Free Probability and Non-Commutative Symmetries

Roland Speicher

Saarland University
Saarbrücken, Germany

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Section 1

The Origin of Freeness: Free Group Factors

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Let us Look on Moments

Free (non-commutative) probability theory investigates

operators on Hilbert spaces

by looking at

moments of those operators

Many methods and concepts for understanding those moments are inspired by analogues from

classical probability theory

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Some Basic Notations

Definition

Let $(\mathcal{A}, \varphi)$ be a **non-commutative probability space**, i.e.,

- $\mathcal{A}$ is a unital algebra
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a unital linear functional, i.e. $\varphi(1) = 1$

Consider **(non-commutative) random variables** $a_1, \dots, a_n \in \mathcal{A}$.
Expressions of the form

$$\varphi(a_{i(1)} \cdots a_{i(k)}) \qquad (k \in \mathbb{N}, 1 \leq i(1), \dots, i(k) \leq n)$$

are called **moments** of $a_1, \dots, a_n$.

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Moments of Generators Determine vN-Algebra

Let $\mathcal{A}$, $\mathcal{B}$ be two von Neumann algebras with

- $\mathcal{A} = \mathrm{vN}(a_1, \dots, a_n)$, and $\mathcal{B} = \mathrm{vN}(b_1, \dots, b_n)$ with selfadjoint generators a_i and b_i
- $\varphi : \mathcal{A} \to \mathbb{C}$ and $\psi : \mathcal{B} \to \mathbb{C}$ are faithful and normal states
- for all $k \in \mathbb{N}$ and $1 \leq i(1), \dots, i(k) \leq n$:

$$\varphi(a_{i(1)} \cdots a_{i(k)}) = \psi(b_{i(1)} \cdots b_{i(k)})$$

Then

$$\mathcal{A} \cong \mathcal{B} \qquad \text{via} \qquad a_i \mapsto b_i \quad (i = 1, \dots, n)$$

Consequence: Moments Can be Uselful

- All questions on operators, which depend only on the generated operator algebra, . . .
  . . . like: spectrum, polar decomposition, existence of hyperinvariant subspaces, inequalities for L^p-norms. . .
  . . . can in principle be answered by the knowldege of the moments of the operators with respect to a faithful normal state
- This insight is in general not very helpful, since moments are usually quite complicated
- However, in many special (and interesting) situations moments have a special structure

This is the realm of free probability theory

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Measure Theory Versus Probability Theory

Difference between measure theory and classical probability theory is essentially given by notion of

independence

Difference between von Neumann algebra theory and free probability theory is essentially given by notion of

freeness or **free independence**

Freeness describes special structure of moments arising from group von Neumann algebras $L(G)$, if G is the free product of subgroups

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Group von Neumann Algebra $L(G)$

Definition

Let G be a discrete group. The corresponding group von Neumann algebra is

$$L(G) := \overline{\mathbb{C}G}^{\text{STOP}}$$

$\uparrow$

representation of the group algebra

acting on the group by left multiplication

If G is i.c.c. (all non-trivial conjugacy classes are infinite), then $L(G)$ is a II_1 factor.
In particular, the neutral element e of G induces a trace τ on $L(G)$, which is faithful and normal, via

$$\tau(a) := \langle ae, e \rangle$$

Hyperfinite and Free Group Factors

G amenable $\Longrightarrow$ $L(G)$ hyperfinite II_1-Faktor

$G = \mathbb{F}_n$ free group on n generators $\Longrightarrow$ $L(\mathbb{F}_n)$ is not hyperfinite (Murray/von Neumann)

Voiculescu's philosophy: The free group factors $L(\mathbb{F}_n)$ are the next interesting class of von Neumann algebras after the hyperfinite one

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The Structure of the Free Group Factors

Free probability theory was created

- in order to understand $L(\mathbb{F}_n)$ and similar von Neumann algebras;
- in particular, to attack the most famous (and still open!!!) problem in this context:

(Isomorphism problem of the free group factors:)

Is it true or false that

$$L(\mathbb{F}_n) \cong L(\mathbb{F}_m) \qquad \text{for } n \neq m \ (n, m \geq 2)$$

Transfering Freeness from G to $L(G)$

$G = G_1 * G_2$	free product of groups
$\downarrow$	
$\mathbb{C}G = \mathbb{C}G_1 * \mathbb{C}G_2$	free product of algebras
$\downarrow ?$	
$L(G) = L(G_1) * L(G_2)$	free product of vN-algebras ???

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Algebraic Freeness of Subgroups

G_1, G_2 are **free** in G (as subgroups) means:

$$\left.\begin{array}{c} g_i \in G_{j(i)} \\ \\ g_i \neq e \quad \forall i \\ \\ j(1) \neq j(2) \neq \cdots \neq j(k) \end{array}\right\} \implies g_1 \cdots g_k \neq e$$

This algebraic formulation can be extented to finite sums (as in $\mathbb{C}G$), but not to infinite sums (as in $L(G)$).

Reformulation in Terms of the Trace

We can reformulate the freeness of the subgroups also in terms of τ:

$$\left.\begin{array}{c} g_i \in G_{j(i)} \\ \\ \tau(g_i) = 0 \quad \forall i \\ \\ j(1) \neq \cdots \neq j(k) \end{array}\right\} \quad \Longrightarrow \quad \tau(g_1 \cdots g_k) = 0$$

This characterisation goes over to finite as well as to infinite sums (note that τ is normal).

This motivated Voiculescu to make the following definition.

The Fundamental Notion: Freeness

Definition (Voiculescu 1985)

Let $\mathcal{A}$ be a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a unital linear functional.
Subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{A}$ are **free (w.r.t. φ)**, if:

$$\left.\begin{array}{c} a_i \in \mathcal{A}_{j(i)} \\ \\ \varphi(a_i) = 0 \quad \forall i \\ \\ j(1) \neq \cdots \neq j(k) \end{array}\right\} \implies \varphi(a_1 \cdots a_k) = 0$$

- Freeness is a special structure of the mixed moments in elements from $\mathcal{A}_1, \dots, \mathcal{A}_n$.
- **This structure should be seen and investigated in analogy to the classical concept of "independence".**

Section 2

Freeness

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Some History

1985 Voiculescu introduces "freeness" in the context of isomorphism problem of free group factors

1991 Voiculescu discovers relation with random matrices (which leads, among others, to deep results on free group factors)

1994 Speicher develops combinatorial theory of freeness, based on "free cumulants"

later ... many new results on operator algebras, eigenvalue distribution of random matrices, and much more ....

Definition of Freeness

Definition

- Let $(\mathcal{A}, \varphi)$ be **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$)
- Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever
  - $a_i \in \mathcal{A}_{j(i)}, \qquad j(i) \in I \quad \forall i,$
  - $j(1) \neq j(2) \neq \cdots \neq j(n)$
  - $\varphi(a_i) = 0 \quad \forall i$
- Random variables $x_1, \dots, x_n \in \mathcal{A}$ are free, if their generated unital subalgebras $\mathcal{A}_i := \text{algebra}(1, x_i)$ are so.

What is Freeness?

Freeness between x and y is an infinite set of equations relating various moments in x and y:

$$\varphi\Big(p_1(x)q_1(y)p_2(x)q_2(y)\cdots\Big) = 0$$

Basic observation: freeness between x and y is actually a **rule for calculating mixed moments** in x and y from the moments of x and the moments of y:

$$\varphi\Big(x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots\Big) = \text{polynomial}\big(\varphi(x^i), \varphi(y^j)\big)$$

Example

If x and y are free, then we have

$$\varphi(x^m y^n) = \varphi(x^m)\cdot\varphi(y^n)$$

Example

$$\varphi\Big(\big(x^m - \varphi(x^m)1\big)\big(y^n - \varphi(y^n)1\big)\Big) = 0,$$

thus

$$\varphi(x^m y^n) - \varphi(x^m \cdot 1)\varphi(y^n) - \varphi(x^m)\varphi(1 \cdot y^n) + \varphi(x^m)\varphi(y^n)\varphi(1 \cdot 1) = 0,$$

and hence $\boldsymbol{\varphi(x^m y^n) = \varphi(x^m) \cdot \varphi(y^n)}$

Example

$$\varphi\Big(\big(x - \varphi(x)1\big) \cdot \big(y - \varphi(y)1\big) \cdot \big(x - \varphi(x)1\big) \cdot \big(y - \varphi(y)1\big)\Big) = 0,$$

which results in

$$\boldsymbol{\varphi(xyxy) = \varphi(xx) \cdot \varphi(y) \cdot \varphi(y) + \varphi(x) \cdot \varphi(x) \cdot \varphi(yy) - \varphi(x) \cdot \varphi(y) \cdot \varphi(x) \cdot \varphi(y)}$$

Example

If x and y are free, then we have

$$\varphi(x^m y^n) = \varphi(x^m) \cdot \varphi(y^n)$$

$$\varphi(x^{m_1} y^n x^{m_2}) = \varphi(x^{m_1+m_2}) \cdot \varphi(y^n)$$

but also

$$\varphi(xyxy) = \varphi(x^2) \cdot \varphi(y)^2 + \varphi(x)^2 \cdot \varphi(y^2) - \varphi(x)^2 \cdot \varphi(y)^2$$

Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables. This is the reason that it is also called "free independence".

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Example

If x and y are free, then we have

$$\varphi(x^m y^n) = \varphi(x^m) \cdot \varphi(y^n)$$

$$\varphi(x^{m_1} y^n x^{m_2}) = \varphi(x^{m_1+m_2}) \cdot \varphi(y^n)$$

but also

$$\varphi(xyxy) = \varphi(x^2) \cdot \varphi(y)^2 + \varphi(x)^2 \cdot \varphi(y^2) - \varphi(x)^2 \cdot \varphi(y)^2$$

Free independence is a rule for calculating mixed moments, analogous to the concept of independence for random variables.
Note: free independence is a different rule from classical independence; free independence occurs typically for **non-commuting random variables**, like operators on Hilbert spaces or (random) matrices.

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Where Does Freeness Show Up?

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices

Section 3

The Emergence of the Combinatorics of Freeness

Motivation for the Combinatorics of Freeness: the Free (and Classical) CLT

Consider $a_1, a_2, \dots \in (\mathcal{A}, \varphi)$ which are

- identically distributed
- centered and normalized: $\varphi(a_i) = 0$ and $\varphi(a_i^2) = 1$
- either classically independent or freely independent

What can we say about

$$S_n := \frac{a_1 + \cdots + a_n}{\sqrt{n}} \quad \xrightarrow{n \to \infty} \quad ???$$

Definition

We say that S_n converges (in distribution) to s if

$$\lim_{n \to \infty} \varphi(S_n^m) = \varphi(s^m) \qquad \forall m \in \mathbb{N}$$

Calculation of Moments of S_n

We have

$$\varphi(S_n^m) = \frac{1}{n^{m/2}}\,\varphi\big[(a_1 + \cdots a_n)^m\big]$$

$$= \frac{1}{n^{m/2}} \sum_{i(1),\dots,i(m)=1}^{n} \varphi[a_{i(1)} \cdots a_{i(m)}]$$

Basic Observation

Note:

$$\varphi[a_{i(1)} \cdots a_{i(m)}] = \varphi[a_{j(1)} \cdots a_{j(m)}]$$

whenever

$$\ker i = \ker j$$

Example

For example, $i = (1, 3, 1, 5, 3)$ and $j = (3, 4, 3, 6, 4)$:

$$\varphi[a_1 a_3 a_1 a_5 a_3] = \varphi[a_3 a_4 a_3 a_6 a_4]$$

because independence/freeness allows to express

$$\varphi[a_1 a_3 a_1 a_5 a_3] = \text{polynomial}\big(\varphi(a_1), \varphi(a_1^2), \varphi(a_3), \varphi(a_3^2), \varphi(a_5)\big)$$
$$\varphi[a_3 a_4 a_3 a_6 a_4] = \text{polynomial}\big(\varphi(a_3), \varphi(a_3^2), \varphi(a_4), \varphi(a_4^2), \varphi(a_6)\big)$$

and $\qquad \varphi(a_1) = \varphi(a_3), \qquad \varphi(a_1^2) = \varphi(a_3^2)$

$$\varphi(a_3) = \varphi(a_4), \qquad \varphi(a_3^2) = \varphi(a_4^2), \qquad \varphi(a_5) = \varphi(a_6)$$

We put

$$\kappa_\pi := \varphi[a_1 a_3 a_1 a_5 a_3] \quad \text{where} \quad \pi := \ker i = \ker j = \{\{1, 3\}, \{2, 5\}, \{4\}\}$$

$\pi \in \mathcal{P}(5)$ is a partition of $\{1, 2, 3, 4, 5\}$.

Calculation of Moments of S_n

Thus

$$\varphi(S_n^m) = \frac{1}{n^{m/2}} \sum_{i(1),\dots,i(m)=1}^{n} \varphi[a_{i(1)} \cdots a_{i(m)}]$$

$$= \frac{1}{n^{m/2}} \sum_{\pi \in \mathcal{P}(m)} \kappa_\pi \cdot \#\{i : \ker i = \pi\}$$

Note:

$$\#\{i : \ker i = \pi\} \quad = \quad n(n-1)\cdots(n-\#\pi-1) \sim n^{\#\pi}$$

So

$$\varphi(S_n^m) \sim \sum_{\pi \in \mathcal{P}(m)} \kappa_\pi \cdot n^{\#\pi - m/2}$$

No Singletons in the Limit

Consider $\pi \in \mathcal{P}(m)$ with singleton:

$$\pi = \{\dots, \{k\}, \dots\},$$

thus

$$\begin{aligned}\kappa_\pi &= \varphi\big(a_{i(1)} \cdots a_{i(k)} \cdots a_{i(m)}\big) \\ &= \varphi\big(a_{i(1)} \cdots a_{i(k-1)} a_{i(k+1)} \cdots a_{i(m)}\big) \cdot \underbrace{\varphi\big(a_{i(k)}\big)}_{=0} \\ &= 0\end{aligned}$$

We used: If $\{x, y\}$ and a are free/independent, then:

$$\varphi(xay) = \varphi(xy)\varphi(a)$$

No Singletons in the Limit

Consider $\pi \in \mathcal{P}(m)$ with singleton:

$$\pi = \{\dots, \{k\}, \dots\},$$

thus

$$\begin{aligned}\kappa_\pi &= \varphi\big(a_{i(1)} \cdots a_{i(k)} \cdots a_{i(m)}\big) \\ &= \varphi\big(a_{i(1)} \cdots a_{i(k-1)} a_{i(k+1)} \cdots a_{i(m)}\big) \cdot \underbrace{\varphi\big(a_{i(k)}\big)}_{=0} \\ &= 0\end{aligned}$$

Thus: $\kappa_\pi = 0$ if π has singleton.

Only Pairings Survive in the Limit

So we have

$$\kappa_\pi \neq 0 \quad \Longrightarrow \quad \pi = \{V_1, \dots, V_r\} \text{ with } \#V_j \geq 2\,\forall j$$
$$\Longrightarrow \quad r = \#\pi \leq \frac{m}{2}$$

So in

$$\varphi(S_n^m) \sim \sum_{\pi \in \mathcal{P}(m)} \kappa_\pi \cdot n^{\#\pi - m/2}$$

only those π survive for $n \to \infty$ with

- π has no singleton, i.e., no block of size 1
- π has exactly $m/2$ blocks

Such π are exactly those, where each block has size 2, i.e.,

$$\pi \in \mathcal{P}_2(m) := \{\pi \in \mathcal{P}(m) \mid \pi \text{ is pairing}\}$$

Limit Moments are Given by Summation over Pairings

Thus we have:

$$\lim_{n\to\infty} \varphi(S_n^m) = \sum_{\pi\in\mathcal{P}_2(m)} \kappa_\pi$$

- This gives in particular: odd moments are zero
  (because no pairings of odd number of elements),
  thus limit distribution is symmetric
- What are the even moments?
  This depends on the κ_π's.
  The actual value of those is now different for the classical and the free case!

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Classical CLT: Assume a_i are Independent

If the a_i commute and are independent, then

$$\kappa_\pi = \varphi(a_{i(1)} \cdots a_{i(2k)}) = 1 \qquad \forall \pi \in \mathcal{P}_2(2k)$$

Example

$$\varphi(a_1 a_2 a_3 a_3 a_2 a_1) = 1 = \varphi(a_1 a_2 a_2 a_3 a_1 a_3)$$

Thus

$$\lim_{n\to\infty} \varphi(S_n^m) = \#\mathcal{P}_2(m) = \begin{cases} 0, & m \text{ odd} \\ (m-1)(m-3)\cdots 5 \cdot 3 \cdot 1, & m \text{ even} \end{cases}$$

Those limit moments are the moments of a Gaussian distribution of variance 1.

Free CLT: Assume a_i are Free

If the a_i are free, then, for $\pi \in \mathcal{P}_2(2k)$,

$$\kappa_\pi = \begin{cases} 0, & \pi \text{ is crossing} \\ 1, & \pi \text{ is non-crossing} \end{cases}$$

Example

- non-crossing π

$$\begin{aligned} \kappa_{\{1,6\},\{2,5\},\{3,4\}} = \varphi(a_1a_2a_3a_3a_2a_1) &= \varphi(a_3a_3)\cdot\varphi(a_1a_2a_2a_1) \\ &= \varphi(a_3a_3)\cdot\varphi(a_2a_2)\cdot\varphi(a_1a_1) \\ &= 1 \end{aligned}$$

- crossing π

$$\kappa_{\{1,5\},\{2,3\},\{4,6\}} = \varphi(a_1a_2a_2a_3a_1a_3) = \varphi(a_2a_2)\cdot\underbrace{\varphi(a_1a_3a_1a_3)}_{0} = 0$$

Free CLT: Assume a_i are Free

Notation

Put

$$NC_2(m) := \{\pi \in \mathcal{P}_2(m) \mid \pi \text{ is non-crossing}\}$$

Thus

$$\lim_{n\to\infty} \varphi(S_n^m) = \#NC_2(m) = \begin{cases} 0, & m \text{ odd} \\ c_k = \frac{1}{k+1}\binom{2k}{k}, & m = 2k \text{ even} \end{cases}$$

Those limit moments are the moments of a semicircular distribution of variance 1,

$$\lim_{n\to\infty} \varphi(S_n^m) = \frac{1}{2\pi}\int_{-2}^{2} t^m \sqrt{4-t^2}\,dt$$

How to Recognize the Catalan Numbers c_k

Notation

Put

$$c_k := \#NC_2(2k).$$

Basic Observation

We have

$$c_k = \sum_{\pi \in NC(2k)} 1 = \sum_{i=1}^{k} \sum_{\pi=\{1,2i\}\cup\pi_1\cup\pi_2} 1 = \sum_{i=1}^{k} c_{i-1}c_{k-i}$$

This recursion, together with $c_0 = 1, c_1 = 1$, determines the sequence of **Catalan numbers**:

$$\{c_k\} = 1, 1, 2, 5, 14, 42, 132, 429, \ldots.$$

Intermezzo:
One Slide on Random Matrices

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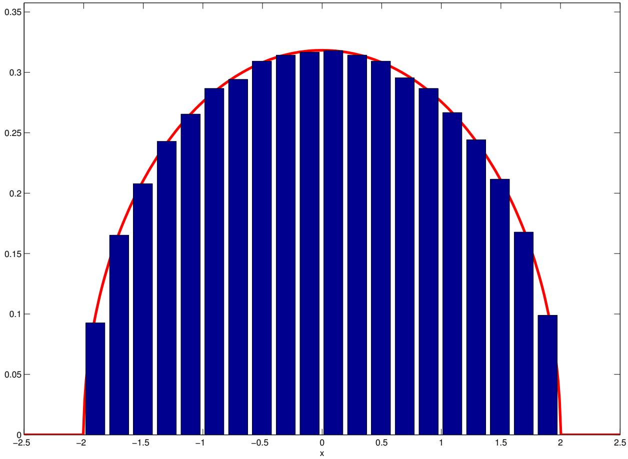
Convergence of Eigenvalue Distribution of Gaussian Random Matrices to Semicircle



Section 4

Free Cumulants

Understanding the Freeness Rule: the Idea of Cumulants

- write moments in terms of other quantities, which we call **free cumulants**
- freeness is much easier to describe on the level of free cumulants: **vanishing of mixed cumulants**
- relation between moments and cumulants is given by summing over **non-crossing or planar partitions**

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Non-Crossing Partitions

Definition

A **partition** of $\{1, \ldots, n\}$ is a decomposition $\pi = \{V_1, \ldots, V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq y), \qquad \bigcup_i V_i = \{1, \ldots, n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.
π is **non-crossing** if we do not have

$$p_1 < q_1 < p_2 < q_2$$

such that p_1, p_2 are in same block, q_1, q_2 are in same block, but those two blocks are different.

$$\mathbf{NC(n)} := \{\text{non-crossing partitions of } \{1,\ldots,n\}\}$$

$NC(n)$ is actually a lattice with refinement order.

Moments and Cumulants

Definition

For unital linear functional

$$\varphi : \mathcal{A} \to \mathbb{C}$$

we define **cumulant functionals** κ_n (for all $n \geq 1$)

$$\kappa_n : \mathcal{A}^n \to \mathbb{C}$$

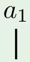
as multi-linear functionals by moment-cumulant relation

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi[a_1, \dots, a_n]$$

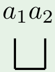
Note: classical cumulants are defined by a similar formula, where only $NC(n)$ is replaced by $\mathcal{P}(n)$

Example ($n = 1$)

$$\varphi(a_1) = \kappa_1(a_1)$$



Example ($n = 2$)



$$\varphi(a_1 a_2) = \kappa_2(a_1, a_2)$$

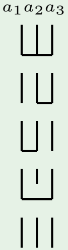
$$+ \kappa_1(a_1)\kappa_1(a_2)$$

and thus

$$\kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$$

Example ($n = 3$)

$$\begin{aligned}\varphi(a_1a_2a_3) &= \kappa_3(a_1,a_2,a_3)\\ &\quad + \kappa_1(a_1)\kappa_2(a_2,a_3)\\ &\quad + \kappa_2(a_1,a_2)\kappa_1(a_3)\\ &\quad + \kappa_2(a_1,a_3)\kappa_1(a_2)\\ &\quad + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3)\end{aligned}$$

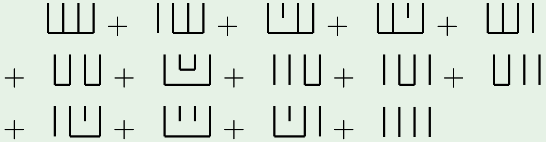


and thus

$$\begin{aligned}\kappa_3(a_1,a_2,a_3) &= \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) - \varphi(a_2)\varphi(a_1a_3)\\ &\qquad - \varphi(a_3)\varphi(a_1a_2) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3)\end{aligned}$$

Example ($n = 4$)

$\varphi(a_1 a_2 a_3 a_4) =$



$$
\begin{aligned}
= \quad & \kappa_4(a_1,a_2,a_3,a_4) + \kappa_1(a_1)\kappa_3(a_2,a_3,a_4) \\
& + \kappa_1(a_2)\kappa_3(a_1,a_3,a_4) + \kappa_1(a_3)\kappa_3(a_1,a_2,a_4) \\
& + \kappa_3(a_1,a_2,a_3)\kappa_1(a_4) + \kappa_2(a_1,a_2)\kappa_2(a_3,a_4) \\
& + \kappa_2(a_1,a_4)\kappa_2(a_2,a_3) + \kappa_1(a_1)\kappa_1(a_2)\kappa_2(a_3,a_4) \\
& + \kappa_1(a_1)\kappa_2(a_2,a_3)\kappa_1(a_4) + \kappa_2(a_1,a_2)\kappa_1(a_3)\kappa_1(a_4) \\
& + \kappa_1(a_1)\kappa_2(a_2,a_4)\kappa_1(a_3) + \kappa_2(a_1,a_4)\kappa_1(a_2)\kappa_1(a_3) \\
& + \kappa_2(a_1,a_3)\kappa_1(a_2)\kappa_1(a_4) + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3)\kappa_1(a_4)
\end{aligned}
$$

Freeness $\hat{=}$ Vanishing of Mixed Cumulants

Theorem (Speicher 1994)

The fact that $x_1, \dots, x_m$ are free is equivalent to the fact that

$$\kappa_n(x_{i(1)}, \dots, x_{i(n)}) = 0$$

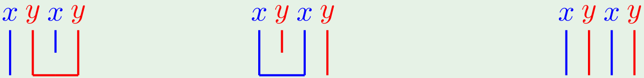
whenever

- $1 \leq i(1), \dots, i(n) \leq m$
- *there are p, q such that $i(p) \neq i(q)$ (in particular, $n \geq 2$)*

Example

If x and y are free then: $\varphi(xyxy) =$

$$\kappa_1(x)\kappa_1(x)\kappa_2(y,y) + \kappa_2(x,x)\kappa_1(y)\kappa_1(y) + \kappa_1(x)\kappa_1(y)\kappa_1(x)\kappa_1(y)$$



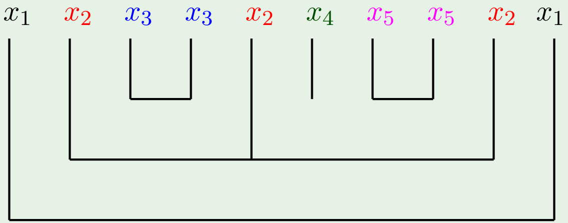
Factorization of Non-Crossing Moments

Example

The iteration of the rule

$$\varphi(axb) = \varphi(ab)\varphi(x) \qquad \text{if } \{a, b\} \text{ and } x \text{ free}$$

leads to the factorization of all "non-crossing" moments in free variables



$$\varphi(x_1 x_2 x_3 x_3 x_2 x_4 x_5 x_5 x_2 x_1)$$
$$= \varphi(x_1 x_1) \cdot \varphi(x_2 x_2 x_2) \cdot \varphi(x_3 x_3) \cdot \varphi(x_4) \cdot \varphi(x_5 x_5)$$

Section 5

Operator-Valued Extension of Free Probability

Definition

Let $\mathcal{B} \subset \mathcal{A}$. A linear map $E : \mathcal{A} \to \mathcal{B}$ is a **conditional expectation** if

$$E[b] = b \qquad \forall b \in \mathcal{B}$$

and

$$E[b_1 a b_2] = b_1 E[a] b_2 \qquad \forall a \in \mathcal{A}, \quad \forall b_1, b_2 \in \mathcal{B}$$

An **operator-valued probability space** consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$

Example (Classical conditional expectation)

Let $\mathfrak{M}$ be a σ-algebra and $\mathfrak{N} \subset \mathfrak{M}$ be a sub-σ-algebra. Then

- $\mathcal{A} = L^\infty(\Omega, \mathfrak{M}, P)$
- $\mathcal{B} = L^\infty(\Omega, \mathfrak{N}, P)$
- $E[\cdot|\mathfrak{N}]$ is the classical conditional expectation from the bigger onto the smaller σ-algebra.

Operator-Valued Freeness

Definition

Consider an operator-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$. The **operator-valued distribution** of $x \in \mathcal{A}$ is given by all operator-valued moments

$$E[xb_1xb_2 \cdots b_{n-1}x] \in \mathcal{B} \qquad (n \in \mathbb{N}, b_1, \ldots, b_{n-1} \in \mathcal{B})$$

Random variables $x_i \in \mathcal{A}$ $(i \in I)$ are **free with respect to E** (or **free with amalgamation over $\mathcal{B}$**) if

$$E[a_1 \cdots a_n] = 0$$

whenever

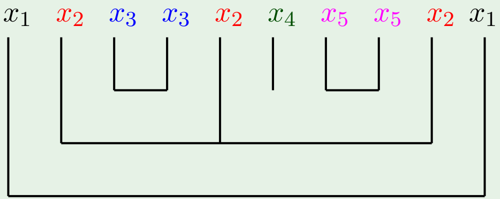
- $a_i \in \mathcal{B}\langle x_{j(i)} \rangle$ are polynomials in some $x_{j(i)}$ with coefficients from $\mathcal{B}$
- $j(1) \neq j(2) \neq \cdots \neq j(n)$
- $E[a_i] = 0$ for all i

Operator-Valued Freeness: NC Moments

Note: random variables x and scalars b from $\mathcal{B}$ do not commute in general!

Example

Still one has factorizations of all non-crossing moments in free variables.



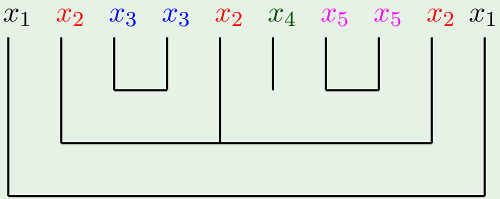
$$E[x_1 x_2 x_3 x_3 x_2 x_4 x_5 x_5 x_2 x_1]$$
$$= E\Big[x_1 \cdot E\big[x_2 \cdot E[x_3 x_3] \cdot x_2 \cdot E[x_4] \cdot E[x_5 x_5] \cdot x_2\big] \cdot x_1\Big]$$

Operator-Valued Freeness: NC Moments

Operator-valued freeness works mostly like ordinary freeness, one only has to take care of the order of the variables; in all expressions they have to appear in their original order!

Example

Still one has factorizations of all non-crossing moments in free variables.



$$E[x_1 x_2 x_3 x_3 x_2 x_4 x_5 x_5 x_2 x_1]$$

$$= E\Big[x_1 \cdot E\big[x_2 \cdot E[x_3 x_3] \cdot x_2 \cdot E[x_4] \cdot E[x_5 x_5] \cdot x_2\big] \cdot x_1\Big]$$

Operator-Valued Freeness: Crossing Moments

For "crossing" moments one has analogous formulas as in scalar-valued case, modulo respecting the order of the variables ...

Example

The formula for free x_1 and x_2

$$\varphi(x_1x_2x_1x_2) = \varphi(x_1x_1)\varphi(x_2)\varphi(x_2) + \varphi(x_1)\varphi(x_1)\varphi(x_2x_2) - \varphi(x_1)\varphi(x_2)\varphi(x_1)\varphi(x_2)$$

has now to be written as

$$E[x_1x_2x_1x_2] = E\big[x_1E[x_2]x_1\big] \cdot E[x_2] + E[x_1] \cdot E\big[x_2E[x_1]x_2\big] - E[x_1]E[x_2]E[x_1]E[x_2]$$

Operator-Valued Free Cumulants

Definition

Consider an operator-valued probability space$E : \mathcal{A} \to \mathcal{B}$.
We define operator-valued **free cumulants**

$$\kappa_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}$$

by

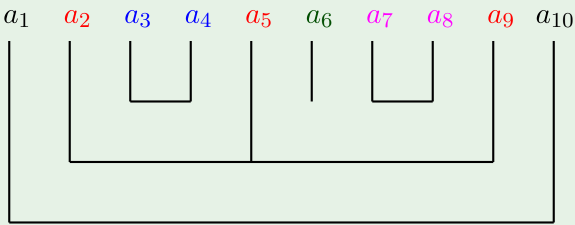
$$E[a_1 \cdots a_n] = \sum_{\pi \in NC(n)} \kappa_\pi^{\mathcal{B}}[a_1, \ldots, a_n]$$

- arguments of $\kappa_\pi^{\mathcal{B}}$ are distributed according to blocks of π
- but now: cumulants are nested inside each other according to nesting of blocks of π

Operator-Valued Free Cumulants

Example

$$\pi = \big\{\{1,10\},\{2,5,9\},\{3,4\},\{6\},\{7,8\}\big\} \in NC(10),$$



$$\kappa_\pi^{\mathcal{B}}[a_1,\dots,a_{10}]$$
$$= \kappa_2^{\mathcal{B}}\Big(a_1 \cdot \kappa_3^{\mathcal{B}}\big(a_2 \cdot \kappa_2^{\mathcal{B}}(a_3,a_4), a_5 \cdot \kappa_1^{\mathcal{B}}(a_6) \cdot \kappa_2^{\mathcal{B}}(a_7,a_8), a_9\big), a_{10}\Big)$$

Vanishing of Mixed Cumulants Characterizes Freeness

Definition

We define operator-valued **free cumulants** $\kappa_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}$ by

$$E[a_1 \cdots a_n] = \sum_{\pi \in NC(n)} \kappa_\pi^{\mathcal{B}}[a_1, \dots, a_n]$$

As in the scalar-valued case the following are equivalent:

- $x_1, \dots, x_m$ are free over $\mathcal{B}$
- for all n, $1 \leq i(1), \dots, i(n) \leq m$ with $i(p) \neq i(q)$ for some p, q, and all $b_1, \dots, b_{n-1} \in \mathcal{B}$ we have

$$\kappa_n^{\mathcal{B}}(x_{i(1)} b_1, x_{i(2)} b_2, \dots, x_{i(n-1)} b_{n-1}, x_{i(n)}) = 0$$

Section 6

Non-Commutative de Finetti Theorem, Quantum Permutation Group and Non-Crossing Partitions

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Classical Exchangeable Random Variables

Consider probability space $(\Omega, \mathfrak{A}, P)$. Denote expectation by φ,

$$\varphi(Y) = \int_\Omega Y(\omega) dP(\omega).$$

Definition

We say that random variables $X_1, X_2, \dots$ are **exchangeable** if their joint distribution is invariant under finite permutations, i.e. if

$$\varphi(X_{i(1)} \cdots X_{i(n)}) = \varphi(X_{\pi(i(1))} \cdots X_{\pi(i(n))})$$

for all $n \in \mathbb{N}$, all $i(1), \dots, i(n) \in \mathbb{N}$, and all permutations π

Example

$$\varphi(X_1^n) = \varphi(X_7^n), \qquad \varphi(X_1^3 X_3^7 X_4) = \varphi(X_8^3 X_2^7 X_9)$$

Tail σ-Algebra

Example

- Independent and identically distributed random variables are exchangeable.
- Note that the X_i might all contain some common component; e.g., if all X_i are the same, then clearly $X, X, X, X, X \ldots$ is exchangeable.

Theorem of de Finetti says that an infinite sequence of exchangeable random variables is independent modulo its common part.

Tail σ-Algebra

Example

- Independent and identically distributed random variables are exchangeable.
- Note that the X_i might all contain some common component; e.g., if all X_i are the same, then clearly $X, X, X, X, X \dots$ is exchangeable.

Formalize common part via **tail σ-algebra** of the sequence $X_1, X_2, \dots$

$$\mathfrak{A}_{\text{tail}} := \bigcap_{i \in \mathbb{N}} \sigma(X_k \mid k \geq i)$$

Denote by E the **conditional expectation onto this tail σ-algebra**

$$E : L^\infty(\Omega, \mathfrak{A}, P) \to L^\infty(\Omega, \mathfrak{A}_{\text{tail}}, P)$$

Classical de Finetti Theorem

Definition

$$\mathfrak{A}_{\text{tail}} := \bigcap_{i \in \mathbb{N}} \sigma(X_k \mid k \geq i)$$

$$E : L^\infty(\Omega, \mathfrak{A}, P) \to L^\infty(\Omega, \mathfrak{A}_{\text{tail}}, P)$$

Theorem (de Finetti 1931, Hewitt, Savage 1955)

The following are equivalent for an infinite sequence of random variables:

- *the sequence is exchangeable*
- *the sequence is independent and identically distributed with respect to the conditional expectation E onto the tail σ-algebra of the sequence*

$$E[X_1^{m(1)} X_2^{m(2)} \cdots X_n^{m(n)}] = E[X_1^{m(1)}] \cdot E[X_2^{m(2)}] \cdots E[X_n^{m(n)}]$$

Non-commutative Random Variables

Replace now

random variables → operators on Hilbert spaces

expectation → state on the algebra generated by those operators

Setting

In the following our setting will be a **non-commutative W^*-probability space** $(\mathcal{A}, \varphi)$, i.e.,

- $\mathcal{A}$ is von Neumann algebra (i.e., weakly closed subalgebra of bounded operators on Hilbert space)
- $\varphi : \mathcal{A} \to \mathbb{C}$ is faithful state on $\mathcal{A}$, i.e.,

$$\varphi(aa^*) \geq 0, \qquad \text{for all } a \in \mathcal{A}$$

$$\varphi(aa^*) = 0 \quad \text{if and only if } a = 0$$

Exchangeable NC Random Variables

Definition

Non-commutative random variables $x_1, x_2, \dots \in \mathcal{A}$ are **exchangeable** if

$$\varphi(x_{i(1)} \cdots x_{i(n)}) = \varphi(x_{\pi(i(1))} \cdots x_{\pi(i(n))})$$

for all $n \in \mathbb{N}$, all $i(1), \dots, i(n) \in \mathbb{N}$, and all permutations π.

Question

Does exchangeability imply anything like independence in this general non-commutative setting?

Answer

Only partially. Exchangeability gives, by work of Koestler, some weak form of independence (special factorization properties), but does not fully determine all mixed moments ... there are too many possibilities out in the non-commutative world, and exchangeability is a too weak condition!

However ...

Invariance under permutations is in a sense also a commutative concept ...
... and should be replaced by a non-commutative counterpart in the non-commutative world!

permutation group $\longrightarrow$ quantum permutation group

Recall: classical permutation group

$$S_k \hat{=} \{k \times k \text{ permutation matrices}\}$$

Dualize

$$C(S_k) = \{f : S_k \to \mathbb{C}; g \mapsto \left((u_{ij}(g)\right)_{i,j=1}^{k}\}$$

Classical Permutation Group S_k

Then $C(S_k)$ is the universal commutative C^*-algebra generated by u_{ij} $(i, j = 1, \dots, k)$, subject to the relations

$$u_{ij}^* = u_{ij} = u_{ij}^2 \quad \forall i, j, \qquad \sum_j u_{ij} = 1 = \sum_j u_{ji} \quad \forall i$$

alg$(u_{ij} \mid i, j = 1, \dots, k)$ is a Hopf algebra (which is dense in $C(S_k)$) with

$$\Delta u_{ij} = \sum_k u_{ik} \otimes u_{kj}$$ coproduct

$$\varepsilon(u_{ij}) = \delta_{ij}$$ co-unit

$$S(u_{ij}) = u_{ji}$$ antipode

Quantum Permutation Group

Definition (Wang 1998)

The quantum permutation group $A_s(k)$ is given by the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \dots, k)$ subject to the relations

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i, j = 1, \dots, k$
- each row and column of $u = (u_{ij})_{i,j=1}^k$ is a partition of unity:

$$\sum_{j=1}^{k} u_{ij} = 1 \quad \forall i \qquad \text{and} \qquad \sum_{i=1}^{k} u_{ij} = 1 \quad \forall j$$

(note: elements within a row or within a column are orthogonal)

$A_s(k)$ is a compact quantum group in the sense of Woronowicz.

Notation

We write: $A_s(k) = C(S_k^+)$

$$S_k^+ \mathrel{\hat{=}} \{\text{quantum permutations}\}$$

$$\mathrel{\hat{=}} \Big\{u = (u_{ij}) \mid u_{ij} \text{ operators on Hilbert space satisfying these relations}\Big\}$$

If

$$u_1 = (u_{ij}^{(1)})_{i,j=1}^k \qquad \text{and} \qquad u_2 = (u_{ij}^{(2)})_{i,j=1}^k$$

are quantum permutations, then so is

$$u_1 \odot u_2 := \Big(\sum_k u_{ik}^{(1)} \otimes u_{kj}^{(2)}\Big)_{i,j=1}^k$$

Quantum Permutations

Example

Examples of $u = (u_{ij})_{i,j=1}^k$ satisfying these relations are:

- permutation matrices
- basic non-commutative example is of the form (for $k = 4$):

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & 1 \end{pmatrix}$$

for (in general, non-commuting) projections p and q

- $S_2^+ = S_2$,
- $S_3^+ = S_3$,
- but $S_k^+ \neq S_k$ for $k \geq 4$

Quantum Exchangeability

Definition

A sequence $x_1, \dots, x_k$ in $(\mathcal{A}, \varphi)$ is **quantum exchangeable** if its distribution does not change under the action of quantum permutations S_k^+, i.e., if we have:
Let a quantum permutation $u = (u_{ij}) \in C(S_k^+)$ act on $(x_1, \dots, x_k)$ by

$$y_i := \sum_j u_{ij} \otimes x_j \qquad \in \qquad C(S_k^+) \otimes \mathcal{A}$$

Then

- $(x_1, \dots, x_k) \in (\mathcal{A}, \varphi)$
- $(y_1, \dots, y_k) \in (C(S_k^+) \otimes \mathcal{A}, \mathrm{id} \otimes \varphi)$

have the same distribution, i.e.,

$$\varphi(x_{i(1)} \cdots x_{i(n)}) \cdot 1_{C(S_k^+)} = \mathrm{id} \otimes \varphi(y_{i(1)} \cdots y_{i(n)})$$

Quantum Exchangeability

Equality of distributions

$$\varphi(x_{i(1)} \cdots x_{i(n)}) \cdot 1_{C(S_k^+)} = \mathrm{id} \otimes \varphi(y_{i(1)} \cdots y_{i(n)})$$

means explicitly that

$$\varphi(x_{i(1)} \cdots x_{i(n)}) \cdot 1 = \sum_{j(1),\dots,j(n)=1}^{k} u_{i(1)j(1)} \cdots u_{i(n)j(n)} \varphi(x_{j(1)} \cdots x_{j(n)})$$

for all $u = (u_{ij})_{i,j=1}^k$ which satisfy the defining relations for $A_s(k)$.

- In particular: quantum exchangeable $\implies$ exchangeable
- Commuting variables are usually not quantum exchangeable

F.I.D. Variables are Quantum Exchangeable

Proposition

Consider $x_1, \dots, x_k \in (\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_1, \dots, x_k$ are quantum exchangeable.

Proof

We have to show equality of moments of x_i's and of y_i's. This is the same, by moment-cumulant formula, as showing for all $n \in \mathbb{N}$ and all $\pi \in NC(n)$

$$\text{id} \otimes \kappa_\pi(y_{i(1)}, \dots, y_{i(n)}) = \kappa_\pi(x_{i(1)}, \dots, i_{i(n)})$$

Consider $n = 3$ and $\pi =$  .Then we have

$$LHS = \sum_{j(1),j(2),j(3)} u_{i(1)j(1)} u_{i(2)j(2)} u_{i(3)j(3)} \cdot \kappa_\pi(x_{j(1)}, x_{j(2)}, x_{j(3)})$$

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Consider $n = 3$ and $\pi =$  .Then we have

$$LHS = \sum_{j(1),j(2),j(3)} u_{i(1)j(1)} \underbrace{u_{i(2)j(2)}}_{\sum_{j(2)} \to 1} u_{i(3)j(3)} \cdot \kappa_2(x_{j(1)}, x_{j(3)}) \cdot \underbrace{\kappa_1(x_{j(2)})}_{\kappa_1(x)}$$

F.I.D. Variables are Quantum Exchangeable

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Consider $n = 3$ and $\pi =$  .Then we have

$$LHS = \sum_{j(1), j(3)} u_{i(1)j(1)} u_{i(3)j(3)} \cdot \kappa_2(x_{j(1)}, x_{j(3)}) \cdot \kappa_1(x)$$

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Consider $n = 3$ and $\pi =$  .Then we have

$$LHS = \sum_{j(1), j(3)} u_{i(1)j(1)} u_{i(3)j(3)} \cdot \underbrace{\kappa_2(x_{j(1)}, x_{j(3)})}_{\delta_{j(1)j(3)} \cdot \kappa_2(x,x)} \cdot \kappa_1(x)$$

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$$\text{id} \otimes \kappa_\pi(y_{i(1)}, \dots, y_{i(n)}) = \kappa_\pi(x_{i(1)}, \dots, i_{i(n)})$$

Consider $n = 3$ and $\pi =$  .Then we have

$$LHS = \sum_{j(1)=j(3)} u_{i(1)j(1)} u_{i(3)j(3)} \cdot \kappa_2(x, x) \cdot \kappa_1(x)$$

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Consider $n = 3$ and $\pi =$  .Then we have

$$LHS = \sum_{j(1)} \underbrace{u_{i(1)j(1)} u_{i(3)j(1)}}_{\delta_{i(1)i(3)} u_{i(1)j(1)}} \cdot \kappa_2(x, x) \cdot \kappa_1(x)$$

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Consider $n = 3$ and $\pi =$  .Then we have

$$LHS = \kappa_2(x_{i(1)}, x_{i(3)}) \cdot \kappa_1(x_{i(2)})$$

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$$\mathrm{id} \otimes \kappa_\pi(y_{i(1)}, \dots, y_{i(n)}) = \kappa_\pi(x_{i(1)}, \dots, i_{i(n)})$$

Consider $n = 3$ and $\pi =$  .Then we have

$$LHS = \kappa_\pi(x_{i(1)}, x_{i(2)}, x_{i(3)}) = RHS$$

Implications of Quantum Exchangeability

Question

What does quantum exchangeability for an infinite sequence $x_1, x_2, \ldots$ imply?

As before, constant sequences are trivially quantum exchangeable, thus we have to take out the common part of all the x_i.

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Implications of Quantum Exchangeability

Question

What does quantum exchangeability for an infinite sequence $x_1, x_2, \ldots$ imply?

Definition

Define the **tail algebra** of the sequence:

$$\mathcal{A}_{\text{tail}} := \bigcap_{i \in \mathbb{N}} \text{vN}(x_k \mid k \geq i),$$

then there exists **conditional expectation** $E : \text{vN}(x_i \mid i \in \mathbb{N}) \to \mathcal{A}_{\text{tail}}$.

Question

Does quantum exchangeability imply an independence like property for this E?

Non-commutative de Finetti Theorem

Theorem (Köstler, Speicher 2009)

The following are equivalent for an infinite sequence of non-commutative random variables:

- *the sequence is quantum exchangeable*
- *the sequence is free and identically distributed with respect to the conditional expectation E onto the tail-algebra of the sequence*

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Proof

Consider first non-crossing moments like $E[x_9x_7x_2x_7x_9] =???$
Because of exchangeability we have

$$??? = \frac{E[x_9x_7x_{10}x_7x_9] + E[x_9x_7x_{11}x_7x_9] + \cdots + E[x_9x_7x_{9+N}x_7x_9]}{N}$$

$$= E[x_9x_7 \cdot \frac{1}{N}\sum_{i=1}^{N} x_{9+i} \cdot x_7x_9]$$

However, by the mean ergodic theorem,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_{9+i} = E[x_9] = E[x_2]$$

Thus

$$E\big[x_9x_7x_2x_7x_9\big] = E\big[x_9x_7E[x_2]x_7x_9\big].$$

Proof

$$E\left[x_9x_7x_2x_7x_9\right] = E\left[x_9x_7E[x_2]x_7x_9\right].$$

Do now the same for $x_7E[x_2]x_7$.

$$E[x_9x_7E[x_2]x_7x_9] = \lim_{N\to\infty} E[x_9\Big(\frac{1}{N}\sum_{i=1}^{N} x_{13+i}E[x_2]x_{13+i}\Big)x_9]$$

$$= E[x_9E[x_7E[x_2]x_7]x_9]$$

So we get

$$\mathbf{E}[\mathbf{x_9x_7x_2x_7x_9}] = \mathbf{E}[\mathbf{x_9E}[\mathbf{x_7E}[\mathbf{x_2}]\mathbf{x_7}]\mathbf{x_9}]$$

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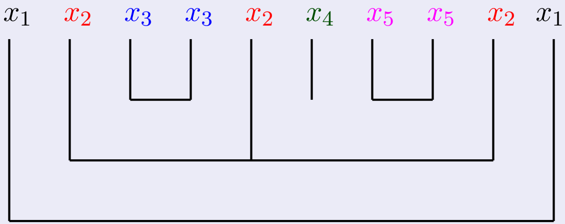
Proof

In the same way one gets factorizations for all non-crossing moments in an iterative way (always work on interval blocks)

$$\pi = \{\{1,10\},\{2,5,9\},\{3,4\},\{6\},\{7,8\}\} \in NC(10),$$



$$\mathbf{E}[\mathbf{x_1 x_2 x_3 x_3 x_2 x_4 x_5 x_5 x_2 x_1}]$$

$$= \mathbf{E}\Big[\mathbf{x_1} \cdot \mathbf{E}\big[\mathbf{x_2} \cdot \mathbf{E}[\mathbf{x_3 x_3}] \cdot \mathbf{x_2} \cdot \mathbf{E}[\mathbf{x_4}] \cdot \mathbf{E}[\mathbf{x_5 x_5}] \cdot \mathbf{x_2}\big] \cdot \mathbf{x_1}\Big]$$

Proof

- Thus exchangeability implies factorizations for all non-crossing terms (Köstler 2008).
- For commuting variables this determines everything.
- However, for non-commuting variables there are many more expressions which cannot be treated like this.

Problem

Basic example: $\mathbf{E}[\mathbf{x_1 x_2 x_1 x_2}] = ???$

To determine those we need quantum exchangeability!

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Proof: determine $E[x_1x_2x_1x_2]$

Assume, for convenience, that $E[x_1] = E[x_2] = 0$.
By quantum exchangeability we have

$$E[x_1x_2x_1x_2] = \sum_{j(1),\dots,j(4)=1}^{k} u_{1j(1)}u_{2j(2)}u_{1j(3)}u_{2j(4)}E[x_{j(1)}x_{j(2)}x_{j(3)}x_{j(4)}]$$

$$= \sum_{j(1)\neq j(2)\neq j(3)\neq j(4)} \cdots$$

$$= \underbrace{\sum_{j(1)=j(3)\neq j(2)=j(4)} u_{1j(1)}u_{2j(2)}u_{1j(3)}u_{2j(4)}}_{\neq 1 \text{ for general } (u_{ij})} E[x_1x_2x_1x_2]$$

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Proof: determine $E[x_1x_2x_1x_2]$

Thus we have: if $E[x_1] = 0 = E[x_2]$, then $E[x_1x_2x_1x_2] = 0$
This implies in general:

$$E[x_1x_2x_1x_2] = E\big[x_1E[x_2]x_1\big] \cdot E[x_2] + E[x_1] \cdot E\big[x_2E[x_1]x_2\big]$$

$$- E[x_1]E[x_2]E[x_1]E[x_2]$$

.... which is the formula for variables which are free with respect to E.

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Proof

In general, one shows in the same way that

$$E\left[p_1(x_{i(1)})p_2(x_{i(2)})\cdots p_n(x_{i(n)})\right] = 0$$

whenever

- $n \in \mathbb{N}$ and $p_1, \dots, p_n \in \mathcal{A}_{\text{tail}}\langle X \rangle$ are polynomials in one variable
- $i(1) \neq i(2) \neq i(3) \neq \cdots \neq i(n)$
- $E[p_j(x_{i(j)})] = 0$ for all $j = 1, \dots, n$

Thus, the x_i are free w.r.t E in the sense of Voiculescu's free probability theory.

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Non-commutative de Finetti Theorem

Theorem (Köstler, Speicher 2009)

The following are equivalent for an infinite sequence of non-commutative random variables:

- *the sequence is quantum exchangeable*
- *the sequence is free and identically distributed with respect to the conditional expectation E onto the tail-algebra of the sequence*

Thus, freeness arises very naturally from symmetry requirements, if one takes the quantum permutation group as the right analogue of the permutation group in the non-commutative world.

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Section 7

More Quantum Symmetries in Free/Non-Commutative Probability

What are Quantum Groups?

- are generalizations of groups G (actually, of $C(G)$)
- are supposed to describe non-classical symmetries
- are Hopf algebras, with some additional structure ...

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What are Quantum Groups?

Deformation of Classical Symmetries: $\mathbf{G} \rightsquigarrow \mathbf{G_q}$

- quantum groups are often deformations G_q of classical groups, depending on some parameter q, such that for $q \to 1$, they go to the classical group $G = G_1$
- G_q and G_1 are incomparable, none is stronger than the other
  - G_1 is supposed to act on commuting variables
  - G_q is the right replacement to act on q-commuting variables

Strengthening of Classical Symmetries: $\mathbf{G} \rightsquigarrow \mathbf{G^+}$

- there are situations where a classical group G has a genuine non-commutative analogue G^+ (no interpolations)
- G^+ is "stronger" than G: $\quad G \subset G^+$
  - G acts on commuting variables
  - G^+ is the right replacement for acting on maximally non-commuting variables

Orthogonal Hopf Algebras

We are interested in quantum versions of **real compact matrix groups**. Think of

- orthogonal matrices or permutation matrices

Such quantum versions are captured by the notion of **orthogonal Hopf algebra**.

Definition

An **orthogonal Hopf algebra** is a C^*-algebra A, given with a system of n^2 self-adjoint generators $u_{ij} \in A$ $(i, j = 1, \dots, n)$, subject to the following conditions:

- The inverse of $u = (u_{ij})$ is the transpose matrix $u^t = (u_{ji})$.
- $\Delta(u_{ij}) = \Sigma_k\, u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.
- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.
- $S(u_{ij}) = u_{ji}$ defines a morphism $S : A \to A^{op}$.

Orthogonal Hopf Algebras

Definition

An **orthogonal Hopf algebra** is a C^*-algebra A, given with a system of n^2 self-adjoint generators $u_{ij} \in A$ $(i,j = 1,\ldots,n)$, subject to the following conditions:

- The inverse of $u = (u_{ij})$ is the transpose matrix $u^t = (u_{ji})$.
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- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.
- $S(u_{ij}) = u_{ji}$ defines a morphism $S : A \to A^{op}$.

- These are compact quantum groups in the sense of Woronowicz.
- In the spirit of non-commutative geometry, we are thinking of $\mathbf{A} = \mathbf{C}(\mathbf{G}^+)$ as the continuous functions, generated by the coordinate functions u_{ij}, on some (non-existing) quantum group G^+, replacing a classical group G.

Definition (Quantum Orthogonal Group O_n^+ (Wang 1995))

The quantum orthogonal group $A_o(n) = C(O_n^+)$ is the universal unital C^*-algebra generated by selfadjoint u_{ij} $(i, j = 1, \dots, n)$ subject to the relation: $u = (u_{ij})_{i,j=1}^n$ is an orthogonal matrix; i.e., for all i, j we have

$$\sum_{k=1}^n u_{ik} u_{jk} = \delta_{ij} \qquad \text{and} \qquad \sum_{k=1}^n u_{ki} u_{kj} = \delta_{ij}$$

Definition (Quantum Permutation Group S_n^+ (Wang 1998))

The quantum permutation group $A_s(n) = C(S_n^+)$ is the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \dots, n)$ subject to the relations

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i, j = 1, \dots, n$
- each row and column of $u = (u_{ij})_{i,j=1}^n$ is a partition of unity:

$$\sum_{j=1}^n u_{ij} = 1 \qquad \text{and} \qquad \sum_{i=1}^n u_{ij} = 1$$

Are there more of those?

$$\begin{array}{ccc} S_n^+ & \subset & O_n^+ \\ \cup & & \cup \\ S_n & \subset & O_n \end{array}$$

Questions

Are there more of those?

$$\begin{array}{ccccc} S_n^+ & \subset & G_n^+ & \subset & O_n^+ \\ \cup & & \cup & & \cup \\ S_n & \subset & G_n & \subset & O_n \end{array}$$

Questions

- Are there more non-commutative versions G_n^+ of classical groups G_n?

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Are there more of those?

$$
\begin{array}{ccc}
S_n^+ & \subset & O_n^+ \\
 & & \diagup \\
\cup & G_n^* & \cup \\
 & \diagup & \\
S_n & \subset & O_n
\end{array}
$$

Questions

- Are there more non-commutative versions G_n^+ of classical groups G_n?
- Actually, are there more nice non-commutative quantum groups G_n^*?

How can we describe and understand intermediate quantum groups

Questions:

- Are there more non-commutative versions G_n^+ of classical groups G_n?
- Actually, are there more nice non-commutative quantum groups G_n^*, stronger than S_n?

$$S_n \subset \mathbf{G_n^*} \subset O_n^+$$

$$C(S_n) \leftarrow \mathbf{C(G_n^*)} \leftarrow C(O_n^+)$$

Deal with quantum groups by looking on their representations!!!

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Section 8

Describing Quantum Groups via Intertwiner Spaces

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Spaces of Intertwiners

Definition

Associated to an orthogonal Hopf algebra $(A = C(G_n^*), (u_{ij})_{i,j=1}^n)$ are the spaces of intertwiners:

$$\mathbf{I}_{G_n^*}(k,l) = \{T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l} \mid Tu^{\otimes k} = u^{\otimes l}T\}$$

where $u^{\otimes k}$ is the $n^k \times n^k$ matrix $(u_{i_1 j_1} \dots u_{i_k j_k})_{i_1\dots i_k, j_1 \dots j_k}$.

$$u \in M_n(A) \qquad u : \mathbb{C}^n \to \mathbb{C}^n \otimes A$$

$$u^{\otimes k} : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k} \otimes A$$

Note: if $T \in \mathbf{I}_{G_n^*}(0,l)$, then $\xi := T1 \in (\mathbb{C}^n)^{\otimes l}$ is a fixed vector unter $u^{\otimes l}$:

$$Tu^{\otimes 0} = u^{\otimes l}T \qquad \Longrightarrow \qquad \xi = T1 = u^{\otimes l}T1 = u^{\otimes l}\xi$$

$\mathbf{I}_{G_n^*}$ is Tensor Category with Duals

Proposition

Collection of vector spaces $\mathbf{I}_{G_n^*}(k,l)$ has the following properties:

- $T, T' \in \mathbf{I}_{G_n^*}$ implies $T \otimes T' \in \mathbf{I}_{G_n^*}$.
- If $T, T' \in \mathbf{I}_{G_n^*}$ are composable, then $TT' \in \mathbf{I}_{G_n^*}$.
- $T \in \mathbf{I}_{G_n^*}$ implies $T^* \in \mathbf{I}_{G_n^*}$.
- $id(x) = x$ is in $\mathbf{I}_{G_n^*}(1,1)$.
- $\xi = \sum e_i \otimes e_i$ is in $\mathbf{I}_{G_n^*}(0,2)$.

Let us check that

$$\xi = \sum e_i \otimes e_i \in \mathbf{I}_{G_n^*}(0,2)$$

Proof: Why is $\xi = \sum_i e_i \otimes e_i \in I_{G_n^*}(0,2)$

We have to see $$\left(u^{\otimes 2}\xi\right)_{i_1,i_2} = \xi_{i_1,i_2}$$

$$\left(u^{\otimes 2}\sum_i e_i \otimes e_i\right)_{i_1,i_2} = \sum_i \sum_{j_1,j_2} u_{i_1 j_1} u_{i_2 j_2} (e_i \otimes e_i)_{j_1,j_2}$$

$$= \sum_i \sum_{j_1,j_2} u_{i_1 j_1} u_{i_2 j_2} \delta_{i j_1} \delta_{i j_2}$$

$$= \sum_i u_{i_1 i} u_{i_2 i} = \delta_{i_1 i_2} = \left(\sum_i e_i \otimes e_i\right)_{i_1,i_2}$$

Tannaka-Krein for compact quantum groups

Theorem (Woronowicz 1988)

The compact quantum group G_n^* can actually be rediscovered from its space of intertwiners.
There is a one-to-one correspondence between:

- *orthogonal Hopf algebras* $C(O_n^+) \to \mathbf{C(G_n^*)} \to C(S_n)$
- *tensor categories with duals* $\mathbf{I}_{O_n^+} \subset \mathbf{I}_{\mathbf{G_n^*}} \subset \mathbf{I}_{S_n}$.

How to Get Intertwiners

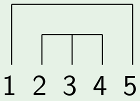
Definition

We denote by $P(k,l)$ the set of partitions of the set with repetitions $\{1,\dots,k,1,\dots,l\}$. Such a partition will be pictured as

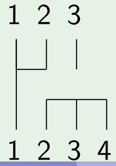
$$p = \begin{array}{c} 1 \dots k \\ \mathcal{P} \\ 1 \dots l \end{array}$$

where $\mathcal{P}$ is a diagram joining the elements in the same block of the partition.

Example



$\in P(0,5)$



$\in P(3,4)$

How to Get Intertwiners

Definition

Associated to any partition $p \in P(k,l)$ is the linear map

$$T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$$

given by

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_p(i,j)\, e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$, and where

$$\delta_p(i,j) = \begin{cases} 1, & \text{if all indices which are connected by } p \text{ are the same} \\ 0, & \text{otherwise} \end{cases}$$

How to Get Intertwiners

Definition

$$T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$$

given by

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_p(i,j)\, e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Example

$$T_{\{|\;|\}}(e_a \otimes e_b) = e_a \otimes e_b$$

$$T_{\{|\!-\!|\}}(e_a \otimes e_b) = \delta_{ab}\, e_a \otimes e_a$$

$$T_{\left\{\substack{\sqcup \\ |\;|}\right\}}(e_a \otimes e_b) = \delta_{ab} \sum_{cd} e_c \otimes e_d$$

Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

Question: What are the intertwiners?

$$\begin{array}{ccccccc} S_n^+ & \subset & O_n^+ & \qquad & \mathbf{I}_{S_n^+} & \supset & \mathbf{I}_{O_n^+} \\ \cup & & \cup & & \cap & & \cap \\ S_n & \subset & O_n & & \mathbf{I}_{S_n} & \supset & \mathbf{I}_{O_n} \end{array}$$

First answer: Intertwiners of S_n

$$\operatorname{span}(T_p | p \in P(k,l)) = \mathbf{I}_{S_n}(k,l)$$

Proof: Why is T_p in $\mathbf{I}_{S_n}$ for all $p \in P$?

Take $u \hat{=} \pi$ permutation matrix, i.e., $ue_i = e_{\pi^{-1}(i)}$. Then

$$\begin{aligned} T_p u^{\otimes k} e_{i_1} \otimes \cdots \otimes e_{i_k} &= T_p e_{\pi^{-1}(i_1)} \otimes \cdots \otimes e_{\pi^{-1}(i_k)} \\ &= \sum_j \delta_p(\pi^{-1}(i_1), \dots, \pi^{-1}(i_k), j_1, \dots, j_l) e_{j_1} \otimes \cdots \otimes e_{j_l} \end{aligned}$$

and

$$\begin{aligned} u^{\otimes l} T_p e_{i_1} \otimes \cdots \otimes e_{i_k} &= u^{\otimes l} \sum_r \delta_p(i_1, \dots, i_k, r_1, \dots, r_l) e_{r_1} \otimes \cdots \otimes e_{r_l} \\ &= \sum_r \delta_p(i_1, \dots, i_k, r_1, \dots, r_l) e_{\pi^{-1}(r_1)} \otimes \cdots \otimes e_{\pi^{-1}(r_l)} \\ &= \sum_j \delta_p(i_1, \dots, i_k, \pi(j_1), \dots, \pi(j_l)) e_{j_1} \otimes \cdots \otimes e_{j_l} \end{aligned}$$

But $\quad \delta_p(\pi^{-1}(i_1), \dots, \pi^{-1}(i_k), j_1, \dots, j_l) = \delta_p(i_1, \dots, i_k, \pi(j_1), \dots, \pi(j_l))$

Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

Let $NC(k,l) \subset P(k,l)$ be the subset of noncrossing partitions.

$$
\begin{array}{ccc}
\mathrm{span}(T_p|p \in NC(k,l)) = \mathbf{I}_{S_n^+}(k,l) & \supset & \mathbf{I}_{O_n^+}(k,l) = \mathrm{span}(T_p|p \in NC_2(k,l)) \\
\cap & & \cap \\
\mathrm{span}(T_p|p \in P(k,l)) = \mathbf{I}_{S_n}(k,l) & \supset & \mathbf{I}_{O_n}(k,l) = \mathrm{span}(T_p|p \in P_2(k,l))
\end{array}
$$

Easy Quantum Groups

Definition (Banica, Speicher 2009)

A quantum group $S_n \subset G_n^* \subset O_n^+$ is called **easy** when its associated tensor category is of the form

$$\begin{array}{c} \mathbf{I}_{S_n} = \mathrm{span}(T_p \mid p \in P) \\ \cup \\ \mathbf{I}_{\mathbf{G}_\mathbf{n}^*} = \mathbf{span}(\mathbf{T_p} \mid \mathbf{p} \in \mathbf{P}_{\mathbf{G}^*}) \\ \cup \\ \mathbf{I}_{O_n^+} = \mathrm{span}(T_p \mid p \in NC_2) \end{array}$$

for a certain collection of subsets $P_{G^*} \subset P$.

What are we interested in?

- classification of easy (and more general) quantum groups
- understanding of meaning/implications of symmetry under such quantum groups; in particular, under quantum permutations S_n^+, or quantum rotations O_n^+
- treating series of such quantum groups (like S_n^+ or O_n^+) as fundamental examples of non-commuting random matrices

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Section 9

Easy Quantum Groups: Classification

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Classification Results for Easy Quantum Groups

Theorem and Definition

The **category of partitions** $P_{G^*} \subset P$ for an easy quantum group G_n^* must satisfy:

- P_{G^*} is stable by tensor product.
- P_{G^*} is stable by composition.
- P_{G^*} is stable by involution.
- P_{G^*} contains the "unit" partition $|$.
- P_{G^*} contains the "duality" partition $\sqcap$.

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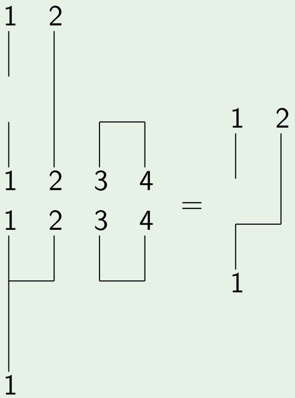
Example of Composition $P(2,4) \times P(4,1) \to P(2,1)$

Example



Are there more of those easy quantum groups?

$$\begin{array}{ccc} S_n^+ & \subset & O_n^+ \\ \cup & & \cup \\ S_n & \subset & O_n \end{array}$$

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Are there more of those easy quantum groups?

$$
\begin{array}{ccccc}
S_n^+ & \subset & G_n^+ & \subset & O_n^+ \\
\cup & & \cup & & \cup \\
S_n & \subset & G_n & \subset & O_n
\end{array}
$$

Questions

- Are there more easy non-commutative versions G_n^+ of easy classical groups G_n?

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Classification Results

Theorem (Banica, Speicher 2009; Weber 2011)

There are

- *7 Categories of Noncrossing Partitions and*

$$
\begin{array}{ccccccc}
\left\{\begin{matrix}\text{singletons}\\ \text{pairings}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons}\\ \text{pairings}\\ (\textit{even part})\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons}\\ \text{pairings}\\ (\textit{resp. parity})\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all}\\ \text{pairings}\end{matrix}\right\} \\
\cap & & \cap & & & & \cap \\
\left\{\begin{matrix}\text{all}\\ \text{partitions}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all partitions}\\ \text{(even part)}\end{matrix}\right\} & \supset & & & \left\{\begin{matrix}\text{blocks of}\\ \text{even size}\end{matrix}\right\}
\end{array}
$$

Classification Results

Theorem (Banica, Speicher 2009; Weber 2011)

There are

- *6 Categories of Partitions containing Basic Crossing*

$$
\begin{array}{ccccccc}
\left\{\begin{matrix}\text{singletons}\\ \text{pairings}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons}\\ \text{pairings}\\ (\textit{even part})\end{matrix}\right\} & \supset & & \supset & \left\{\begin{matrix}\text{all}\\ \text{pairings}\end{matrix}\right\} \\
\cap & & \cap & & & & \cap \\
\left\{\begin{matrix}\text{all}\\ \text{partitions}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all partitions}\\ \text{(even part)}\end{matrix}\right\} & \supset & & & \left\{\begin{matrix}\text{blocks of}\\ \text{even size}\end{matrix}\right\}
\end{array}
$$

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Classification Results

Theorem (Banica, Speicher 2009; Weber 2011)

...and thus there are

- *7 free easy quantum groups* $S_n^+ \subset G_n^+ \subset O_n^+$ *and*

$$\begin{array}{ccccccc}
B_n^+ & \subset & B_n'^+ & \subset & B_n^{\#+} & \subset & O_n^+ \\
\cup & & \cup & & & & \cup \\
S_n^+ & \subset & S_n'^+ & & \subset & & H_n^+
\end{array}$$

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Classification Results

Theorem (Banica, Speicher 2009; Weber 2011)

...and thus there are

- *6 classical easy groups* $S_n \subset G_n \subset O_n$

B_n	$\subset$	B'_n	$\subset$		$\subset$	O_n
$\cup$		$\cup$				$\cup$
S_n	$\subset$	S'_n		$\subset$		H_n

The easy classical groups

The easy classical groups are:

- O_n
- S_n
- $H_n = \mathbb{Z}_2 \wr S_n$: the hyperoctahedral group, consisting of monomial matrices with ± 1 nonzero entries.
- $B_n \simeq O_{n-1}$: the bistochastic group, consisting of orthogonal matrices having sum 1 in each row and each column.
- $S'_n = \mathbb{Z}_2 \times S_n$: permutation matrices multiplied by ± 1.
- $B'_n = \mathbb{Z}_2 \times B_n$: bistochastic matrices multiplied by ± 1.

Are there more of those easy quantum groups?

$$
\begin{array}{ccc}
S_n^+ & \subset & O_n^+ \\
\cup & & \cup \\
S_n & \subset & O_n
\end{array}
$$

Question

Are there more of those easy quantum groups?

$$
\begin{array}{ccccc}
S_n^+ & & \subset & & O_n^+ \\
 & & & / & \\
\cup & & G_n^* & & \cup \\
 & / & & & \\
S_n & & \subset & & O_n
\end{array}
$$

Question

- Are there more easy non-commutative quantum groups G_n^*, stronger than S_n?

Classification of Easy Quantum Groups

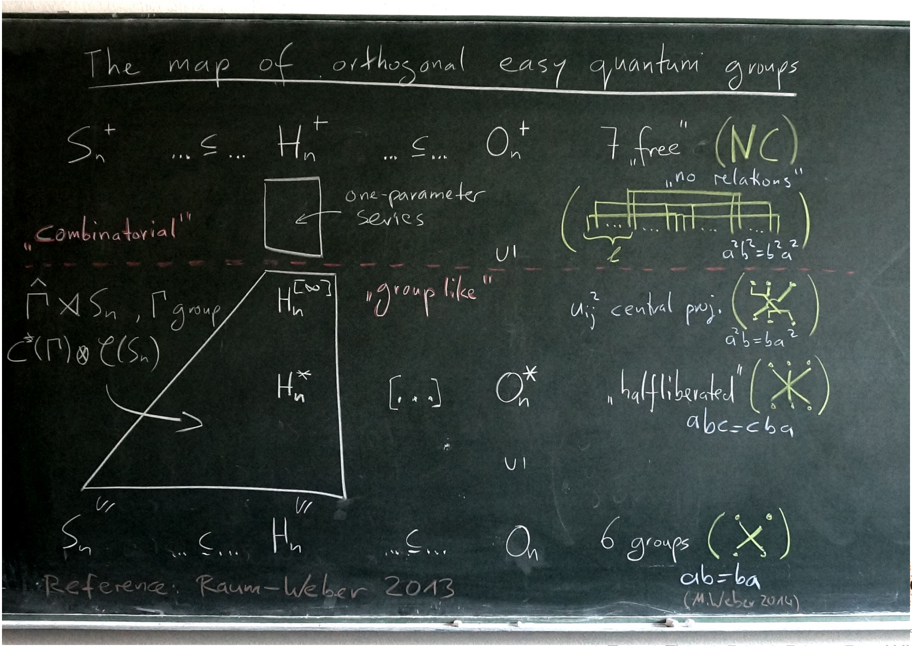
- $\exists!\ 7$ free easy QG's (categories noncrossing)
  [Banica, Speicher 09, Weber 13; (Banica, Bichon, Collins 07)]

- $\exists!\ 6$ easy groups (categ. containing $\times \in P(2,2)$, $u_{ij}u_{kl} = u_{kl}u_{ij}$)
  [Banica, Speicher 09]

- $\exists!\ 3$ half-liberated easy QG's & one infinite series
  (categories containing $* \in P(3,3)$, $u_{ij}u_{kl}u_{st} = u_{st}u_{kl}u_{ij}$)
  [Banica, Curran, Speicher 10, Weber 13]

- $\exists!\ 13$ *non-hyperoctahedral* easy QG's
  ($\sim$ categories containing singletons as blocks)
  [Banica, Curran, Speicher 10, Weber 13]

- *hyperoctahedral* case: [Raum, Weber 12 & 13]


The map of orthogonal easy quantum groups
S_n^+ ... ⊆ ... H_n^+ ... ⊆ ... O_n^+
7 „free" (NC)
„no relations"
„Combinatorial"
one-parameter series
$a^2b^2=b^2a^2$
$\hat{\Gamma} \rtimes S_n$, Γ group
$C^*(\Gamma) \otimes \mathcal{C}(S_n)$
$H_n^{[\infty]}$
„group like"
u_{ij}^2 central proj.
$a^2b=ba^2$
H_n^*
[...]
O_n^*
„halfliberated"
abc=cba
S_n ... ⊆ ... H_n ... ⊆ ... O_n
6 groups
ab=ba
(M. Weber 2014)
Reference: Raum-Weber 2013

de Finetti Theorems

Theorem (de Finetti 1931, Hewitt, Savage 1955)

The following are equivalent for an infinite sequence of classical, commuting random variables:

- *the sequence is exchangeable (i.e., invariant under all S_n)*
- *the sequence is independent and identically distributed with respect to the conditional expectation E onto the tail σ-algebra of the sequence*

Theorem (Köstler, Speicher 2008)

The following are equivalent for an infinite sequence of non-commutative random variables:

- *the sequence is quantum exchangeable (i.e., invariant under all S_n^+)*
- *the sequence is free and identically distributed with respect to the conditional expectation E onto the tail-algebra of the sequence*

Section 10

Haar State and Non-Commutative Random Matrices

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Non-Commutative Random Matrices

- there exists, as for any compact quantum group, a unique Haar state on the easy quantum groups, thus one can integrate/average over the quantum groups
- actually: for the easy quantum groups, there exist nice and "concrete" formula for the calculation of this state:

$$\int_{G_n^*} u_{i_1 j_1} \cdots u_{i_k j_k} du = \sum_{\substack{p,q \in P_{G^*}(k) \\ p \leq \ker i \\ q \leq \ker j}} W_n(p,q),$$

where W_n is inverse of

$$G_n(p,q) = n^{|p \vee q|}.$$

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Non-Commutative Random Matrices

- this allows the calculation of distributions of functions of our non-commutative random matrices G_n^*, in the limit $n \to \infty$
- in particular, in analogy to Diaconis&Shashahani, we have results about the asymptotic distribution of $\text{Tr}(u^k)$
- note: in the classical case, knowledge about traces of powers of the matrices is the same as knowledge about the eigenvalues of the matrices

Weingarten Formula for Easy Quantum Groups

Denote by $D = (D(k))_{k\in\mathbb{N}}$ the category of partitions for the easy quantum group G_n^*; where $D(k) := D(0,k)$. Then

$$\int_{G_n^*} u_{i_1 j_1} \cdots u_{i_k j_k} du = \sum_{\substack{p,q\in D(k)\\ p\leq \ker i\\ q\leq \ker j}} W_n(p,q),$$

where $W_{k,n} = (W_n(p,q))_{p,q\in D(k)} = G_{k,n}^{-1}$ is the inverse of the Gram matrix

$$G_{k,n} = (G_n(p,q))_{p,q\in D(k)} \qquad \text{where} \qquad G_n(p,q) = n^{|p\vee q|}.$$

Note: $p \vee q$ is always the supremum in the lattice of all partitions; i.e., $p \vee q$ is not necessarily in D

Weingarten Formula for Easy Quantum Groups

Example: Integrate $u_{21}u_{23}$. Then $i = (2,2)$, $j = (1,3)$, hence

$$\ker i = \sqcup\ , \qquad \ker j = |\ |$$

and thus

$$\int_{G_n} u_{21}u_{23}du = W(\sqcup, |\ |) + W(|\ |, |\ |)$$

Similarly,

$$\int_{G_n} u_{23}u_{23}du = W(\sqcup, \sqcup) + W(\sqcup, |\ |) + W(|\ |, \sqcup) + W(|\ |, |\ |)$$

Asymptotics of the Weingarten Formula

$$\int_{G_n^*} u_{i_1 j_1} \cdots u_{i_k j_k} du = \sum_{\substack{p,q \in D(k) \\ p \leq \ker i \\ q \leq \ker j}} W_n(p,q),$$

where $W_{k,n} = (W_n(p,q))_{p,q \in D(k)} = G_{k,n}^{-1}$ is the inverse of the Gram matrix

$$G_{k,n} = (G_n(p,q))_{p,q \in D(k)} \qquad \text{where} \qquad G_n(p,q) = n^{|p \vee q|}.$$

We have the asymptotics

$$W_n(p,q) = O(n^{|p \vee q| - |p| - |q|})$$

Distribution of Traces of Powers

Let G be an easy quantum group. Consider $s \in \mathbb{N}$, $k_1, \dots, k_s \in \mathbb{N}$, $k := \sum_{i=1}^{s} k_i$, and denote

$$\gamma := (1, 2, \dots, k_1)(k_1 + 1, k_1 + 2, \dots, k_1 + k_2) \cdots (\cdots, k) \in S_k$$

Then we have, for any n such that G_{kn} is invertible:

$$\int_{G_n} \mathrm{Tr}(u^{k_1}) \dots \mathrm{Tr}(u^{k_s})\, du = \#\{p \in D(k) \mid p = \gamma(p)\} + O(1/n).$$

If G is a classical easy group, then this formula is exact, without any lower order corrections in n.

Proof

$$
\begin{aligned}
I &:= \int_G \mathrm{Tr}(u^{k_1}) \dots \mathrm{Tr}(u^{k_s})\, du \\
&= \sum_{i_1 \dots i_k} \int_G (u_{i_1 i_2} \dots u_{i_{k_1} i_1}) \dots (u_{i_{k-k_s+1} i_{k-k_s+2}} \dots u_{i_k i_{k-k_s+1}}) \\
&= \sum_{i_1 \dots i_k} \int_G u_{i_1 i_{\gamma(1)}} \dots u_{i_k i_{\gamma(k)}} \\
&= \sum_{i_1 \dots i_k = 1}^{n} \sum_{\substack{p,q \in D_k \\ p \le \ker \mathbf{i},\, q \le \ker \mathbf{i} \circ \gamma}} W_{kn}(p,q) \\
&= \sum_{i_1 \dots i_k = 1}^{n} \sum_{\substack{p,q \in D_k \\ p \le \ker \mathbf{i},\, \gamma(q) \le \ker \mathbf{i}}} W_{kn}(p,q)
\end{aligned}
$$

Proof

$$\begin{aligned} I &= \sum_{i_1\dots i_k=1}^{n} \sum_{\substack{p,q\in D_k \\ p\leq \ker \mathbf{i}, \gamma(q)\leq \ker \mathbf{i}}} W_{kn}(p,q) \\ &= \sum_{p,q\in D_k} \sum_{\substack{i_1\dots i_k=1 \\ p\leq \ker \mathbf{i}, \gamma(q)\leq \ker \mathbf{i}}}^{n} W_{kn}(p,q) \\ &= \sum_{p,q\in D_k} n^{|p\vee\gamma(q)|} W_{kn}(p,q) \\ &= \sum_{p,q\in D_k} n^{|p\vee\gamma(q)|} n^{|p\vee q|-|p|-|q|}(1+O(1/n)). \end{aligned}$$

The leading order of $n^{|p\vee\gamma(q)|+|p\vee q|-|p|-|q|}$ is n^0, which is achieved if and only equivalently $p=q=\gamma(q)$.

Proof

In the classical case, instead of using the approximation for $W_{nk}(p,q)$, we can write $n^{|p\vee\gamma(q)|}$ as $G_{nk}(\gamma(q),p)$.
(Note that this only makes sense if we know that $\gamma(q)$ is also an element in D_k; and this is only the case for the classical partition lattices.)
Then one can continue as follows:

$$I=\sum_{p,q\in D_k} G_{nk}(\gamma(q),p)W_{kn}(p,q)=\sum_{q\in D_k}\delta(\gamma(q),q)=\#\{q\in D_k|q=\gamma(p)\}.$$

□

The Distribution of $u_r := \lim_{n\to\infty} \mathrm{Tr}(u^r)$

Variable	O_n	O_n^+
u_1	real Gaussian	semicircular
u_2	real Gaussian	semicircular
u_r $(r \geq 3)$	real Gaussian	circular

Variable	S_n	S_n^+
u_1	Poisson	free Poisson
$u_2 - u_1$	Poisson	semicircular
$u_r - u_1$ $(r \geq 3)$	sum of Poissons	circular

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Something to Remember

Whereas $\mathrm{Tr}(u)$ and $\mathrm{Tr}(u^2)$ are selfadjoint, this is not true for $\mathrm{Tr}(u^3)$ in the general non-commutative situation!

$$u_1 = \sum u_{ii} = u_1^*$$

$$u_2 = \sum u_{ij} u_{ji} = \sum u_{ji} u_{ij} = u_2^*$$

$$u_3 = \sum u_{ij} u_{jl} u_{li} \neq \sum u_{li} u_{jl} u_{ij} = u_3^*$$

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Non-Commutative Random Matrices

- this allows the calculation of distributions of functions of our non-commutative random matrices G_n^*, in the limit $n \to \infty$
- in particular, in analogy to Diaconis&Shashahani, we have results about the asymptotic distribution of $\text{Tr}(u^k)$
- note: in the classical case, knowledge about traces of powers of the matrices is the same as knowledge about the eigenvalues of the matrices

The Final Question

What actually are eigenvalues of a non-commutative matrix?

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The Final Question

What actually are eigenvalues of a non-commutative matrix?

"Whereof one cannot speak, thereof one must be silent"

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