Sharp Bounds for Sums Associated to Graphs of Matrices

at the

WaldenFest

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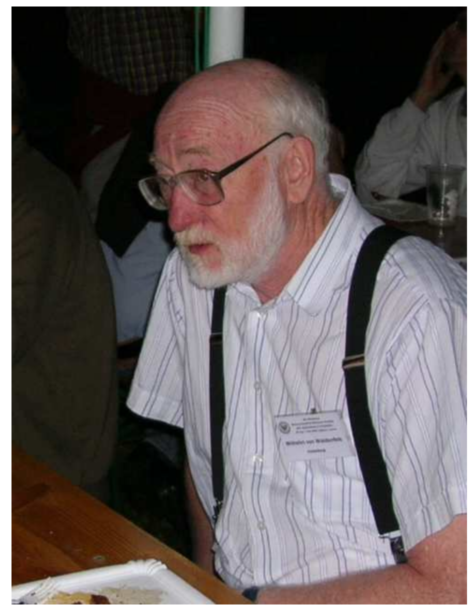
Universität des Saarlandes
Saarbrücken, Germany

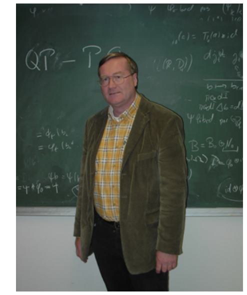
Academic family tree of Wilhelm von Waldenfels


König


Weise


König


Waldenfels


Schürmann


Speicher

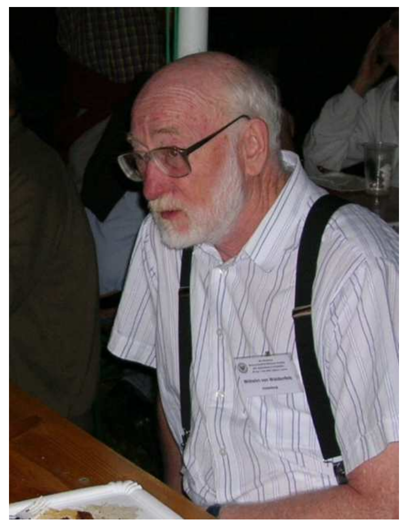

Skeide

Academic family tree of Wilhelm von Waldenfels

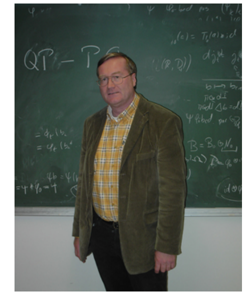
König

Weise

König



Waldenfels



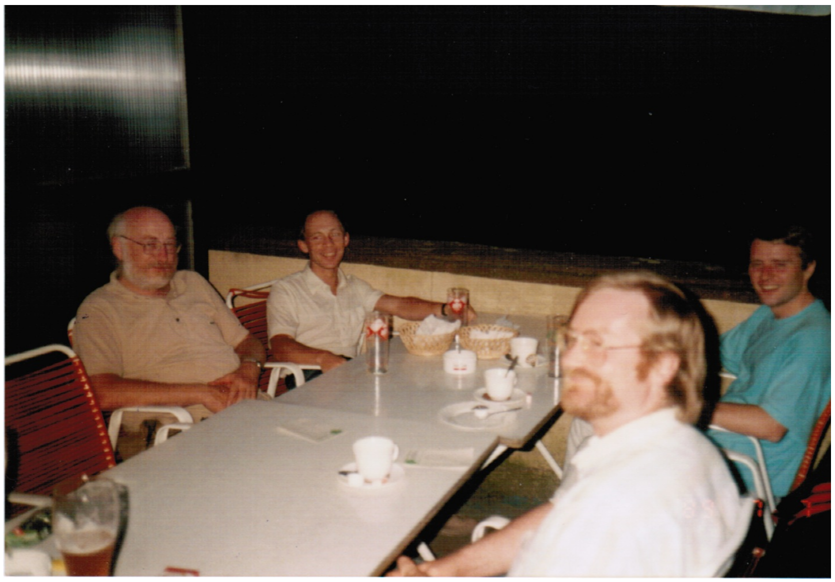
Schürmann

Speicher and "sons"

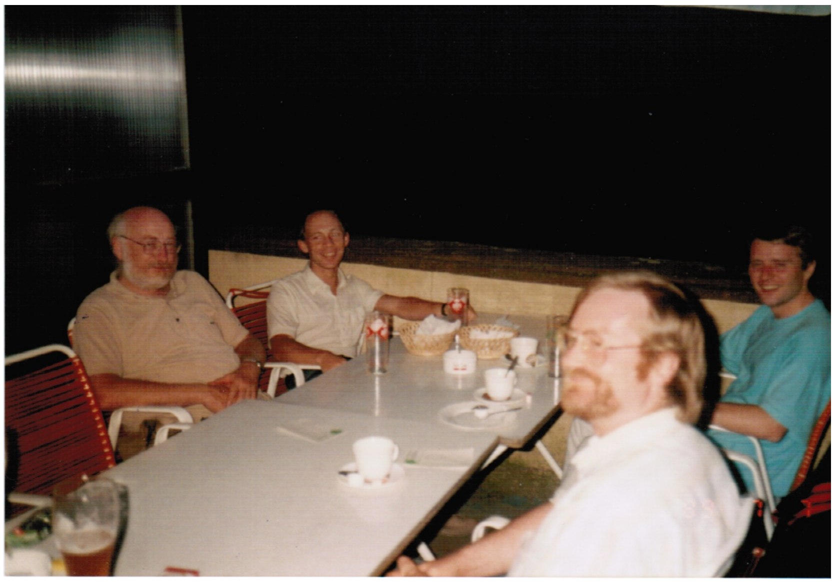
Skeide

Heidelberg ... at the good old times ... last century

Some things we learned

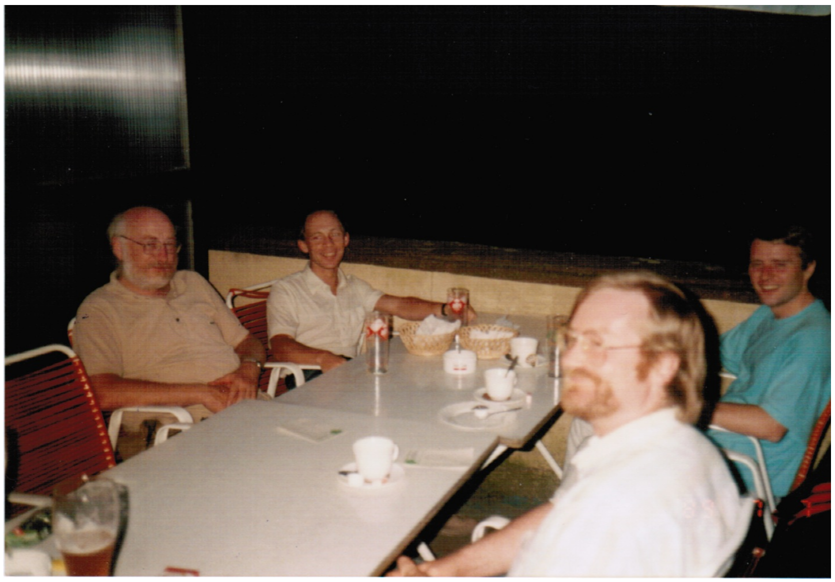


Some things we learned

... thinking is hard work ....

Some things we learned



... look for a (hidden) conceptual structure

behind your complicated concrete problem ....

Question

What is the asyptotic behaviour in N of

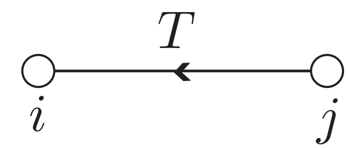
$$\sum_{\substack{j_1,\ldots,j_{2m} \\ \text{some constraints on} \\ \text{equality of indices}}}^{N} t^{(1)}_{j_1 j_2} t^{(2)}_{j_3 j_4} \cdots t^{(m)}_{j_{2m-1} j_{2m}}$$

with given matrices

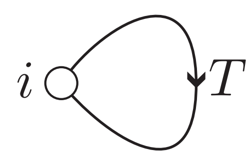
$$T_k = \left(t^{(k)}_{ij} \right)_{i,j=1}^{N}$$

Examples

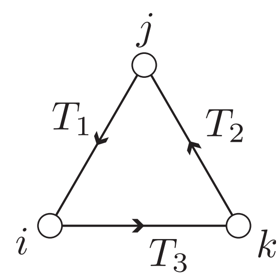
$$\sum_{i,j=1}^{N} t_{ij}$$



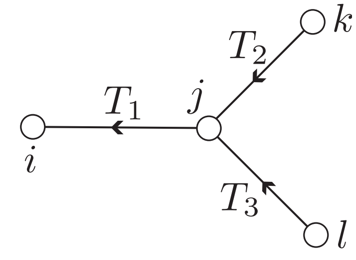
$$\sum_{i=1}^{N} t_{ii}$$



$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$

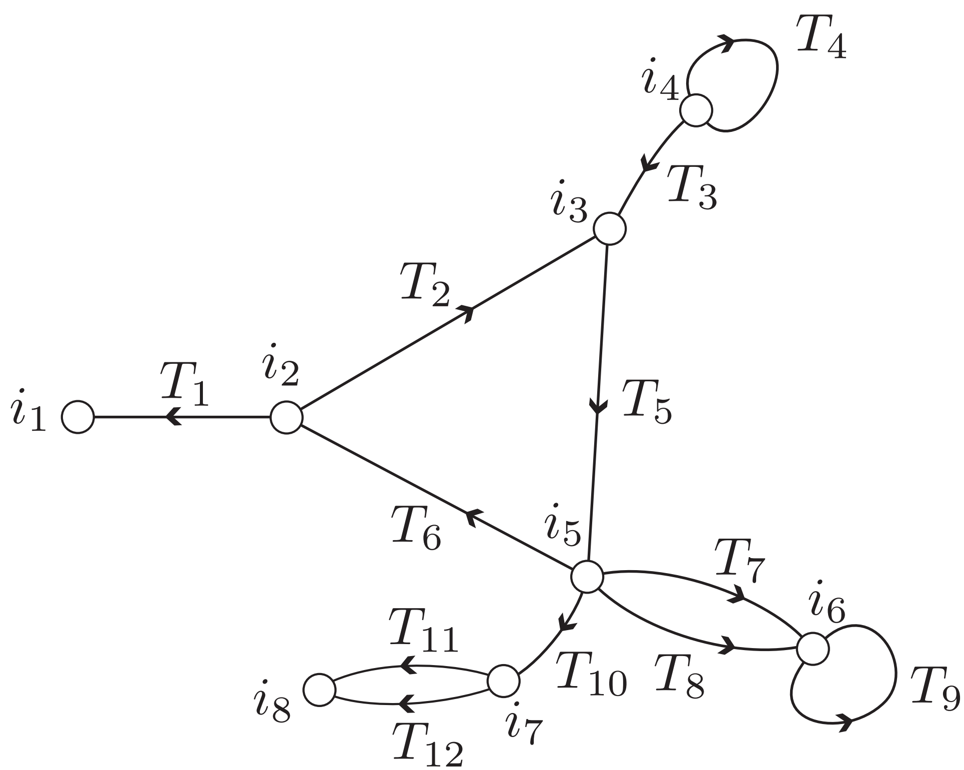


$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$



That's easy? Okay, so look on this:

$$\sum_{i_1,\ldots,i_8=1}^{N} t^{(1)}_{i_1 i_2} t^{(2)}_{i_3 i_2} t^{(3)}_{i_3 i_4} t^{(4)}_{i_4 i_4} t^{(5)}_{i_5 i_3} t^{(6)}_{i_2 i_5} t^{(7)}_{i_6 i_5} t^{(8)}_{i_6 i_5} t^{(9)}_{i_6 i_6} t^{(10)}_{i_7 i_5} t^{(11)}_{i_8 i_7} t^{(12)}_{i_8 i_7}$$



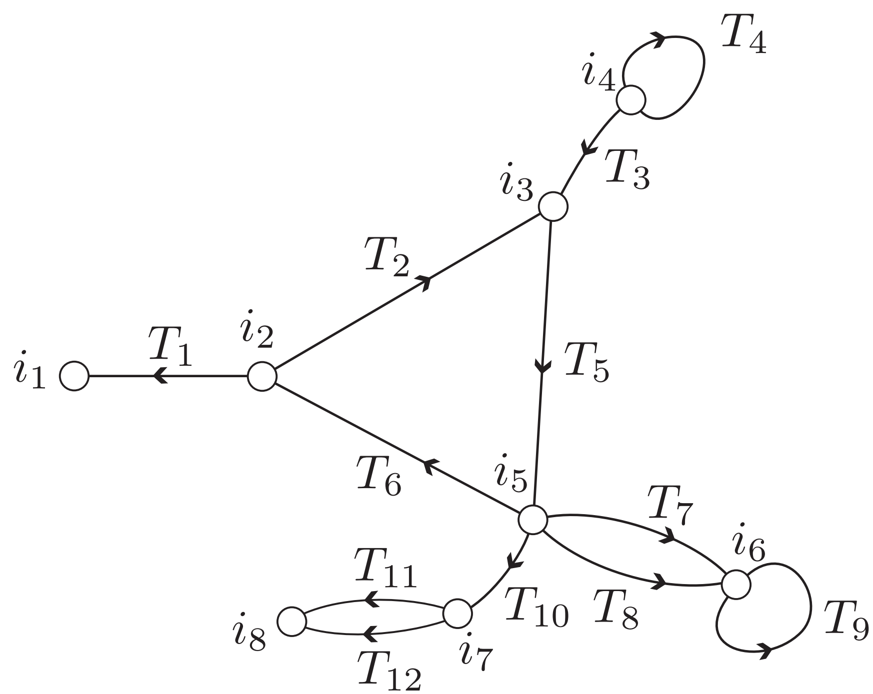
Problem

For a given directed graph G (multiple edges and loops allowed), with matrices attached to the edges, denote the corresponding sum by $S_G(N)$

Question: What is the optimal bound $r(G)$ in

$$|S_G(N)| \leq N^{r(G)} \prod_{k=1}^{m} \|T_k\|$$

Optimal asymptotics in N?



$$\left|\sum_1^N t^{(1)}_{i_1 i_2} t^{(2)}_{i_3 i_2} t^{(3)}_{i_3 i_4} t^{(4)}_{i_4 i_4} t^{(5)}_{i_5 i_3} t^{(6)}_{i_2 i_5} t^{(7)}_{i_6 i_5} t^{(8)}_{i_6 i_5} t^{(9)}_{i_6 i_6} t^{(10)}_{i_7 i_5} t^{(11)}_{i_8 i_7} t^{(12)}_{i_8 i_7}\right| \leq N^{???} \cdot \prod_{i=1}^{12} \|T_i\|$$

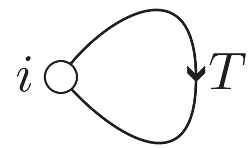
Motivation

such sums appear and have to be asymptotically controlled in calculations of moments of products of random (Wigner) matrices and deterministic matrices

- Yin + Krishnaiah, 1983
- Bai (+ Silverstein), 1999 (2006)
- Mingo + Speicher, 2012 JFA
  (asymptotic freeness of Wigner and deterministic matrices)
- Male, 2012 $\rightarrow$ "traffics"

Example

$$\sum_{i=1}^{N} t_{ii}$$

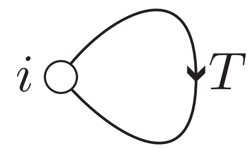


Then

$$|\sum_{i=1}^{N} t_{ii}| \leq \sum_{i=1}^{N} \|T\| = N\|T\|$$

Example

$$\sum_{i=1}^{N} t_{ii}$$



Then

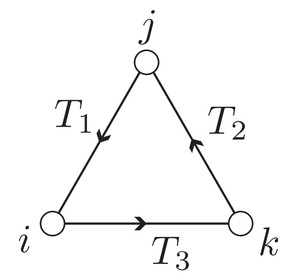
$$|\sum_{i=1}^{N} t_{ii}| \leq \sum_{i=1}^{N} \|T\| = N\|T\|$$

or

$$|\sum_{i=1}^{N} t_{ii}| = |\mathsf{Tr}(T)| = N|\mathsf{tr}(T)| \leq N\|T\|$$

Example

$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$

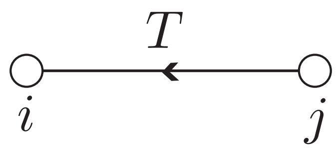


$$|\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}| = |\mathrm{Tr}(T_1 T_2 T_3)| = N|\mathrm{tr}(T_1 T_2 T_3)|$$

$$\leq N\|T_1 T_2 T_3\| \leq N\|T_1\|\|T_2\|\|T_3\|$$

Example

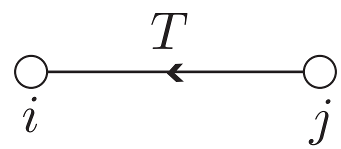
$$\sum_{i,j=1}^{N} t_{ij}$$



trivial estimate: $\quad |\sum_{i,j=1}^{N} t_{ij}| \leq \sum_{i,j=1}^{N} \|T\| = N^2 \|T\|$

But we can do better ....

$$\sum_{i,j=1}^{N} t_{ij}$$



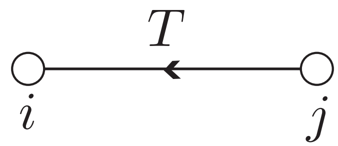
... with using the vectors

$$e_i := (0, \ldots, \underset{i}{1}, \ldots, 0), \qquad e := (1, 1, \ldots, 1)$$

we can write

$$\sum_{i,j=1}^{N} t_{ij} = \sum_{i,j=1}^{N} \langle e_i, Te_j \rangle = \langle e, Te \rangle$$

$$\sum_{i,j=1}^{N} t_{ij}$$



... with using the vectors

$$e_i := (0, \ldots, \underset{i}{1}, \ldots, 0), \qquad e := (1, 1, \ldots, 1)$$

we can write

$$\sum_{i,j=1}^{N} t_{ij} = \sum_{i,j=1}^{N} \langle e_i, T e_j \rangle = \langle e, Te \rangle$$

and thus

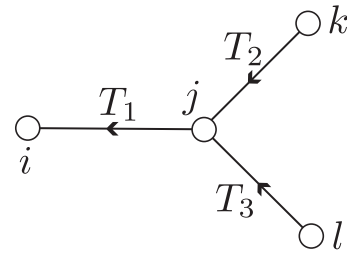
$$\left| \sum_{i,j=1}^{N} t_{ij} \right| \leq |\langle e, Te \rangle| \leq \|e\|^2 \|T\| = N \|T\|$$

since

$$\|e\| = \sqrt{N}$$

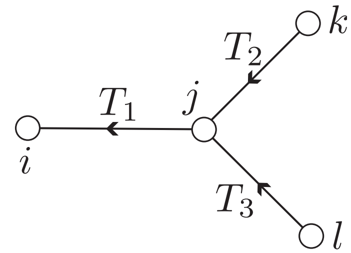
Example

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$



Example

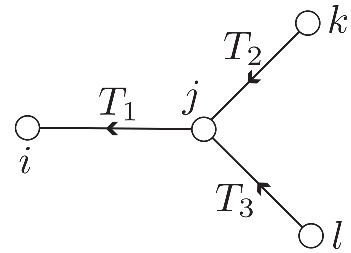
$$\sum_{i,j,k,l=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{jl}$$



$$\sum_{i,j,k,l=1}^{N} t^{(1)}_{ij} t^{(2)}_{jk} t^{(3)}_{jl} = \sum_{i,j,k,l} \langle e_i, T_1 e_j \rangle \underbrace{\langle e_j, T_2 e_k \rangle \langle e_j, T_3 e_l \rangle}$$

Example

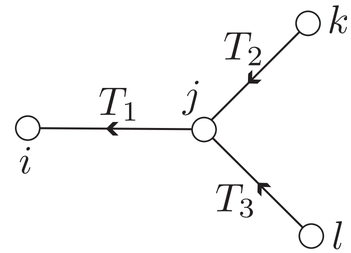
$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$



$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)} = \sum_{i,j,k,l} \langle e_i, T_1 e_j \rangle \underbrace{\langle e_j, T_2 e_k \rangle \langle e_j, T_3 e_l \rangle}_{\langle e_j \otimes e_j, T_2 \otimes T_3 e_k \otimes e_l \rangle}$$

Example

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$

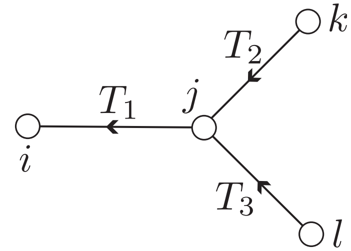


$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)} = \sum_{i,j,k,l} \langle e_i, T_1 e_j \rangle \underbrace{\langle e_j, T_2 e_k \rangle \langle e_j, T_3 e_l \rangle}_{\langle e_j \otimes e_j, T_2 \otimes T_3 e_k \otimes e_l \rangle}$$

$$= \langle e, T_1 \underbrace{\left(\sum_j |e_j\rangle\langle e_j \otimes e_j| \right)}_{V:\mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N} T_2 \otimes T_3, e \otimes e \rangle$$

$$= \langle e, T_1 V (T_2 \otimes T_3)\, e \otimes e \rangle$$

$$\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}$$



$$\begin{aligned}
\left|\sum_{i,j,k,l=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{jl}^{(3)}\right| &= |\langle e, T_1 V (T_2 \otimes T_3)\, e \otimes e\rangle| \\
&\leq \|e\| \cdot \underbrace{\|e \otimes e\|}_{\|e\|\cdot\|e\|} \cdot \|T_1\| \cdot \underbrace{\|V\|}_{=1} \cdot \|T_2 \otimes T_3\| \\
&= N^{3/2} \cdot \|T_1\| \cdot \|T_2\| \cdot \|T_3\|
\end{aligned}$$

since

$$V : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N, \qquad e_i \otimes e_j \mapsto \begin{cases} e_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

is an isometry, and thus $\|V\| = 1$.

General structure

In this example, our sum $S(N)$ is given as inner product

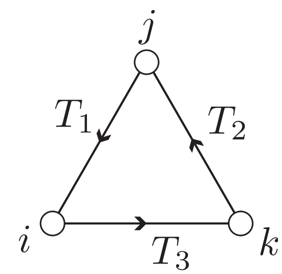
$$S(N) \qquad \hat{=} \qquad \langle \text{output}, \textit{stuff}\ \text{input} \rangle$$

where

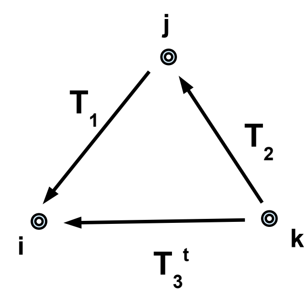
- each input and each output vertex contributes factor $N^{1/2}$
- internal vertices do not contribute, summation over them corresponds to matrix multiplication or, more general, partial isometries

Note that one can also modify

$$\sum_{i,j,k=1}^{N} t_{ij}^{(1)} t_{jk}^{(2)} t_{ki}^{(3)}$$



to an input-output form:



General strategy

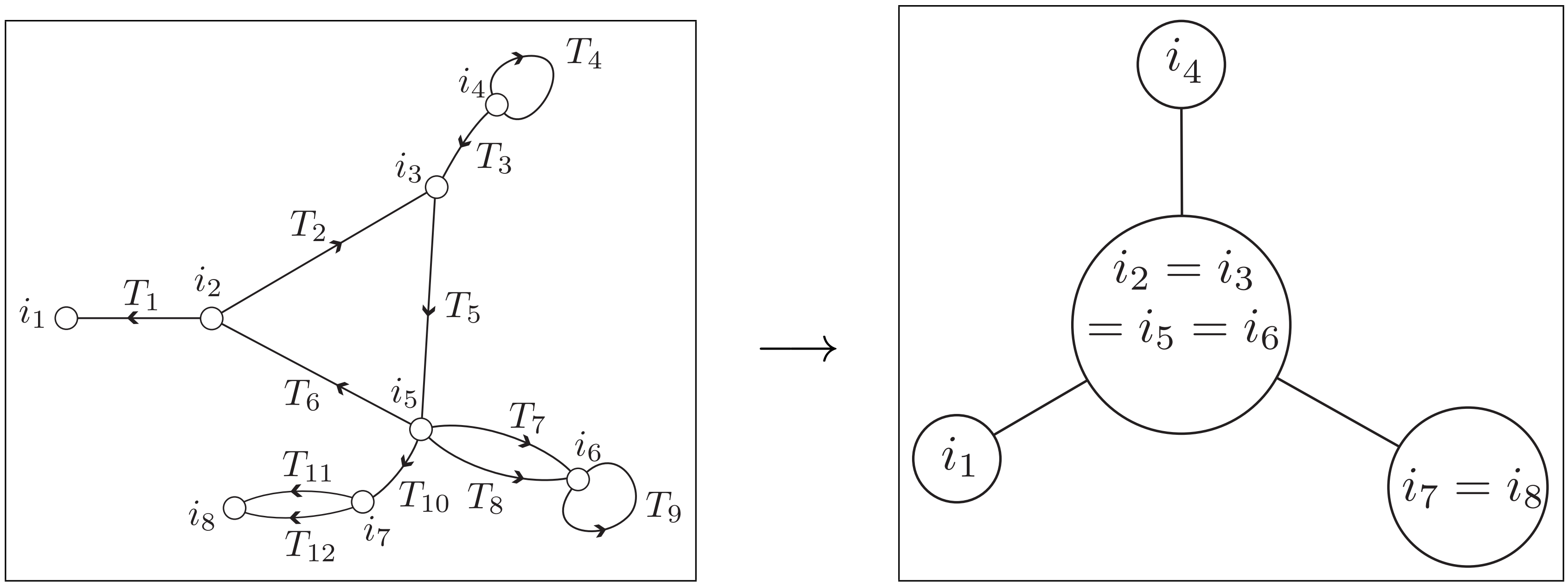
- Modify given graph to equivalent input-output graph
- then each input and output vertex gives factor $N^{1/2}$

Main problems:

- Can we always do such a modification?
- If yes, how can we recognize how many input/output vertices we need?

Asymptotics determined by structure of $\mathfrak{F}(G)$

$\mathfrak{F}(G) =$ forrest of two-edge-connected components



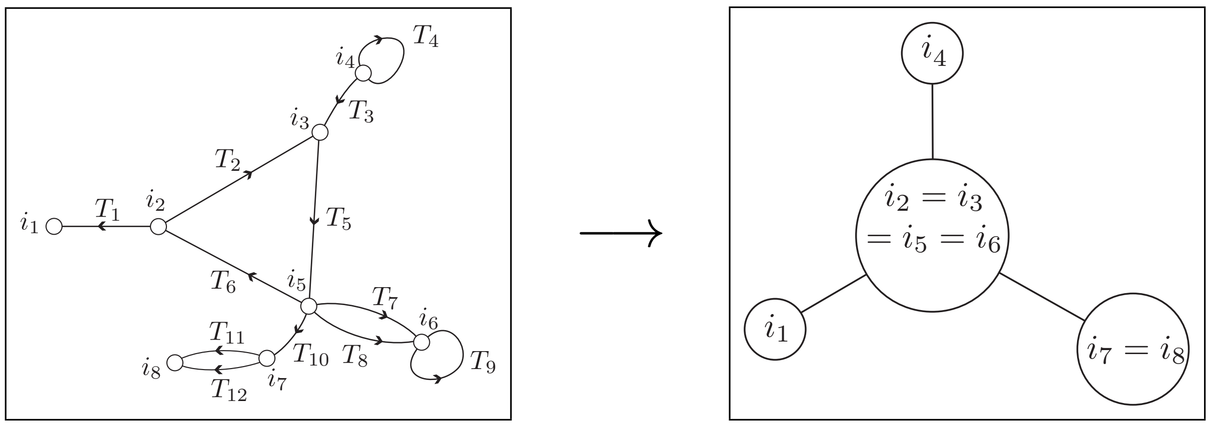
Theorem [Mingo, Speicher, JFA 2012]

We have the following optimal estimate:

$$\Big|\sum_{i:V\to[N]}\prod_{e\in E} t^{(e)}_{i_{t(e)},i_{s(e)}}\Big| \le N^{\frac{1}{2}\cdot\#\text{leaves of }\mathfrak{F}(G)}\cdot\prod_{e\in E}\|T_e\|$$

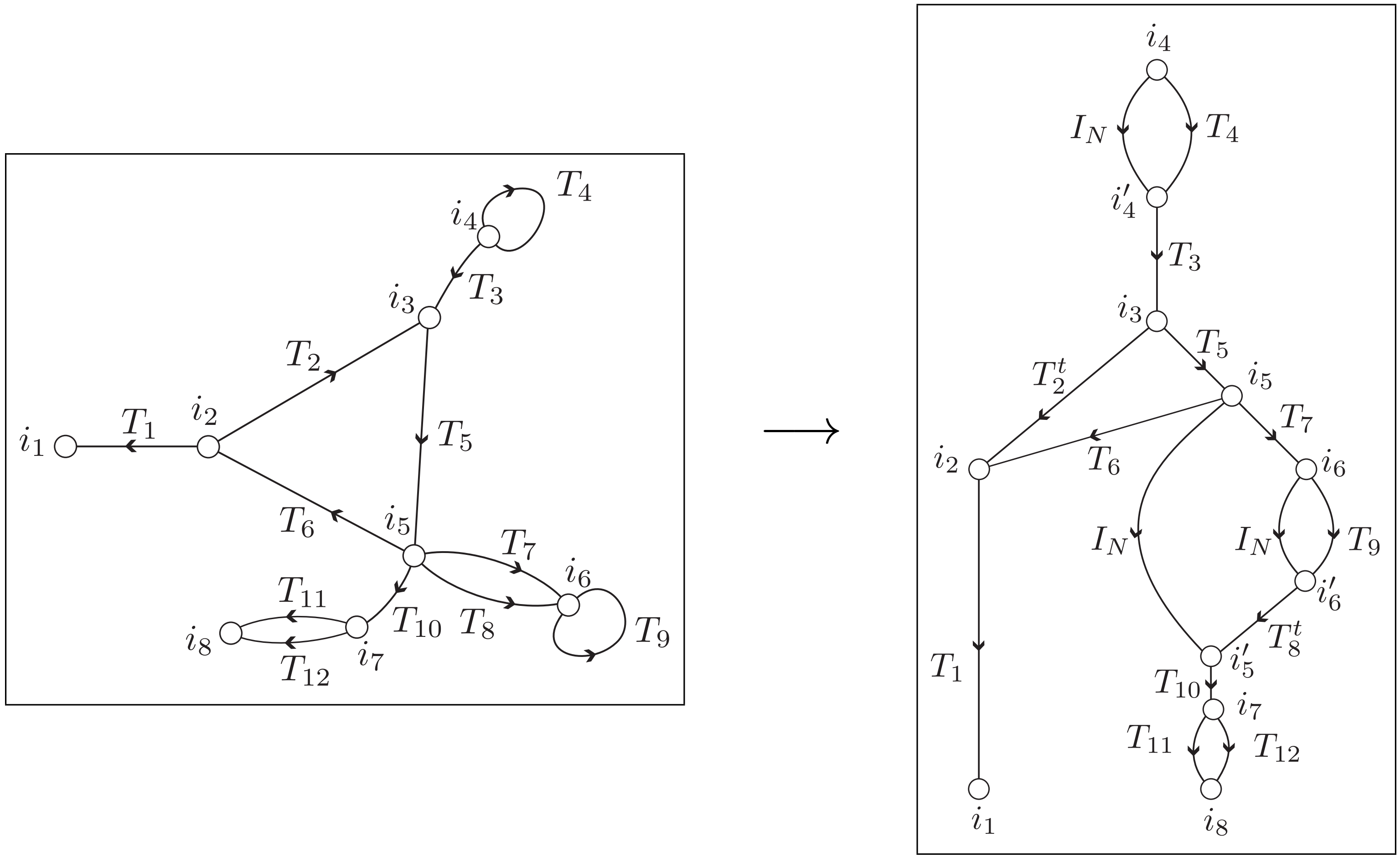
(Trivial leaves, i.e., only vertices of trivial trees, count twice!)

Optimal asymptotics in N!

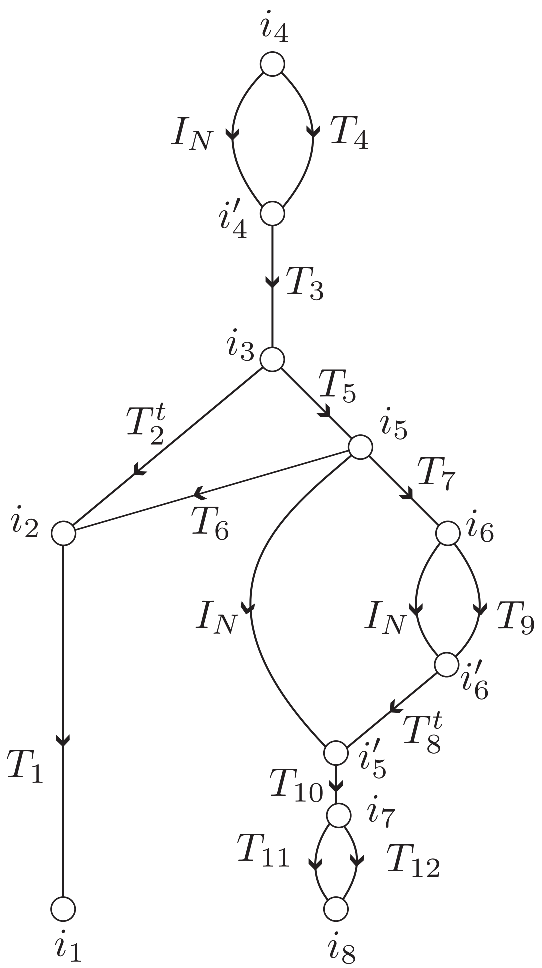


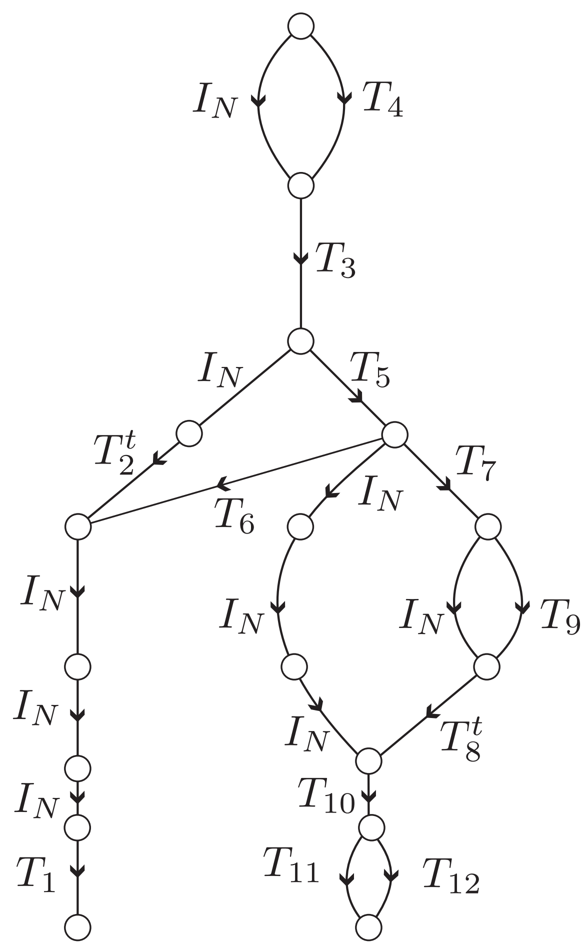
$$|\sum_1^N t^{(1)}_{i_1 i_2} t^{(2)}_{i_3 i_2} t^{(3)}_{i_3 i_4} t^{(4)}_{i_4 i_4} t^{(5)}_{i_5 i_3} t^{(6)}_{i_2 i_5} t^{(7)}_{i_6 i_5} t^{(8)}_{i_6 i_5} t^{(9)}_{i_6 i_6} t^{(10)}_{i_7 i_5} t^{(11)}_{i_8 i_7} t^{(12)}_{i_8 i_7}| \le N^{\mathbf{3/2}} \cdot \prod_{i=1}^{12} \|T_i\|$$

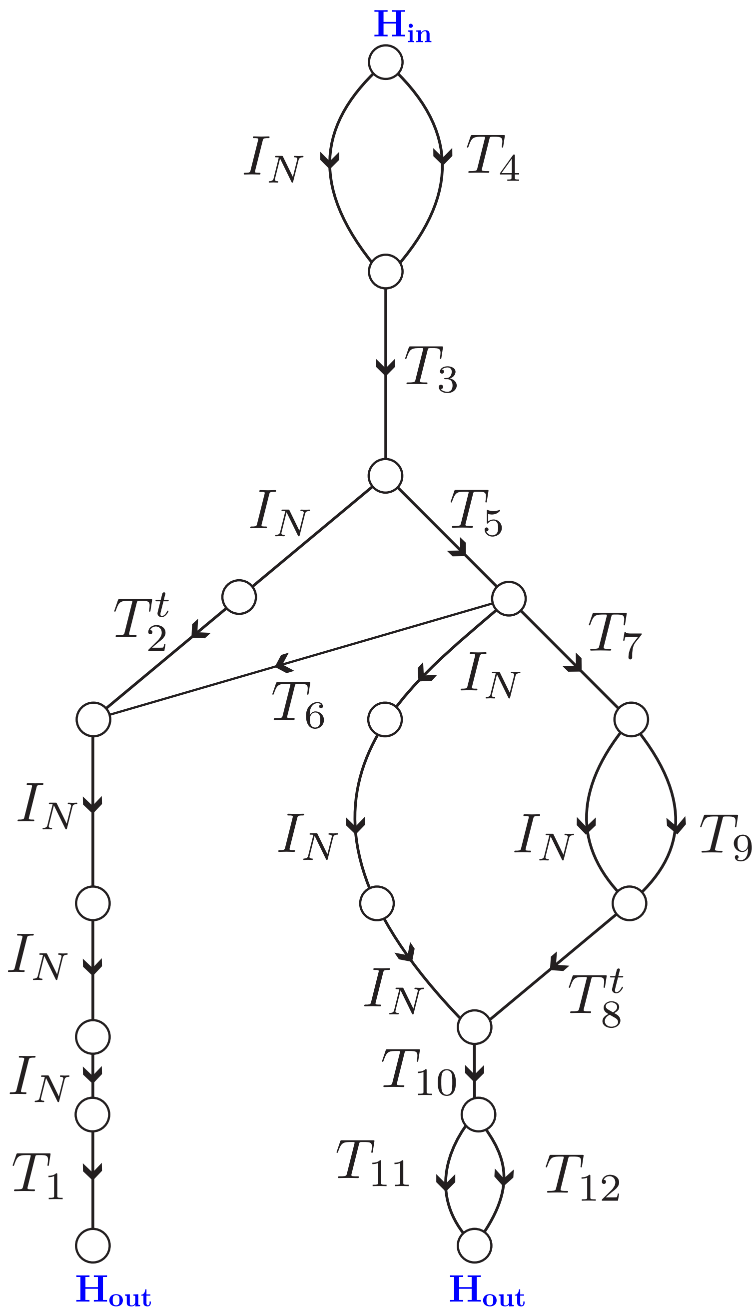
Replace by equivalent input-output problem



Adding additional vertices







$$V_{in,2} : \mathbf{H_{in}} \to H \otimes H$$

$$I_N \otimes T_4$$

$$V_{2,1} : H \otimes H \to H$$

$$T_3$$

$$V_{1,2} : H \to H \otimes H$$

$$I_N \otimes T_5$$

$$V_{1,1} \otimes V_{1,3} : H \otimes H) \to H \otimes (H \otimes H \otimes H)$$

$$T_2^t \otimes T_6 \otimes I_N \otimes T_7$$

$$V_{2,1} \otimes V_{1,1} \otimes V_{1,2} : (H \otimes H) \otimes H \otimes H \to H \otimes H \otimes (H \otimes H)$$

$$I_N \otimes I_N \otimes I_N \otimes T_9$$

$$V_{1,1} \otimes V_{1,1} \otimes V_{2,1} : H \otimes H \otimes (H \otimes H) \to H \otimes H \otimes H$$

$$I_N \otimes I_N \otimes T_8^t$$

$$V_{1,1} \otimes V_{2,1} : H \otimes (H \otimes H) \to H \otimes H$$

$$I_N \otimes T_{10}$$

$$V_{1,1} \otimes V_{1,2} : H \otimes H \to H \otimes (H \otimes H)$$

$$T_1 \otimes T_{11} \otimes T_{12}$$

$$V_{1,out} \otimes V_{2,out} : H \otimes (H \otimes H) \to \mathbf{H_{out}} \otimes \mathbf{H_{out}}$$

Theorem [Mingo, Speicher, JFA 2012]

We have the following optimal estimate:

$$\Big|\sum_{i:V\to[N]}\prod_{e\in E} t^{(e)}_{i_{t(e)},i_{s(e)}}\Big| \le N^{\frac{1}{2}\cdot\#\text{leaves of }\mathfrak{F}(G)}\cdot\prod_{e\in E}\|T_e\|$$

(Trivial leaves, i.e., only vertices of trivial trees, count twice!)

Happy Birthday, Wilhelm!!!