Dan-Virgil Voiculescu: visionary operator algebraist and creator of free probability theory

Roland Speicher

Saarland University
Saarbrücken, Germany

June 14, 1949+70

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- since 1987 at UC Berkeley

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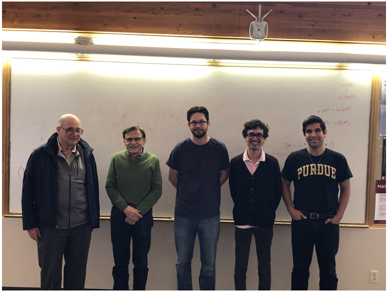

Dan-Virgil Voiculescu, Sorin Popa, Jesse Peterson, Thomas Sinclair, Roy Araiza. IPAM, UCLA, May 2018.

PhD students: Popa, Sorin (1983), Pimsner, Mihai (1984), Timotin, Dan (1986), Bernier, David (1991), Nistor, Victor (1991), Dykema, Kenneth (1993), Germain, Emmanuel (1994), Nica, Alexandru (1994), Shlyakhtenko, Dimitri (1997), Walker, Trent (1997), Hsu, Eric (1998), Anshelevich, Michael (2000), Jung, Kenley (2004), Curran, Stephen (2010), Merberg, Adam (2015), Liu, Weihua (2016)

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von Neumann algebras	free probability theory	random matrices

Operators, C^*-algebras, von Neumann algebras

- single operators: kind of complicated matrices
- operator algebras: families of operators
  - C^*-algebras: large families of operators
  - von Neumann algebras: very large families of operators

Operators, C^*-algebras, von Neumann algebras

- single operators: kind of complicated matrices
  bounded linear operator on Hilbert space
- operator algebras: families of operators
  - C^*-algebras: large families of operators
    $*$-algebras of bounded operators which are closed in the operator norm topology ("uniform convergence")
  - von Neumann algebras: very large families of operators
    $*$-algebras of bounded operators which are closed in weak operator topology ("point-wise convergence")

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We corresponded before we met: he kept solving problems that were too hard for the rest of us (such, for instance, as whether the reducible operators on Hilbert space form a norm dense set). He has left Romania since then, but he is still solving hard problems. And, the alphabet being what it is, this is the last of my pictures from the Vancouver Congress.

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beginnings of free probability:

- "Symmetries of some reduced free product C^*-algebras" (1985)

Timisoara-Herculane, ~ 1980

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History

- 1936-1943: fundamental work by Murray and von Neumann
- 1976: deep results on classification by Alain Connes (Fields Medal 1982)
- around 1982: work of Vaughan Jones on subfactors and connection to knot theory (Fields Medal 1990)

The free group factor(s)

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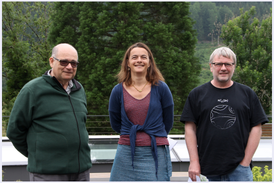

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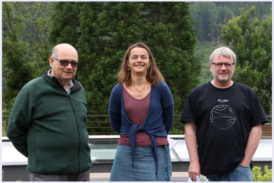

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Which is it?

Why is this so hard

It might not look so hard to see that they are the same, even if they look different on the surface

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- There we have strong tools (K-theory) for distinguishing different C^*-algebras
- Pimsner-Voiculescu (1982): all $C^*_r(F_n)$ are different!

Why is this so hard for von Neumann algebras

It might not look so hard to see that they are the same, even if they look different on the surface . . . but von Neumann algebras go deeper

eigth times: Alec Guinness in *Kind Hearts and Coronets*

Why is this so hard for von Neumann algebras

actually: von Neumann algebras can look **very** different, but still be the same deep inside ... like Dr. Who

Dr. Who, in 13 different incarnations

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- we have only very few tools to distinguish von Neumann algebras
- none of them applies to the free group factors
- Voiculescu took, starting in the 1980's, a quite different approach ("non-commutative probability") to deal with the free group factors

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... no, not really ...

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Usual probability: throwing a coin

heads

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- Free group factors are in a sense also a complicated kind of matrices,
- and we can also calculate for them averages of all possible kind of functions

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Conclusion

- So let us consider our free group factors also as realizations of a kind of non-commutative probability theory ...
- and take some inspiration from classical probability theory to understand them better

Our inspiration from classical probability: It is the distribution that matters!!!

Relevance of distribution

Very different concrete situations have the same distribution. For statistical purposes only the latter is important, actual details of realization are irrelevant.

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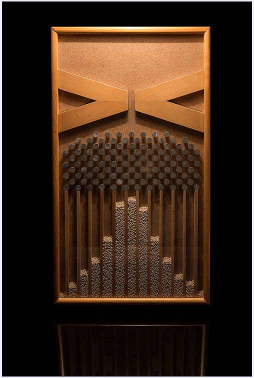
Normal distribution describes approximately:

- distribution of Galton box



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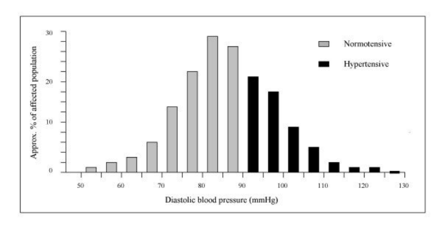
- distribution of blood pressure



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- our free group factors $L(F_n)$ are coming with a faithful trace τ
- all relevant properties of $L(F_n)$ can be recovered from the distribution of a set of generators $e_1, \dots, e_n$ of $L(F_n)$; i.e., from all moments

$$\tau(e_{i(1)} \cdots e_{i(k)}) \qquad \forall k, i(1), \dots, i(k)$$

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- instead of investigating the technical syntax of our sentences formed by the various $L(F_n)$ we try now to understand their semantics

Free group factors speak in different languages . . . so their distributions are hard to understand

- $L(F_2)$: du balsiongie chongbun.
- $L(F_3)$: toH, wej maS jIH.
- . . .

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- … (……)

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Fundamental observation

- many random matrices show for $N \to \infty$ almost surely a *deterministic* behavior;
- thus the randomness helps us to calculate things, but the result does not depend on it

Wigner random matrices (Wigner 1955)

Definition

A **Wigner random matrix** $X_N = \frac{1}{\sqrt{N}}(x_{ij})_{i,j=1}^{N}$

- is symmetric: $X_N^* = X_N$, i.e. $x_{ij} = x_{ji}$ for all i, j
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$$\begin{pmatrix}
1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
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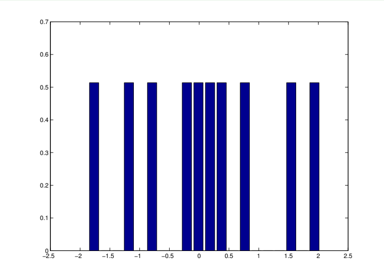
Example: eigenvalue distribution for $N = 10$



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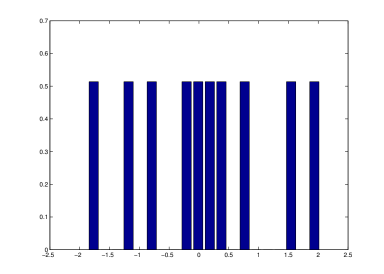
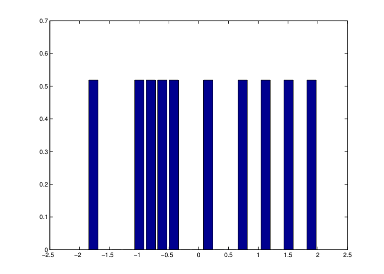
- is symmetric: $X_N^* = X_N$, i.e. $x_{ij} = x_{ji}$ for all i, j
- entries $\{x_{ij} \mid 1 \le i \le j \le N\}$ are chosen according to independent coin tosses: head $= +1$, tail $= -1$

Example: eigenvalue distribution for $N = 10$





Wigner random matrices (Wigner 1955)

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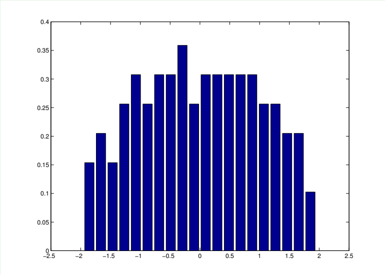
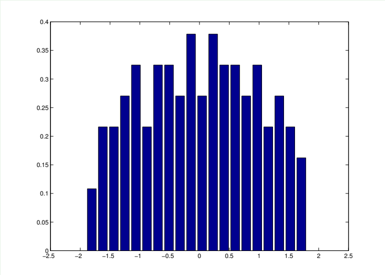
Example: eigenvalue distribution for $N = 100$





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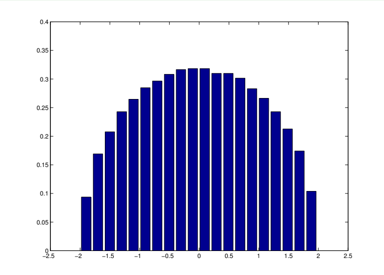

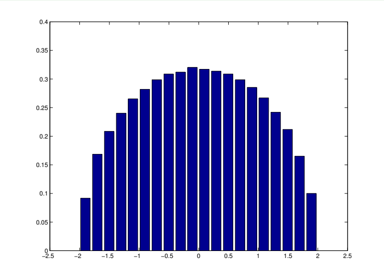

von Neumann algebras and random matrices

- we can translate some parts of the free group factors language into the random matrix language

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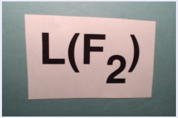

random matrix

$$\begin{pmatrix}
1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
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1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\
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-1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
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1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1
\end{pmatrix}$$

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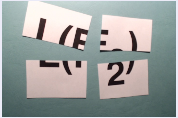

random matrix

$$\begin{pmatrix}
1 & -1 & -1 & 1 & -1 & \| & 1 & -1 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & \| & 1 & -1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & \| & 1 & 1 & 1 & -1 & 1 \\
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-1 & 1 & -1 & 1 & -1 & \| & -1 & -1 & -1 & 1 & 1 \\
= & = & = & = & = & = & = & = & = & = & = \\
1 & 1 & 1 & 1 & -1 & \| & 1 & 1 & -1 & 1 & 1 \\
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-1 & 1 & -1 & -1 & 1 & \| & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & \| & 1 & 1 & -1 & -1 & 1
\end{pmatrix}$$

For example, when we cut them into pieces . . . it is easier to work with the pieces of random matrices.

Results about free group factors

Realizing the free group factors with the help of random matrices gave some new insights and allowed to derive new deep results about free group factors

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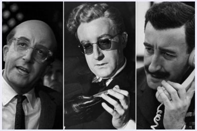
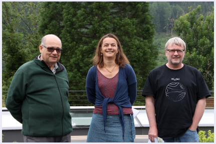
$L(F_2)$, $L(F_3)$, $L(F_4)$ could be ...



all the same
Peter Sellers
(in *Dr. Strangelove*)

some the same,
some different



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Voiculescu, Guionnet,
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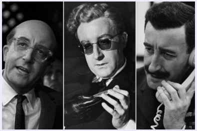
- it can not be that some are the same and some are different

$L(F_2)$, $L(F_3)$, $L(F_4)$ ... are all the same or all different



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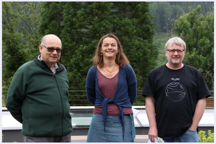

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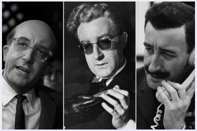

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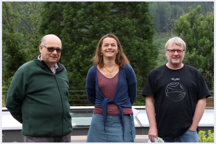

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- free group factors do not have Cartan subalgebras (counter example to old conjecture)

Why is this all so exciting?

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operator algebras

complicated versions of infinite dimensional matrices; with a lot of interesting structure – but not easy to deal with

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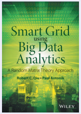

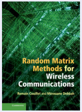

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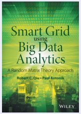

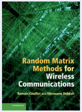

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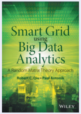

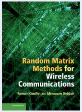

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(next talk on Monday)

And no: there is no free lunch nor free dinner in free probability ...

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... unless you are the inventor of the subject and have your 70th birthday!

Happy Birthday, Dan!

Dan and Ioana at Pisa, in front of a statue of Dini