Linearization and Brown measure

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Section 1

Uffe's Legacy in Free Probability Theory

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On Voiculescu's R- and S-Transforms for Free Non-Commuting Random Variables

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1 Introduction

Voiculescu introduced in 1986-87 (cf. [8], [9]) the R- and S-transforms for the distribution of "random variables" in a non-commutative algebra $\mathcal{A}$ with a fixed probability measure (i.e., a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ for which $\varphi(1) = 1$), and proved the addition formula for the R-transform

$$R_{\mu_{a+b}}(z) = R_{\mu_a}(z) + R_{\mu_b}(z), \tag{1.1}$$

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Brown's Spectral Distribution Measure for *R*-Diagonal Elements in Finite von Neumann Algebras

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In 1983 L. G. Brown introduced a spectral distribution measure for non-normal elements in a finite von Neumann algebra $\mathscr{M}$ with respect to a fixed normal faithful tracial state τ. In this paper we compute Brown's spectral distribution measure in case T has a polar decomposition $T=UH$ where U is a Haar unitary and U and H are $*$-free. (When $\operatorname{Ker} T=\{0\}$ this is equivalent to that (T, T^*) is an R-diagonal pair in the sense of Nica and Speicher.) The measure μ_T is expressed explicitly in terms of the S-transform of the distribution μ_{T^*T} of the positive operator T^*T. In case T is a circular element, i.e., $T=(X_1+iX_2)\sqrt{2}$ where (X_1, X_2) is a free semicircular system, then $\operatorname{sp} T=D$, the closed unit disk, and μ_T has constant density $1\ \pi$ on D.

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A new application of random matrices: $\text{Ext}(C^*_{\text{red}}(F_2))$ is not a group

By Uffe Haagerup and Steen Thorbjørnsen*

Dedicated to the memory of Gert Kjærgård Pedersen

Abstract

In the process of developing the theory of free probability and free entropy, Voiculescu introduced in 1991 a random matrix model for a free semicircular system. Since then, random matrices have played a key role in von Neumann algebra theory (cf. [V8], [V9]). The main result of this paper is the following extension of Voiculescu's random matrix result: Let $(X_1^{(n)}, \ldots, X_r^{(n)})$ be a system of r stochastically independent $n \times n$ Gaussian self-adjoint random matrices as in Voiculescu's random matrix paper [V4], and let $(x_1, \ldots, x_r)$ be

Brown measure

Haagerup and Larsen

⟶

invariant subspaces

- with H. Schultz
- with K. Dykema

Linearization

Haagerup and Thorbjørnsen

⟶

operator algebraic applications

- $Ext(C^*_{red}(FF_2))$ not group with S. Thorbjørnsen
- no projections in $C^*(F_2)$ with S. Thorbjørnsen, H. Schultz

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Brown measure

Haagerup and Larsen

$\longrightarrow$

Linearization

Haagerup and Thorbjørnsen

$\longrightarrow$

new ideas and concepts for dealing with limits of random matrices and functions of free random variables

- Brown measure as candidate for asymptotic eigenvalue distribution of non-Hermitean random matrices
- limits of norms of polynomials in independent random matrices
- calculation of distribution of functions of free variables

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Section 2

Linearization

Voiculescu, Kirchberg, Pisier, etc.

non-linear problem → operator-valued linear problem

$$p(x_1, \ldots, x_m) \to \alpha_0 \otimes 1 + \alpha_1 \otimes x_1 + \cdots \alpha_m \otimes x_m$$

Example

Is xy invertible? We have

$$\begin{pmatrix} xy & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & x \\ y & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

Hence xy is invertible if and only if

$$\begin{pmatrix} 0 & x \\ y & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes y$$

is invertible

Theorem (Haagerup, Thorbjørnsen (+Schultz); Anderson)

Every polynomial $p(x_1,\dots,x_m)$ *has a (non-unique) linearization* $\hat{p} = \alpha_0 \otimes 1 + \alpha_1 \otimes x_1 + \cdots + \alpha_m \otimes x_m$ *such that*

$$(z-p)^{-1} = [(\Lambda(z)-\hat{p})^{-1}]_{1,1}, \quad \textit{where} \quad \Lambda(z) = \begin{pmatrix} z & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

and hence also

$$G_p(z) = \varphi((z-p)^{-1}) = [\varphi \otimes 1(\Lambda(z)-\hat{p})^{-1}]_{1,1} = [G_{\hat{p}}(\Lambda(z))]_{1,1}.$$

- $G_p(z) = \varphi[(z-p)^{-1}]$ is the Cauchy transform of p
- $G_{\hat{p}}(b) = \varphi \otimes 1[(b-\hat{p})^{-1}]$ is the operator-valued Cauchy transform of $\hat{p}$

Historical remark

Note that, as we only became aware in recent years, this linearization trick is also a well-known idea in many other mathematical communities, known under various names like

- Higman's trick (Higman "The units of group rings", 1940)
- recognizable power series (automata theory, Kleene, Schützenberger)
- linearization by enlargement (ring theory, Cohn)
- system realization (control theory, Helton, Vinnikov)

So maybe Uffe's trick is not so new, but his applications in operator algebras, free probability and random matrices are nevertheless amazing ...

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Theorem (Haagerup, Thorbjørnsen)

For $X_1^{(N)}, \dots, X_r^{(N)}$ independent Gaussian random matrices, and $s_1, \dots, s_r$ free semicircular elements we have for any polynomial p in r non-commuting variables almost surely

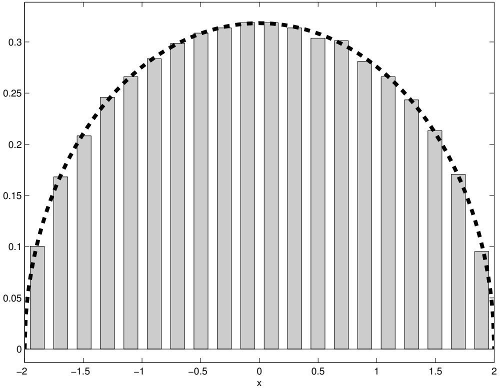
$$\lim_{N\to\infty} \|p(X_1^{(N)}, \dots, X_r^{(N)})\| = \|p(s_1, \dots, s_r)\|$$

Generalizations

This has been generalized to many other situations

- real and symplectic case (Schultz)
- other types of random matrices
  - Wigner or Wishart matrices (Capitaine, Donati-Martin; Anderson)
  - Haar unitary random matrices (Collins, Male)
  - including deterministic matrices (Male)
- "exact separation of eigenvalues" (Haagerup, Schultz, Thorbjørnsen)
- non-commutative rational functions instead of polynomials (Yin)

One-matrix case: classical random matrix case



$s = l + l^*,$

l one-sided shift on $\bigoplus_{n \geq 0} \mathbb{C} e_n$

$\varphi(a) = \langle a e_0, e_0 \rangle$

$\mu = \varphi \circ E_s$

- $X^{(N)} \to s$ in distribution (Wigner 1955)
- $\|X^{(n)}\| \to \|s\| = 2$ (Füredi, Komlós 1981)

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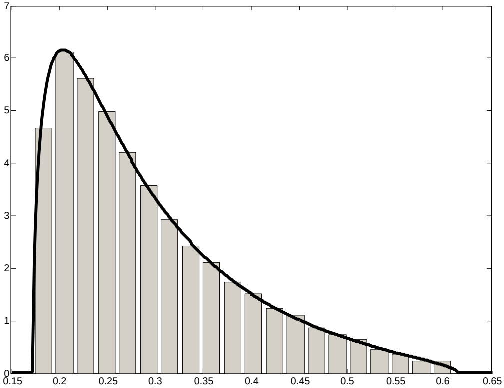
Multi-matrix case: non-commutative case



$$s_1 = l_1 + l_1^*,\ s_2 = l_2 + l_2^*$$

- $r(X_1^{(N)}, X_2^{(N)}) \to r(s_1, s_2)$

  in distribution (Voiculescu)

- $\|r(X_1^{(N)}, X_2^{(N)})\|$

  $\to \|r(s_1, s_2)\|$ (HT)

$$(1-x_1)^{-1} + (1-x_1)^{-1} x_2 \big((1-x_1) - x_2(1-x_1)^{-1} x_2\big)^{-1} x_2 (1-x_1)^{-1}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1-x_1 & -x_2 \\ -x_2 & 1-x_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Question

Can we actually calculate

- the distribution/norm of $r(s_1, s_2)$

Solution: again by linearization (Belinschi, Mai, Speicher; Helton)

- we need the distribution of $r(s_1, s_2)$
- this is determined by the operator-valued distribution of its linearization

$$\hat{r}(s_1, s_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 - s_1 & -s_2 \\ 0 & -s_2 & 1 - s_1 \end{pmatrix}$$

Question

Can we actually calculate

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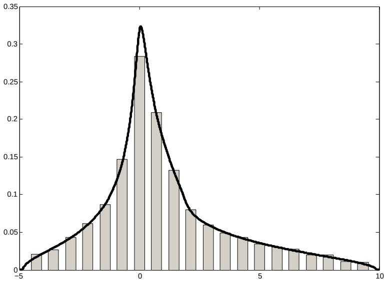
Solution: again by linearization (Belinschi, Mai, Speicher; Helton)

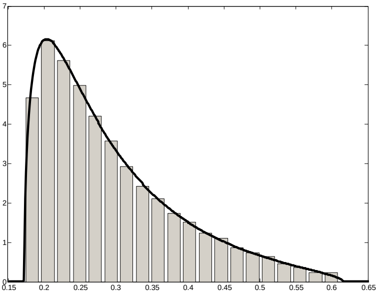
- we need the distribution of $r(s_1, s_2)$
- this is determined by the operator-valued distribution of its linearization
- but this is now an additive (operator-valued) problem

$$\hat{r}(s_1, s_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes s_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes s_2$$

- for this we have analytic theory of operator-valued free convolution

This allows to calculate (by also involving some numerical solving of fixed point equations for the Cauchy transforms) the solid lines in the pictures





$$p(x_1, x_2) = x_1 x_2 + x_2 x_1 + x_1^2$$

$$r(x_1, x_2)$$

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Relevance of p being selfadjoint

Note that whether p is selfadjoint or not

- is not an issue for the norm, as

$$\|p\|^2 = \|pp^*\|$$

  $\rightarrow$ the Haagerup-Thorbjørnsen Theorem is for arbitrary polynomials

- is an issue for convergence of distribution, if we want to understand the latter as a probability measure (like: eigenvalue distribution for matrices)

  $\rightarrow$ theory of Belinschi-Mai-Speicher is only for selfadjoint polynomials

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Section 3

Brown Measure

Idea of Brown measure μ_a for operator a

Normal case

Let $x \in (\mathcal{A}, \varphi)$ be normal operator (e.g., selfadjoint or unitary). Then there exists uniquely determined probability measure μ_x on $\mathbb{C}$ such that

$$\varphi(x^n x^{*m}) = \int_{\mathbb{C}} z^n \bar{z}^m d\mu_x(z) \qquad \forall m, n \geq 0$$

General, not necessarily normal, case

Let $a \in (\mathcal{A}, \varphi)$ be general operator. Its distribution in algebraic sense is given by collection of all $*$-moments

$$\varphi(a^{i_1} \cdots a^{i_n}), \qquad n \in \mathbb{N}, i_1, \dots, i_n \in \{1, *\}$$

This information cannot be encoded by a probability measure on $\mathbb{C}$, but still we would like to have such a μ_a which captures as much information about all $*$-moments of a as possible.

Rough idea

In principle, we want to define the Brown measure μ_a by the requirement that μ_a and a have the same Cauchy transform

$$\int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu_a(z) = \varphi(\frac{1}{\lambda - a}).$$

- For normal a it suffices to have this outside the spectrum of a, for $\lambda \notin \sigma(a)$; there both sides make sense and are analytic functions in λ
- For non-normal a one needs also the information inside the spectrum; the left-hand side makes then still sense almost surely (as a non-analytic function); but the right hand side is problematic

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Rough idea

- Instead of
$$\int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu_a(z) = \varphi(\frac{1}{\lambda - a})$$
we consider the integrated version
$$\int_{\mathbb{C}} \ln|\lambda - z| d\mu_a(z) = \varphi(\ln|\lambda - a|) = \log \Delta(|a - \lambda|)$$
where Δ is the Fuglede-Kadison determinant (which makes sense in a tracial von Neumann algebra setting)
- The latter formulation makes sense for any $\lambda \in \mathbb{C}$ and defines μ_a uniquely.

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History

- L. Brown 1986: Lidskii's theorem in the type II case
- Haagerup, Larsen 2000

Impact on operator algebraic, free probabilitiy and random matrix side

- calculation of Brown measures for known and also new classes of operators
  - R-diagonal operators and variations (Haagerup, Larsen; Haagerup, Schultz; Biane, Lehner)
  - DT operators (Dykema, Haagerup)
- investigations on invariant subspaces (Dykema, Haagerup; Haagerup, Schultz)
- Brown measure gives candidate for limit of eigenvalue distribution of non-normal random matrices
  - single ring theorem (Guionnet, Krishnapur, Zeitouni)

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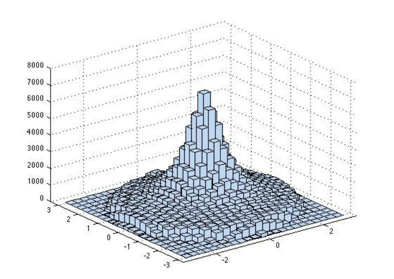
Limits of non-normal random matrices

Conjecture

Let $X_1^{(N)}, \dots, X_r^{(N)}$ be r independent Gaussian $N \times N$ random matrices and $s_1, \dots, s_r$ be r free semicirculars. Then, for any polynomial in r non-commuting variables we have that the Brown measue (i.e., the eigenvalue distribution) of $p(X_1^{(N)}, \dots, X_r^{(N)})$ converges, almost surely, weakly to the Brown measure of $p(s_1, \dots, s_r)$.

Example ($p(x_1, x_2, x_3) = x_1x_2x_3 + 2x_2x_3x_1 + x_3x_1x_2$)



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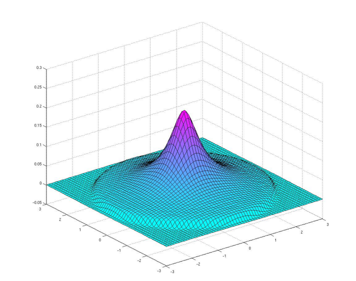

Section 4

Brown Measure and Linearization

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Hermitization method

Idea: transform non-normal problem into operator-valued selfadjoint problem

We have

$$\int_{\mathbb{C}} \log|\lambda - z| d\mu_a(z) = \log \Delta(|a-\lambda|) = \int_0^\infty \log(t) d\mu_{|a-\lambda|}(t).$$

Hence in order to calculate the Brown measure of a

- we need to calculate the distribution of all hermitian operators $|a-\lambda|$,
- which can be gotten from the operator-valued distribution of

$$A = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

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Hermitization method

History

- folklore in random matrix theory (going back to Girko?) for eigenvalue distribution
- contact with free probability and operator-valued description on formal level by physicists
  - Zee, Feinberg
  - Janik, Nowak, Papp, Zahed
- first rigorous calculations using operator-valued description by
  - Aagaard, Haagerup (for quasi-nilpotent DT-operator)
- general rigorous theory by
  - Belinschi, Sniady, Speicher

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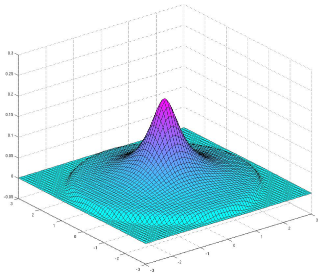
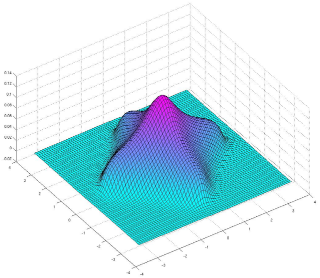
This allows then the calculation of the Brown measure of any polynomial (or even non-commutative rational function) in free variables





Thanks, Uffe, for the wonderful mathematics!

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