Multiplication of free random variables and the S-transform: the case of vanishing mean

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Definition [Voiculescu]: Let x be a random variable with $\varphi(x) \neq 0$. Then its S-**transform** S_x is defined as follows. Let χ denote the inverse under composition of the series

$$\psi(z) := \sum_{n=1}^{\infty} \varphi(x^n) z^n = \varphi(x) z + \varphi(x^2) z^2 + \cdots ,$$

then

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Note: We need $\varphi(x) \neq 0$ in order to be able to invert the formal power series ψ!

Alternative Characterization of S [Nica + Speicher]: Let

$$C(z) = \sum_{n=1}^{\infty} \kappa_n z^n = \kappa_1 z + \kappa_2 z^2 + \cdots$$

be the free cumulant generating series. Denote by $C^{<-1>}$ its inverse under composition. Then

$$S_x(z) = \frac{1}{z} \cdot C^{<-1>}(z).$$

Again, we need $\kappa_1 = \varphi(x) \neq 0$ in order to invert $C(z)$.

Theorem [Voiculescu]:

If x and y are free random variables such that $\varphi(x) \neq 0$ and $\varphi(y) \neq 0$, then we have

$$S_{xy}(z) = S_x(z) \cdot S_y(z).$$

Note: we are interested in cases where the considered moments are actually moments of probability measures on $\mathbb{R}$.

Consider $x = x^*$ and $y = y^*$ then

$$\varphi(x^n) = \int t^n d\mu_x(t), \qquad \varphi(y^n) = \int t^n d\mu_y(t)$$

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But how about xy? This is in general not selfadjoint, thus there is no apriori reason that there exists μ_{xy} with

$$\varphi((xy)^n) = \int t^n d\mu_{xy}(t).$$

However, if $x \geq 0$ then $\sqrt{x}$ exists and

$$\text{moments of } xy \quad = \quad \text{moments of } \sqrt{x}y\sqrt{x}$$

Thus there exists $\mu_{\sqrt{x}y\sqrt{x}}$ with

$$\int t^n d\mu_{\sqrt{x}y\sqrt{x}} = \varphi((\sqrt{x}y\sqrt{x})^n) = \varphi((xy)^n).$$

We call

$$\mu_{\sqrt{x}y\sqrt{x}} =: \mu_x \boxtimes \mu_y$$

the **multiplicative free convolution of** μ_x **and** μ_y.

Thus

$$\boxtimes : \mathrm{Prob}(\mathbb{R}) \times \mathrm{Prob}(\mathbb{R}_+) \to \mathrm{Prob}(\mathbb{R}).$$

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Often one restricts to considering $\boxtimes$ as binary operation

$$\boxtimes : \text{Prob}(\mathbb{R}_+) \times \text{Prob}(\mathbb{R}_+) \to \text{Prob}(\mathbb{R}_+).$$

In the latter case, we have $\varphi(x) = 0$ only for $x = 0$, i.e., $\mu_x = \delta_0$; thus if we exclude this uninteresting case, $S_x = S_{\mu_x}$ is always defined and we can use

$$S_{\mu \boxtimes \nu}(z) = S_\mu(z) \cdot S_\nu(z)$$

to calculate $\mu \boxtimes \nu$ for $\mu, \nu \in \text{Prob}(\mathbb{R}_+)$.

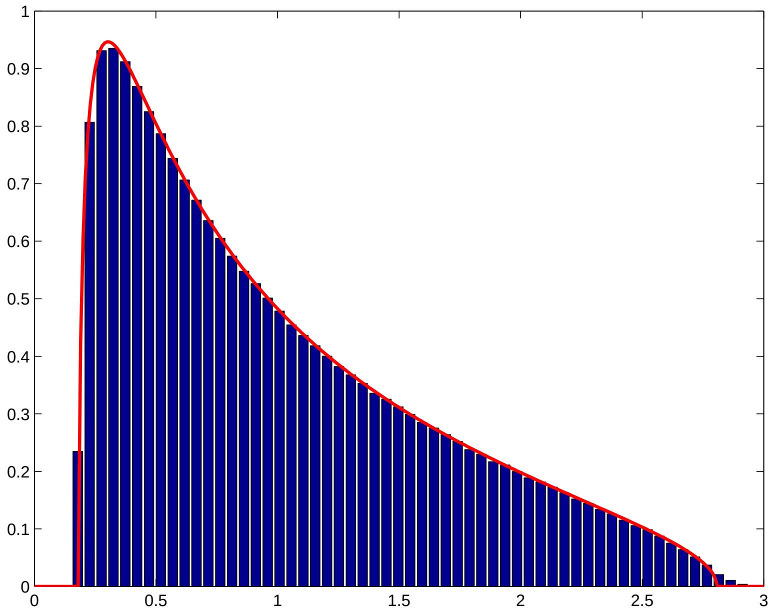
Note:

Many random matrices are becoming asymptotically free for $N \to \infty$, and thus S-transform is useful tool for calculating asymptotic eigenvalue distribution of product of random matrices.

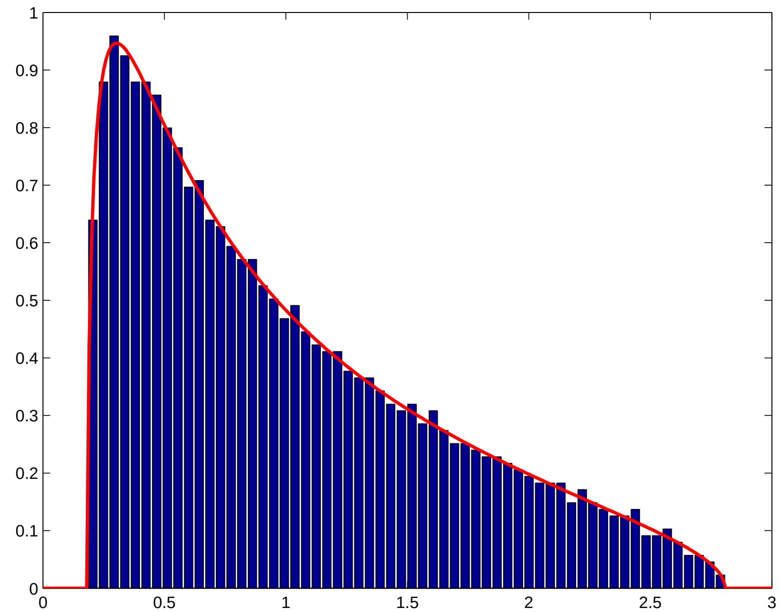
Example

free Poisson $\boxtimes$ free Poisson,

i.e., Wishart $\times$ Wishart



averaged Wishart x Wishart; N=100



... one realization ... N=2000

But what about situations

$$\text{Prob}(\mathbb{R}) \boxtimes \text{Prob}(\mathbb{R}_+)$$

For example:

semicircle $\boxtimes$ free Poisson

i.e., Gaussian $\times$ Wishart

Problem: Usual formulation with S-transform does not apply, since

- the S-transform of the semicircle does not exist as power series in z!

- the formula $S_{xy}(z) = S_x(z)S_y(z)$ is only proved for $\varphi(x) \neq 0 \neq \varphi(y)$

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However: This is not really a problem!!

Consider μ=semicircle.

Then

$$\kappa_n = \begin{cases} 1, & n = 2 \\ 0, & n \neq 2 \end{cases}, \qquad \text{hence} \qquad C(z) = z^2$$

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Note: actually, we are loosing uniqueness, we could also take

$$S(z) = -\frac{1}{\sqrt{z}}$$

This is true in general for μ with vanishing mean, if we exclude $\mu = \delta_0$. Then $\kappa_2 = \varphi(x^2) \neq 0$, thus

$$C(z) = \kappa_2 z^2 + \kappa_3 z^3 + \cdots$$

and thus

$$C^{<-1>}(z) = \text{power series in } \sqrt{z},$$

$$S(z) = \frac{1}{\sqrt{z}} \times \text{power series in } \sqrt{z}$$

Again, we have two possible choices for $C^{<-1>}$ and thus for S.

Definition [Rao + Speicher]: Let x be a random variable with $\varphi(x) = 0$ and $\varphi(x^2) \neq 0$. Then its two S-**transforms** S_x **and** $\tilde{S}_x$ are defined as follows. Let χ and $\tilde{\chi}$ denote the two inverses under composition of the series

$$\psi(z) := \sum_{n=1}^{\infty} \varphi(x^n) z^n = \varphi(x^2)z^2 + \varphi(x^3)z^3 + \cdots,$$

then

$$S_x(z) := \chi(z) \cdot \frac{1+z}{z} \qquad \text{and} \qquad \tilde{S}_x(z) := \tilde{\chi}(z) \cdot \frac{1+z}{z}.$$

Both S_x and $\tilde{S}_x$ are formal series in $\sqrt{z}$ of the form

$$\gamma_{-1} \frac{1}{\sqrt{z}} + \sum_{k=0}^{\infty} \gamma_k z^{k/2}$$

Theorem [Rao + Speicher]: Let x and y be free random variables such that $\varphi(x) = 0$, $\varphi(x^2) \neq 0$ and $\varphi(y) \neq 0$. By S_x and $\tilde{S}_x$ we denote the two S-transforms of x. Then

$$S_{xy}(z) = S_x(z) \cdot S_y(z) \qquad \text{and} \qquad \tilde{S}_{xy}(z) = \tilde{S}_x(z) \cdot S_y(z)$$

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Proof: Modify the combinatorial proof of [Nica + Speicher] to adjust it to this situation.

Meaning: To calculate $\mu \boxtimes \nu$ for

$$\mu \in \text{Prob}(\mathbb{R}), \qquad \nu \in \text{Prob}(\mathbb{R}_+)$$

- just calculate with the S-transforms as usual;
- don't bother about the choice of the sign, it will not matter in the end!

Example: semicircle $\boxtimes$ free Poisson

$$\text{semicircle } \mu: \qquad S_\mu(z) = \frac{1}{\sqrt{z}}$$

$$\text{free Poisson } \gamma: \qquad S_\gamma(z) = \frac{1}{z+1}.$$

Thus

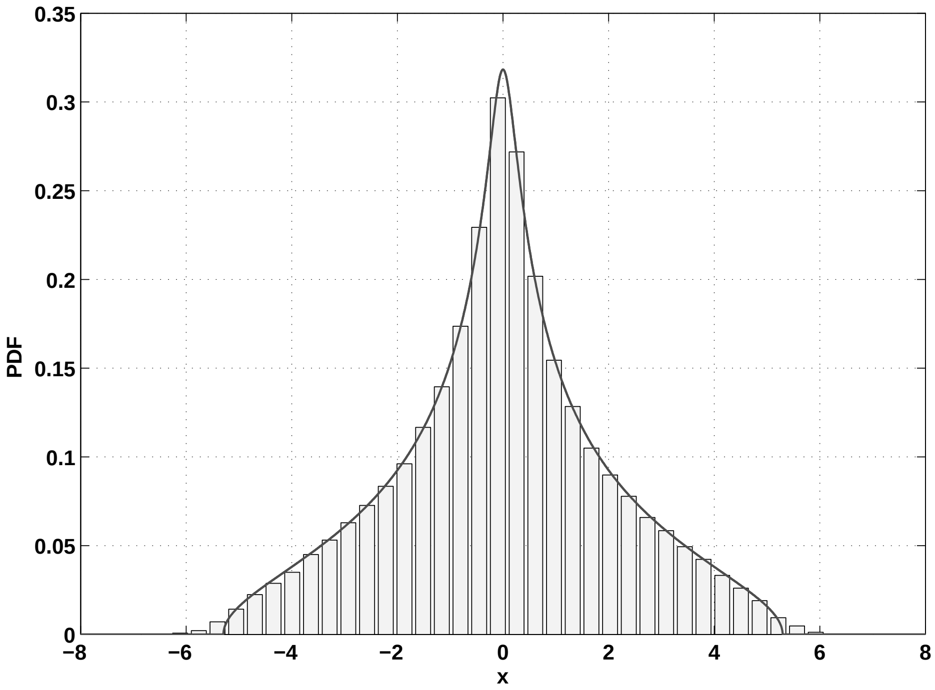
$$S_{\mu\boxtimes\gamma}(z) = \frac{1}{\sqrt{z}(z+1)},$$

which leads to the following algebraic equation for the Cauchy transform g of the probability measure $\mu \boxtimes \gamma$

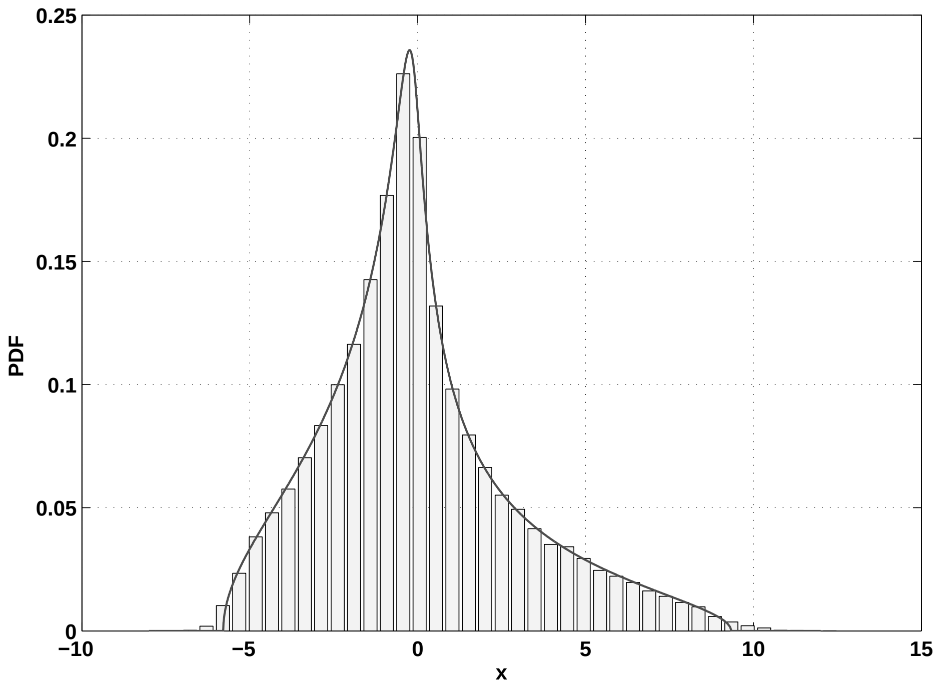
$$g^4 z^2 - zg + 1 = 0.$$

Examples:

Wigner x Wishart



shifted Wishart x Wishart



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- note, however, that for all $n \in \mathbb{N}$

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thus we could say formally

$$\mu \boxtimes \nu = \delta_0$$

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  thus we could say formally

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- but even this trivial statement is not captured correctly by the S-transform

Example: semicircle ⊠ semicircle

$$S_\mu(z) = \frac{1}{\sqrt{z}},$$

- thus

$$S_\mu(z) \cdot S_\mu(z) = \frac{1}{z}, \qquad \text{which yields} \qquad \psi(z) = \frac{1}{z} - 1.$$

This is not a moment series.

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  This is not a moment series.

- $\psi_{\delta_0}(z) = 0$;
  thus there is no corresponding inverse and no S-transform

Summary: To calculate $\mu \boxtimes \nu$ for

$$\mu \in \text{Prob}(\mathbb{R}), \qquad \nu \in \text{Prob}(\mathbb{R}_+)$$

- just calculate with the S-transforms as usual;

- don't bother about the choice of the sign, it will not matter in the end!