On Atoms and Zero Divisors (Polynomial and Rational)

Roland Speicher
(joint work with Tobias Mai and Moritz Weber)

Saarland University
Saarbrücken, Germany

supported by ERC Advanced Grant
"Non-Commutative Distributions in Free Probability"

Section 1

Motivation

Motivation

Smooth functions of smooth operators

We want to understand (distributions of) functions of non-commuting operators.

Idea: If we consider

- "smooth" functions f
- "generic" operators $x_1, \ldots, x_n$,

we expect "regular" behaviour for $f(x_1, \ldots, x_n)$.

Classes of "smooth" functions

polynomials $\subset$ rational functions $\subset$ analytic functions

European Research Council
Established by the European Commission

Motivation

Notations

Let (M, τ) be a W^*-probability space with a faithful normal tracial state τ. For selfadjoint $x_1, \dots, x_n \in M$, Voiculescu defined various numerical quantities:

- free entropy $\chi(x_1, \dots, x_n)$, $\chi^*(x_1, \dots, x_n)$
- free Fisher information $\Phi^*(x_1, \dots, x_n)$
- more refined free dimension versions $\delta(x_1, \dots, x_n)$, $\delta^*(x_1, \dots, x_n)$

What are "generic" operators?

A promising condition seems to be (in a tracial setting)

$$\delta^*(x_1, \dots, x_n) = n \qquad \text{maximal possible value.}$$

This should also correspond to the fact that there are many possibilities of approximating $(x_1, \dots, x_n)$ in distribution by matrices.

Polynomials

So for such $x_1, \dots, x_n$ with $\delta^*(x_1, \dots, x_n) = n$ we expect that

- $\mathrm{vN}(x_1, \dots, x_n)$ is like a free group factor $L(\mathbb{F}_n)$
- $\mathbb{C}\langle x_1, \dots, x_n \rangle$ is like the group algebra $\mathbb{C}\mathbb{F}_n$

and, in particular, we expect

- there are no non-trivial polynomial relations between the $x_1, \dots, x_n$, i.e., for any $0 \neq P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$:

$$P(x_1, \dots, x_n) \neq 0$$

- there are no non-trivial zero-divisors, i.e., for any $0 \neq p = P(x_1, \dots, x_n) \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ and any $w \in \mathrm{vN}(x_1, \dots, x_n)$ we have:

$$pw = 0 \qquad \Longrightarrow \qquad w = 0$$

Theorem (Mai, Speicher, Weber 2014/15; Shlyakhtenko 2014)

Assume $\delta^*(x_1, \dots, x_n) = n$ *and* $\mathbb{C}\langle X_1, \dots, X_n\rangle \ni P \neq 0$. *Then*

- *there are no algebraic relations,* $P(x_1, \dots, x_n) \neq 0$
- *there are no zero-divisors: for any* $u \in \mathrm{vN}(x_1, \dots, x_n)$ *we have*

$$P(x_1, \dots, x_n)u = 0 \quad \Rightarrow \quad u = 0$$

Remark

- Note that $\delta^*(x_1, \dots, x_n) = n$ is satisfied, when the variables are free and each variable is "smooth" (e.g., semicircular). In this case, the theorem was proved before by Shlyakhtenko and Skoufranis in 2013.
- The crucial point here is that the relevant feature is not freeness, but maximal entropy dimension.
- There are now more results in this direction by Charlesworth and Shlyakhtenko . . . see talk of Dima

Idea of Proof

- $x_1, \ldots, x_n$ "generic" implies that we have a good theory of non-commutative derivatives ∂_i on our operators;
- thus we can take derivatives of relations on them.
- So we can argue:

$$\begin{aligned} &\text{If a property holds for polynomial } P(x_1, \ldots, x_n) \\ &\implies \text{property holds also for } \partial_i P(x_1, \ldots, x_n) \\ &\quad \vdots \\ &\implies \text{property holds also for (non-trivial) constant polynomial} \end{aligned}$$

which is obviously not true in our cases. □

European Research Council
Established by the European Commission

A random matrix perspective on zero-divisors

Zero-divisors of p = atoms of distributions μ_p

The distribution of a selfadjoint operator $p = p^*$ in (M, τ) corresponds to a probability measure μ_p on $\mathbb{R}$, determined by

$$\tau(p^k) = \int t^k d\mu_p(t).$$

Zero-divisors are given by projections onto the eigenspaces of p; hence the question about zero-divsisor is the same as asking whether μ_p has atoms.

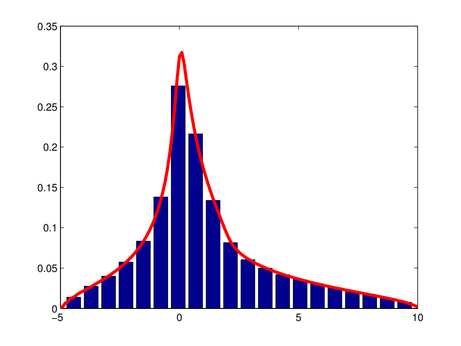
Absence of atoms for asymptotic eigenvalue distribution

$P(x_1, \ldots, x_n)$ arises quite canonically, in distribution, as the limit of polynomials in n random matrices; the absence of zero-divisors means in particular, that this asymptotic eigenvalue distribution has no atoms

Example

$P(X, Y) = XY + YX + X^2$

for independent X, Y; X is Gaussian and Y is Wishart



$p(x, y) = xy + yx + x^2$

for free x, y; x is semicircular and y is Marchenko-Pastur

Section 2

Staying rational in the non-commutative world

European Research Council
Established by the European Commission

More general functions?

Question

Is this absence of relations and zero-divisors also true for more general functions?

- Dabrowski: no analytic relations for power series which converge on polydisc
- Mai: no zero-divisors for limits of "homogeneous polynomials" in infinitely many free semicirculars (i.e., for Wigner integrals in finite chaos)
- how about general non-commutative analytic functions???
- an interesting class of intermediate functions is given by

  non-commutative rational functions

Non-commutative rational functions (starting in the 1960's)

Schützenberger, Amitsur, Cohn, Bergman, Malcomson, Reutenauer, Vinnikov, Kaliuzhnyi-Verbovetskyi, etc ...

What is a non-commutative rational function?

A non-commutative rational function is

- an equivalence class of non-commutative rational expressions
- which is the same as a linearization/realization in terms of linear matrices

Example

Consider

- $r_1(X_1, X_2, X_3, X_4) := X_1^{-1} + X_2^{-1} X_2 (X_4 - X_3 X_1^{-1} X_2)^{-1} X_3 X_1^{-1}$
- $r_2(X_1, X_2, X_3, X_4) := (X_1 - X_2 X_4^{-1} X_3)^{-1}$
- $r(X_1, X_2, X_3, X_4) := \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

Then

- r_1, r_2, and r represent the same rational function
- each have different domains, r has the maximal domain

What is a non-commutative rational function?

A non-commutative rational function is

- an equivalence class of non-commutative rational expressions
- which is the same as a linearization/realization in terms of linear matrices

Example

Consider

- $r_1(X_1, X_2, X_3, X_4) := X_1^{-1} + X_2^{-1} X_2 (X_4 - X_3 X_1^{-1} X_2)^{-1} X_3 X_1^{-1}$
- $r_2(X_1, X_2, X_3, X_4) := (X_1 - X_2 X_4^{-1} X_3)^{-1}$
- $r(X_1, X_2, X_3, X_4) := \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

Then

- being rational in the non-commutative world requires nestings
- one cannot do with less

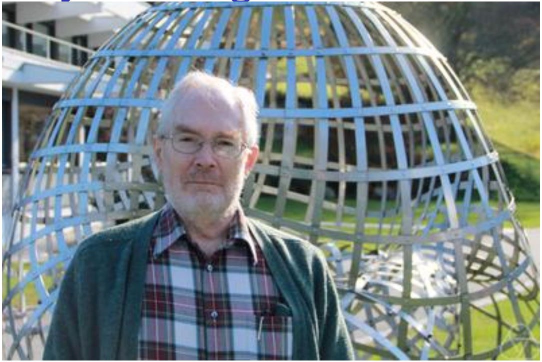
The rationality of George A. Elliott: Being rational in the non-commutative world requires nestings

Towards a theory of classification

George A. Elliott

Perhaps, rather than labels for isomorphism classes, what one really wants, given a category, is a functor—distinguishing isomorphism classes—from this category into some other, simpler, category. (In other words, still labels for objects in the given category, but with isomorphic objects no longer required to have the same label—just isomorphic labels!) (And maps between objects reflected by maps—or at least formal arrows—between labels—but reflected faintly, in the sense that certain maps will coalesce.)

The rationality of George A. Elliott



- statements of George have to be nested (unless trivial)
- there might be different ways of saying the same, but there is a minimal number of nestings which are necessary to convey the idea
- it might help (maybe not for George, but for the rest of us) to have a linearized matrix version of George's ideas

Notation

There exists a universal skew field of fractions of $\mathbb{C}\langle X_1, \ldots, X_n\rangle$, the *universal free field*, denoted by

$$\mathbb{C}\langle\!\langle X_1, \ldots, X_n \rangle\!\rangle$$

Example

If we have a non-commutative rational function $r(X_1, \ldots, X_n)$, then we want to plug in operators $x_1, \ldots, x_n$. But be careful!
In $\mathbb{C}\langle\!\langle X, Y \rangle\!\rangle$ we have

$$X(YX)^{-1}Y = 1.$$

However, if u is a proper isometry, i.e. $u^*u = 1$, but $uu^* \neq 1$, then we have

$$u(u^*u)^{-1}u^* = uu^* \neq 1,$$

thus the above identity from the universal free field is not true for $X = u$ and $Y = u^*$, though all inverses makes sense.

However, this is the only obstruction.

Definition

An algebra $\mathcal{A}$ is *stably finite* (aka *weakly finite*), if we have for all n and all $a, b \in M_n(\mathcal{A})$ that

$$ab = 1 \quad \Rightarrow \quad ba = 1.$$

Theorem (Cohn)

Let $\mathcal{A}$ be a stably finite algebra. Consider two rational expressions r_1, r_2 with

$$r_1(X_1, \dots, X_n) = r_2(X_1, \dots, X_n) \quad \textit{in} \quad \mathbb{C}\langle\!\langle X_1, \dots, X_n \rangle\!\rangle.$$

Then we have

$$r_1(x_1, \dots, x_n) = r_2(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in \mathcal{A}$ for which both sides make sense.

Note that stably finite C^*-algebras and finite von Neumann algebras are stably finite in the above sense.

What is the domain of a rational function?

Represent a rational function $r(X_1, \dots, X_n)$ in matrix form, with (linear) polynomials p_{ij}

$$\begin{pmatrix} a_1 & \dots & a_k \end{pmatrix} \begin{pmatrix} p_{11}(X_1, \dots, X_n) & \dots & p_{1k}(X_1, \dots, X_n) \\ \vdots & \ddots & \vdots \\ p_{k1}(X_1, \dots, X_n) & \dots & p_{kk}(X_1, \dots, X_n) \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}$$

The domain is then given by $x_1, \dots, x_n \in \mathcal{A}$ (for a stably finite algebra $\mathcal{A}$) for which the matrix

$$\begin{pmatrix} p_{11}(x_1, \dots, x_n) & \dots & p_{1k}(x_1, \dots, x_n) \\ \vdots & \ddots & \vdots \\ p_{k1}(x_1, \dots, x_n) & \dots & p_{kk}(x_1, \dots, x_n) \end{pmatrix} \in M_k(\mathcal{A})$$

is invertible.

Zero-Divisors for NC Rational Functions

Conjecture

Consider $x_1, \dots, x_n$ in a tracial W^*-probability space and

$$\mathbb{C}\langle\!\langle X_1, \dots, X_n \rangle\!\rangle \ni r \neq 0.$$

If $\delta^*(x_1, \dots, x_n) = n$ and if $x_1, \dots, x_n$ is in the domain of r then we believe that

- $r(x_1, \dots, x_n) \neq 0$
- $r(x_1, \dots, x_n)$ has no zero-divisors

Note that our proof for polynomials does not work here; as ∂_i does not reduce the order of a rational function in any obvious way.

European Research Council
Established by the European Commission

Some more conjectures/hopes

Note that in the case of polynomials we can write:

$$\delta^*(x_1,\dots,x_n) = n \implies \mathbb{C}\langle x_1,\dots,x_n\rangle \cong \mathbb{C}\langle X_1,\dots,X_n\rangle$$

So it would be nice to have in the same way

$$\delta^*(x_1,\dots,x_n) = n \implies \mathbb{C}(\!\langle x_1,\dots,x_n\rangle\!) \cong \mathbb{C}(\!\langle X_1,\dots,X_n\rangle\!)$$

But: $\mathbb{C}(\!\langle x_1,\dots,x_n\rangle\!)$ does not make sense as a skew field, since not every operator $r(x_1,\dots,x_n) \neq 0$ is invertible.
Actually, it is,

- if we allow unbounded operators
- affiliated to a II_1-von Neumann algebra
- and our operators do not have zero-divisors

Every operator affiliated to a finite von Neumann algebra which has no zero-divisor is invertible as an unbounded operator.

Some more conjectures/hopes

Note that in the case of polynomials we can write:

$$\delta^*(x_1,\dots,x_n)=n \implies \mathbb{C}\langle x_1,\dots,x_n\rangle \cong \mathbb{C}\langle X_1,\dots,X_n\rangle$$

So it would be nice to have in the same way

$$\delta^*(x_1,\dots,x_n)=n \implies \mathbb{C}\Lbag x_1,\dots,x_n\Rbag \cong \mathbb{C}\Lbag X_1,\dots,X_n\Rbag$$

Unbounded hopes

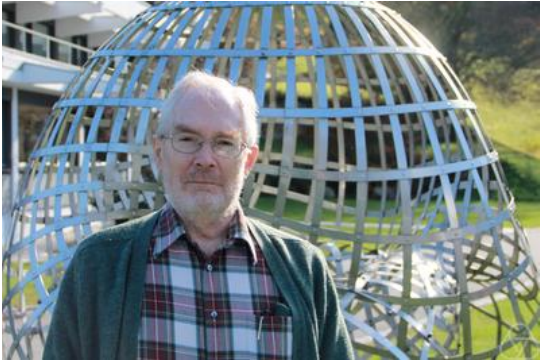
So if we could show the absence of algebraic relations and of zero-divisors for rational functions in the unbounded operators affiliated to vN$(x_1,\dots,x_n)$ then we would get

$$\mathbb{C}\Lbag x_1,\dots,x_n\Rbag \cong \mathbb{C}\Lbag X_1,\dots,X_n\Rbag$$

Happy Birthday, George!

Congratulations

In this spirit I wish George 70 more years of being rational und unbounded in the non-commutative world .... see his talk on Friday, maybe!



European Research Council
Established by the European Commission