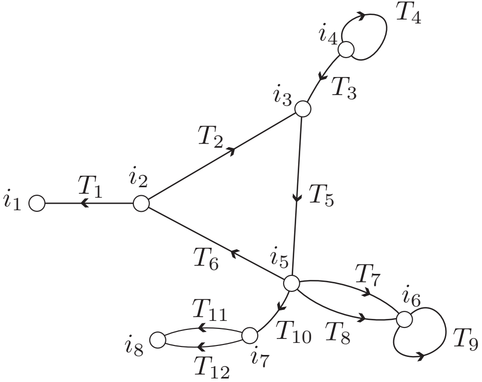
But how about more complicated graphs? This does not look like an input-output graph.



- we might have several loops

General structure: input-output graph

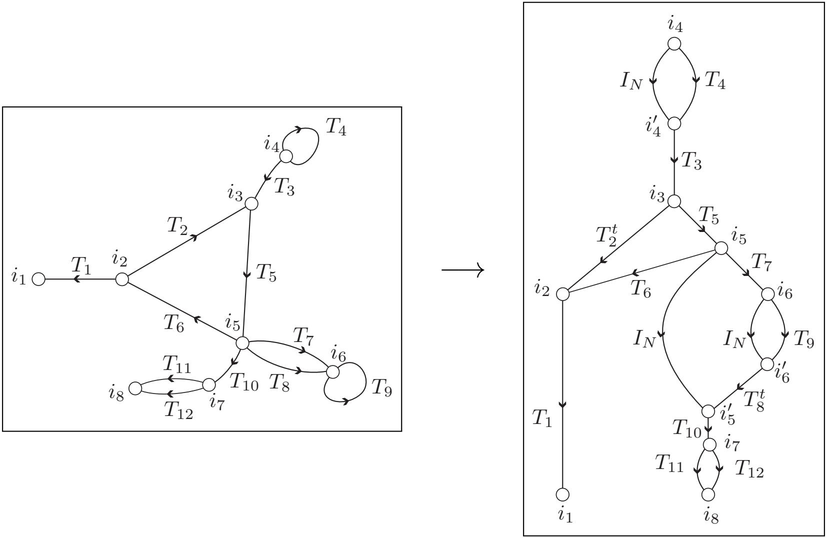
But how about more complicated graphs? This does not look like an input-output graph.



- we might have several loops
- there might be no overall consistent flow direction in the graph

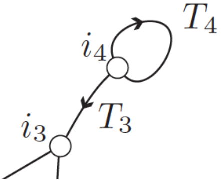
Replace by equivalent input-output problem

Change the graph without changing the graph sum

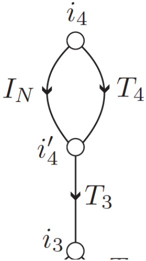


Replace by equivalent input-output problem

- split a vertex with a loop into 2 vertices and identify them via an additional edge with the identity matrix

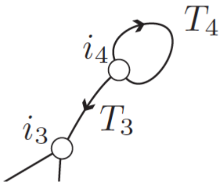


$\longrightarrow$

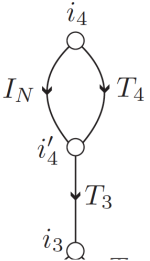


Replace by equivalent input-output problem

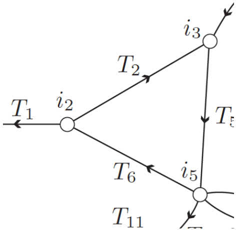
- split a vertex with a loop into 2 vertices and identify them via an additional edge with the identity matrix



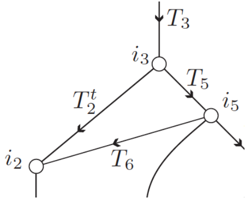
$\longrightarrow$

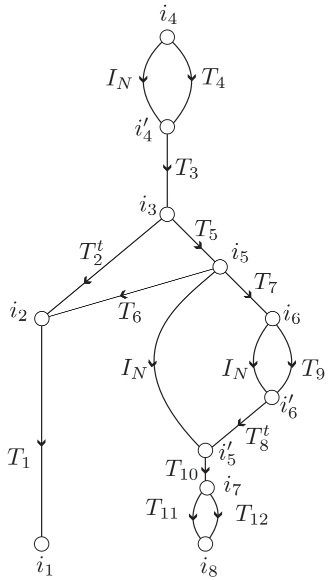


- change the direction of some edges and replace T by its transpose

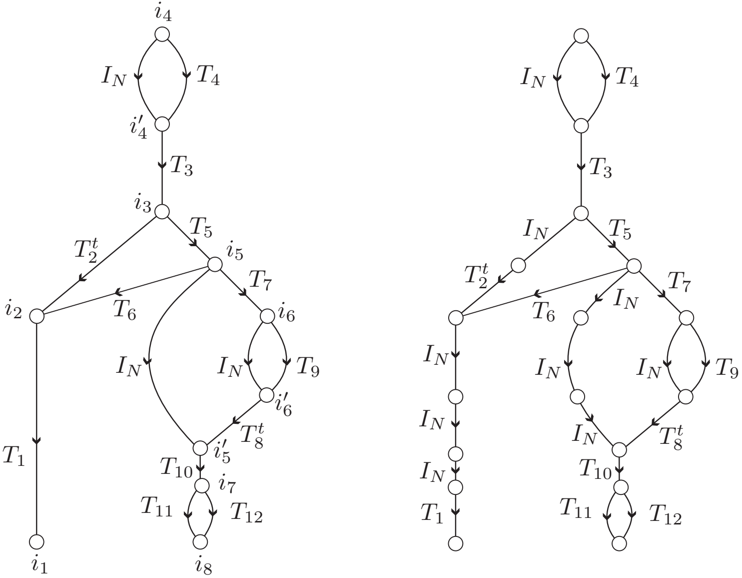


$\longrightarrow$

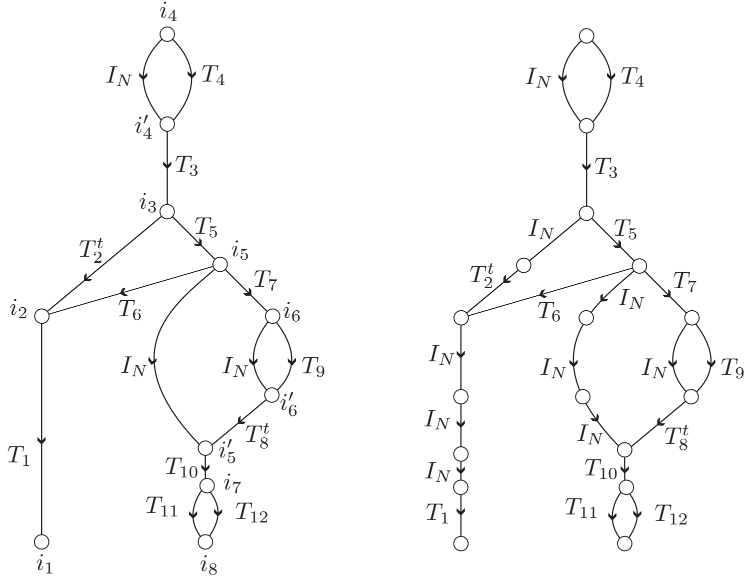



i_4
I_N
T_4
i_4'
T_3
i_3
T_5
T_2^t
i_5
T_7
i_2
T_6
i_6
I_N
I_N
T_9
i_6'
T_8^t
T_1
i_5'
T_{10}
i_7
T_{11}
T_{12}
i_1
i_8

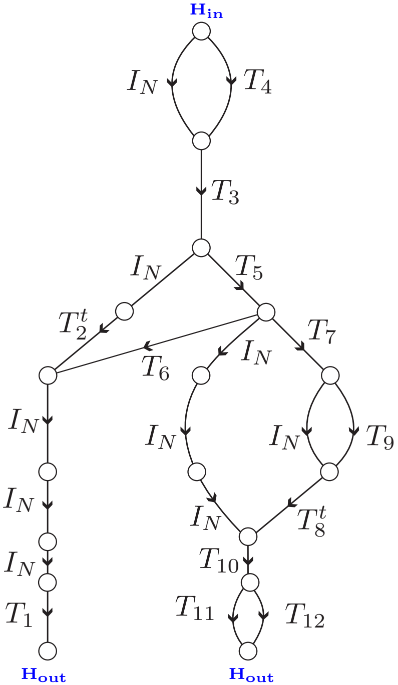
Adding additional vertices gives flow diagram



Adding additional vertices gives flow diagram



$S_G(N) = \langle e_{in}, \text{[product of many operators of norm } \|T_i\| \text{ or 1]}\, e_{out} \otimes e_{out} \rangle$



$$V_{in,2} : \mathbf{H_{in}} \to H \otimes H$$

$$I_N \otimes T_4$$

$$V_{2,1} : H \otimes H \to H$$

$$T_3$$

$$V_{1,2} : H \to H \otimes H$$

$$I_N \otimes T_5$$

$$V_{1,1} \otimes V_{1,3} : H \otimes H) \to H \otimes (H \otimes H \otimes H)$$
$$T_2^t \otimes T_6 \otimes I_N \otimes T_7$$

$$V_{2,1} \otimes V_{1,1} \otimes V_{1,2} : (H \otimes H) \otimes H \otimes H \to H \otimes H \otimes (H \otimes H)$$

$$I_N \otimes I_N \otimes I_N \otimes T_9$$

$$V_{1,1} \otimes V_{1,1} \otimes V_{2,1} : H \otimes H \otimes (H \otimes H) \to H \otimes H \otimes H$$

$$I_N \otimes I_N \otimes T_8^t$$

$$V_{1,1} \otimes V_{2,1} : H \otimes (H \otimes H) \to H \otimes H$$
$$I_N \otimes T_{10}$$
$$V_{1,1} \otimes V_{1,2} : H \otimes H \to H \otimes (H \otimes H)$$

$$T_1 \otimes T_{11} \otimes T_{12}$$

$$V_{1,out} \otimes V_{2,out} : H \otimes (H \otimes H) \to \mathbf{H_{out}} \otimes \mathbf{H_{out}}$$

We have thus seen:

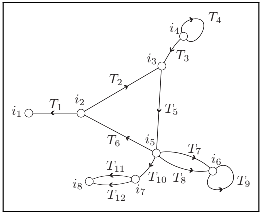
$$r(G) \leq \frac{\text{number of input/output vertices in equivalent input-output graph}}{2}$$

We have thus seen:

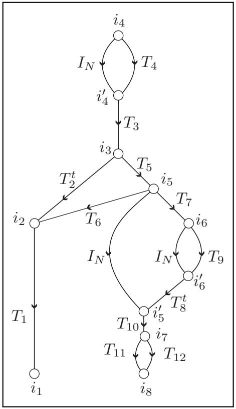
$$r(G) \leq \frac{\text{number of input/output vertices in equivalent input-output graph}}{2}$$

- But what is the optimal equivalent input-output graph?
- Is the above choice somehow a canonical one?

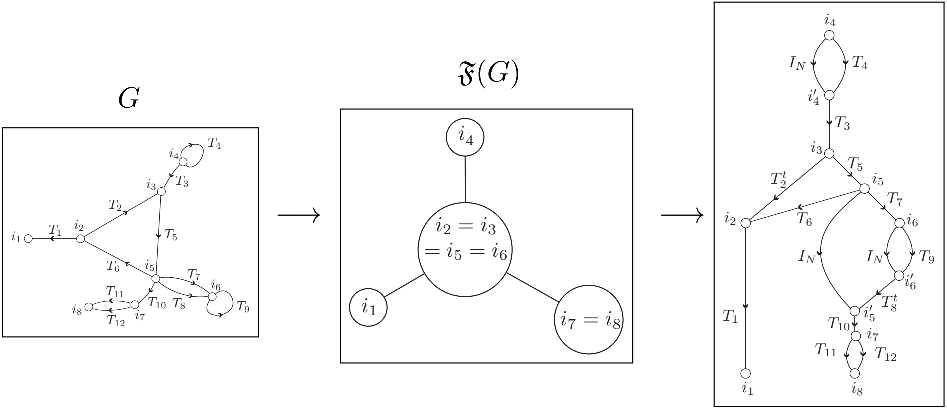
Why this equivalent input-output graph?



$\longrightarrow$ $\longrightarrow$

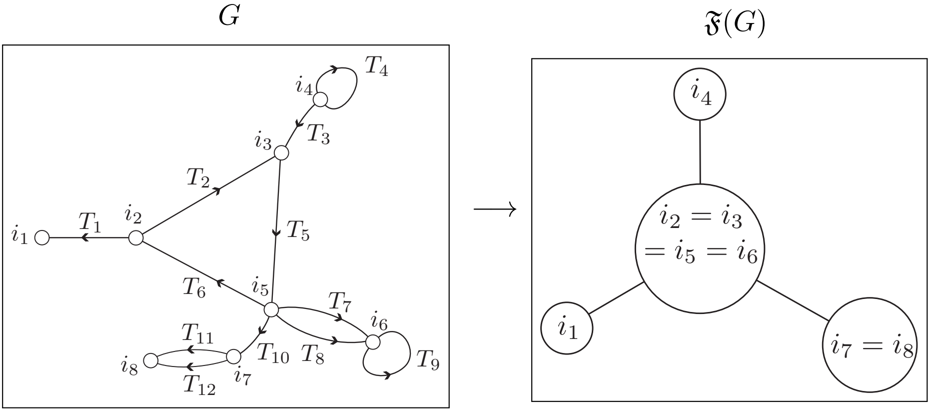


Why this equivalent input-output graph? Because the relevant structure is given by the forrest of two-edge-connected components $\mathfrak{F}(G)$



Asymptotics determined by structure of $\mathfrak{F}(G)$

$\mathfrak{F}(G)$ = forrest of two-edge-connected components of G



The optimal bound

Theorem (Mingo, Speicher, JFA 2012)

We have the following optimal estimate:

$$\left|\sum_{i:V\to[N]}\prod_{e\in E} t^{(e)}_{i_{t(e)},i_{s(e)}}\right| \leq N^{\frac{1}{2}\cdot\#\text{leaves of }\mathfrak{F}(G)} \cdot \prod_{e\in E}\|T_e\|$$

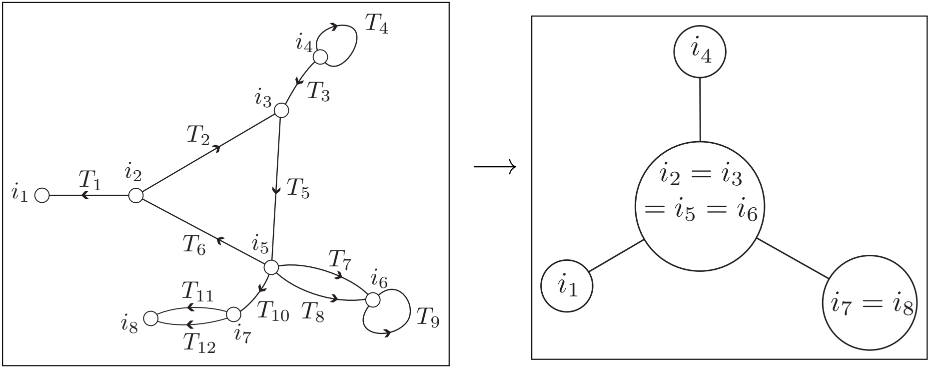
thus:

$$\mathbf{r}(\mathbf{G}) = \#\text{leaves of }\mathfrak{F}(G)$$

(Trivial leaves count twice!)

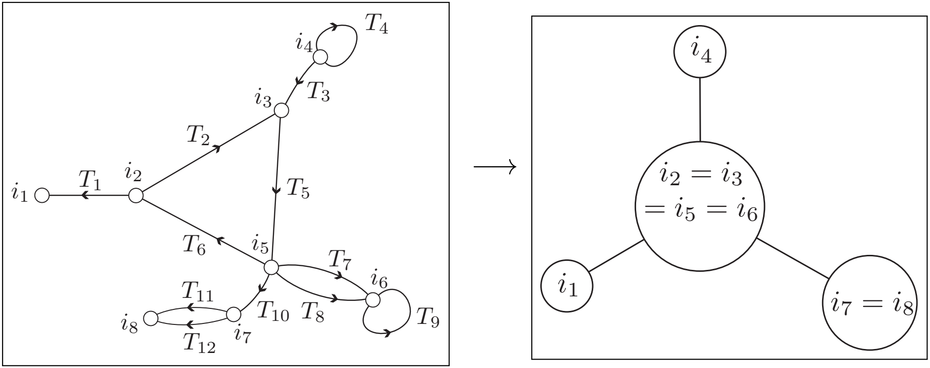
Optimal asymptotics in N

G $\mathfrak{F}(G)$, $\mathbf{r(G) = 3/2}$



Optimal asymptotics in N

G $\qquad\qquad$ $\mathfrak{F}(G), \quad \mathbf{r(G) = 3/2}$



$$\left|\sum_{1}^{N} t^{(1)}_{i_1 i_2} t^{(2)}_{i_3 i_2} t^{(3)}_{i_3 i_4} t^{(4)}_{i_4 i_4} t^{(5)}_{i_5 i_3} t^{(6)}_{i_2 i_5} t^{(7)}_{i_6 i_5} t^{(8)}_{i_6 i_5} t^{(9)}_{i_6 i_6} t^{(10)}_{i_7 i_5} t^{(11)}_{i_8 i_7} t^{(12)}_{i_8 i_7}\right| \leq N^{\mathbf{3/2}} \cdot \prod_{i=1}^{12} \|T_i\|$$

The optimal bound

Theorem (Mingo, Speicher, JFA 2012)

We have the following optimal estimate:

$$\Big|\sum_{i:V\to[N]}\prod_{e\in E} t^{(e)}_{i_{t(e)},i_{s(e)}}\Big| \leq N^{\frac{1}{2}\cdot\#\text{leaves of }\mathfrak{F}(G)}\cdot\prod_{e\in E}\|T_e\|$$

thus:

$$\mathbf{r(G)} = \#\text{leaves of }\mathfrak{F}(G)$$

(Trivial leaves count twice!)

Possible extensions

Can we get similar estimates for

Possible extensions for vectors

Can we get similar estimates for

- sums for vectors

$$\sum_{\substack{j_1,\dots,j_m \\ \text{some constraints on} \\ \text{equality of indices}}} t^{(1)}_{j_1} t^{(2)}_{j_2} \cdots t^{(m)}_{j_m}$$

  - this seems to be trivial, but I am not really sure what the best description is

Possible extensions for vectors or for tensors

Can we get similar estimates for

- sums for vectors

$$\sum_{\substack{j_1,\ldots,j_m \\ \text{some constraints on} \\ \text{equality of indices}}} t^{(1)}_{j_1} t^{(2)}_{j_2} \cdots t^{(m)}_{j_m}$$

  - this seems to be trivial, but I am not really sure what the best description is

- sums for tensors

$$\sum_{\substack{i_1,j_1,k_1,\ldots,i_m,j_m,k_m \\ \text{some constraints on} \\ \text{equality of indices}}} t^{(1)}_{i_1 j_1 k_1} t^{(2)}_{i_2 j_2 k_2} \cdots t^{(m)}_{i_m j_m k_m}$$

Possible extensions for vectors or for tensors

Can we get similar estimates for

- sums for vectors

$$\sum_{\substack{j_1,\ldots,j_m \\ \text{some constraints on} \\ \text{equality of indices}}} t^{(1)}_{j_1} t^{(2)}_{j_2} \cdots t^{(m)}_{j_m}$$

  - this seems to be trivial, but I am not really sure what the best description is

- sums for tensors

$$\sum_{\substack{i_1,j_1,k_1,\ldots,i_m,j_m,k_m \\ \text{some constraints on} \\ \text{equality of indices}}} t^{(1)}_{i_1 j_1 k_1} t^{(2)}_{i_2 j_2 k_2} \cdots t^{(m)}_{i_m j_m k_m}$$

  - what are the relevant quantities for tensors

Possible extensions for vectors or for tensors

Can we get similar estimates for

- sums for vectors

$$\sum_{\substack{j_1,\dots,j_m \\ \text{some constraints on} \\ \text{equality of indices}}} t^{(1)}_{j_1} t^{(2)}_{j_2} \cdots t^{(m)}_{j_m}$$

  - this seems to be trivial, but I am not really sure what the best description is
- sums for tensors

$$\sum_{\substack{i_1,j_1,k_1,\dots,i_m,j_m,k_m \\ \text{some constraints on} \\ \text{equality of indices}}} t^{(1)}_{i_1 j_1 k_1} t^{(2)}_{i_2 j_2 k_2} \cdots t^{(m)}_{i_m j_m k_m}$$

  - what are the relevant quantities for tensors
  - is there any good diagrammatic way of encoding such problems

Possible extensions for vectors or for tensors

Can we get similar estimates for

- sums for vectors

$$\sum_{\substack{j_1,\ldots,j_m \\ \text{some constraints on} \\ \text{equality of indices}}} t^{(1)}_{j_1} t^{(2)}_{j_2} \cdots t^{(m)}_{j_m}$$

  - this seems to be trivial, but I am not really sure what the best description is

- sums for tensors

$$\sum_{\substack{i_1,j_1,k_1,\ldots,i_m,j_m,k_m \\ \text{some constraints on} \\ \text{equality of indices}}} t^{(1)}_{i_1 j_1 k_1} t^{(2)}_{i_2 j_2 k_2} \cdots t^{(m)}_{i_m j_m k_m}$$

  - what are the relevant quantities for tensors
  - is there any good diagrammatic way of encoding such problems

Thank You!