On Deterministic Equivalents from a Free Probabilistic Perspective

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(joint work with Carlos Vargas)

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Section 1

Potential Shortcomings of Free Probability Approaches to Wireless Problems

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Potential Shortcomings

Many random matrix models can be treated by free probability methods!

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- doing concrete calculations for more complicated models

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Potential Shortcomings and Their Solutions

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- matrices without asymptotic distribution
  make them converge (or do deterministic equivalents)
- doing concrete calculations for more complicated models
  try harder (or use operator-valued free probability)

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Section 2

Rectangular Matrices Versus Square Matrices

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Free probability can deal with usual square matrices, but in engineering problems the channel matrices are quite often rectangular.

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Solution

Rectangular matrices can in principle be treated within usual free probability by introducing additional projections and cutting down square matrices to the wanted rectangular ones.

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Free probability can deal with usual square matrices, but in engineering problems the channel matrices are quite often rectangular.

Example

Let X be an $n \times m$ Gaussian random matrix (say, $n \leq m$)

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} & \cdots & x_{1m} \\ \ldots & \cdots & \cdots & \cdots & \ldots \\ x_{n1} & \cdots & x_{nn} & \cdots & x_{nm} \end{pmatrix}$$

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Let X be an $n \times m$ Gaussian random matrix (say, $n \leq m$)
Then enlarge X to an $m \times m$ matrix $\tilde{X}$ by filling in 0's:

$$\tilde{X} = \begin{pmatrix} x_{11} & \cdots & x_{1n} & \cdots & x_{1m} \\ \ldots & \cdots & \cdots & \cdots & \ldots \\ x_{n1} & \cdots & x_{nn} & \cdots & x_{nm} \\ 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

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This is not good because $\tilde{X}$ is not a Gaussian random matrix, thus we do not have asymptotic freeness results for it.

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Example

Let X be an $n \times m$ Gaussian random matrix (say, $n \leq m$)
Thus write $\tilde{X}$ in terms of a full $m \times m$ Gaussian random matrix

$$\tilde{X} = \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_{11} & \cdots & x_{1n} & \cdots & x_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ x_{n1} & \cdots & x_{nn} & \cdots & x_{nm} \\ x_{..1} & \cdots & x_{..n} & \cdots & x_{..m} \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1} & \cdots & x_{mn} & \cdots & x_{mm} \end{pmatrix}$$

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This is good because the full Gaussian random matrix is now asymptotically free from deterministic matrices, in particular from the projection matrix.

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Solution

Rectangular matrices can in principle be treated within usual free probability by introducing additional projections and cutting down square matrices to the wanted rectangular ones.
For a systematic treatment of this use the theory of

rectangular free probability

of Benaych-Georges.

Section 3

Wigner Matrices Versus Gaussian Matrices

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A crucial ingredient in most calculations from the free probability side is the asymptotic freeness between Gaussian random matrices and deterministic matrices.

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Problem

However, in applications, quite often one does not have Gaussian matrices but just Wigner matrices (i.e., entries are i.i.d., but their distribution is not necessarily Gaussian).

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The main problem is that this question was neglected for quite a while in the free probability community and explicit references to this are only very recent, and still not easy to localize:

- book of Anderson, Guionnet, Zeitouni: Theorem 5.4.5
- book of Mingo, Speicher (to appear)

Section 4

Asymptotics Versus Deterministic Equivalents

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Quite often, one has a random matrix problem for (large) size N, but the limit $N \to \infty$ is not adequate, because there is no canonical limit for some of the involved matrices

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Solution (Girko; Couillet, Hoydis, Debbah; Hachem, Loubaton, Najim)

Deterministic Equivalent: Replace the random Stieltjes transform g_N of the problem for N by a deterministic transform $\tilde{g}_N$ such that

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Deterministic Equivalent: Replace the random Stieltjes transform g_N of the problem for N by a deterministic transform $\tilde{g}_N$ such that

- $\tilde{g}_N$ is calculable, usually as the fixed point solution of some system of equations
- the difference between g_N and $\tilde{g}_N$ goes, for $N \to \infty$, to 0 (even though g_N itself might not converge)

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Deterministic Equivalent

- Replace the original unsolvable problem by another problem which is
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  - close to the original problem (at least for large N)

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- Essentially one tries to close the system of equations for the Stieltjes transforms by keeping as much data as possible of the original situation
- Replacement and solving is done in one step

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Free Deterministic Equivalent (Speicher, Vargas)

- We will replace the original problem by another one on the level of operators in a quite precise way, essentially by prescribing
  - replace Gaussian random matrices by semicircular variables
  - replace matrices which are asymptotically free by free variables

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- The free deterministic equivalent is then a well-defined function in free variables
- That the free deterministic equivalent is close to the original model (for large N) is essentially the same calculation as showing asymptotic freeness
- One can then try to solve for the distribution of this replacement in a second step

Free Deterministic Equivalent (Speicher, Vargas)

Example

Consider $A_N = T_N + X_N$ where

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We replace now A_N by $a_N = t_N + s$, where

- s is a semicircular element
- t_N is an operator which has the same distribution as T_N
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In this case, the distribution of a_N is given by the free convolution of the distribution of t_N and the distribution of s,

$$\mu_{A_N} \sim \mu_{a_N} = \mu_{t_N+s} = \mu_{t_N} \boxplus \mu_s = \mu_{T_N} \boxplus \mu_s$$

Section 5

Scalar-Valued Versus Operator-Valued Free Probability

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Can We Calculate Free Deterministic Equivalents?

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For this, use the combination of

- the linearization trick
- and recent advances on the analytic description of operator-valued free convolution

The Linearization Philosophy

In order to understand polynomials in non-commuting variables, it suffices to understand matrices of **linear** polynomials in those variables.

History (in operator algebras)

- Voiculescu 1987: motivation
- Haagerup, Thorbjørnsen 2005: largest eigenvalue
- Anderson 2012: the selfadjoint version ("Schur complement")

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History (in other fields)

The same idea has been used in other fields under different names (like "descriptor system" in control theory), for example:

- Schützenberger 1961: automata theory
- Helton, McCullough, Vinnikov 2006: symmetric descriptor realization

Definition

Consider a polynomial p in non-commuting variables x and y.
A **linearization** of p is an $N \times N$ matrix (with $N \in \mathbb{N}$) of the form

$$\hat{p} = \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix},$$

- u, v, Q are matrices of the following sizes: u is $1 \times (N-1)$; v is $(N-1) \times N$; and Q is $(N-1) \times (N-1)$
- u, v, Q are polynomials in x and y, each of degree ≤ 1
- Q is invertible and we have $p = -uQ^{-1}v$

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Theorem (Schützenberger; Helton, McCullough, Vinnikov; Anderson)

- For each p there exists a linearization $\hat{p}$ (with an explicit algorithm for finding those)
- If p is selfadjoint, then this $\hat{p}$ is also selfadjoint

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Example

A selfadjoint linearization of

$$p = xy + yx + x^2 \qquad \text{is} \qquad \hat{p} = \begin{pmatrix} 0 & x & \frac{x}{2} + y \\ x & 0 & -1 \\ \frac{x}{2} + y & -1 & 0 \end{pmatrix}$$

because we have

$$\begin{pmatrix} x & \frac{x}{2} + y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} x \\ \frac{x}{2} + y \end{pmatrix} = -(xy + yx + x^2)$$

What is a Linearization Good for?

We have then

$$\hat{p} = \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix} = \begin{pmatrix} 1 & uQ^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q^{-1}v & 1 \end{pmatrix}$$

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and thus (under the condition that Q is invertible):

$$p \text{ invertible} \qquad \Longleftrightarrow \qquad \hat{p} \text{ invertible}$$

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What is a Linearization Good for?

More general, for $z \in \mathbb{C}$ put $b = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$ and then

$$b - \hat{p} = \begin{pmatrix} z & -u \\ -v & -Q \end{pmatrix} = \begin{pmatrix} 1 & uQ^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z - p & 0 \\ 0 & -Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q^{-1}v & 1 \end{pmatrix}$$

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$$z - p \text{ invertible} \qquad \Longleftrightarrow \qquad b - \hat{p} \text{ invertible}$$

and actually

$$(b-\hat{p})^{-1} = \left[\begin{pmatrix} 1 & uQ^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z-p & 0 \\ 0 & -Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q^{-1}v & 1 \end{pmatrix} \right]^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ -Q^{-1}v & 1 \end{pmatrix} \begin{pmatrix} (z-p)^{-1} & 0 \\ 0 & -Q^{-1} \end{pmatrix} \begin{pmatrix} 1 & -uQ^{-1} \\ 0 & 1 \end{pmatrix}$$

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$$= \begin{pmatrix} (z-p)^{-1} & -(z-p)^{-1}uQ^{-1} \\ -Q^{-1}v(z-p)^{-1} & Q^{-1}v(z-p)^{-1}uQ^{-1} - Q^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (z-p)^{-1} & * \\ * & * \end{pmatrix}$$

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$$= \begin{pmatrix} (z-p)^{-1} & * \\ * & * \end{pmatrix}$$

and we can get the Cauchy transform $G_p(z) = \varphi((z-p)^{-1})$ of p as the (1,1)-entry of the matrix-valued Cauchy-transform of $\hat{p}$

$$G_{\hat{p}}(b) = \text{id} \otimes \varphi((b-\hat{p})^{-1}) = \begin{pmatrix} \varphi((z-p)^{-1}) & \cdots \\ \cdots & \cdots \end{pmatrix}$$

Why is $\hat{p}$ better than p?

The selfadjoint linearization $\hat{p}$ is now the sum of two selfadjoint operator-valued variables

$$\hat{p} = \hat{x} + \hat{y} = \begin{pmatrix} 0 & x & \frac{x}{2} \\ x & 0 & 0 \\ \frac{x}{2} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & -1 \\ y & -1 & 0 \end{pmatrix}$$

where

- we know the operator-valued distribution of $\hat{x}$ and the operator-valued distribution of $\hat{y}$
- and $\hat{x}$ and $\hat{y}$ are operator-valued freely independent!

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- and $\hat{x}$ and $\hat{y}$ are operator-valued freely independent!

This is now a problem about operator-valued free convolution. This we can do.

Analytic Description of Operator-Valued Free Convolution

Definition

Consider an operator-valued probability space $E : \mathcal{A} \to \mathcal{B}$.
For a random variable $x \in \mathcal{A}$, we define the **operator-valued Cauchy transform**:

$$G(b) := E[(b - x)^{-1}] \qquad (b \in \mathcal{B}).$$

For $x = x^*$, this is well-defined and a nice analytic map on the operator-valued upper halfplane:

$$\mathbb{H}^+(\mathcal{B}) := \{b \in \mathcal{B} \mid \frac{b - b^*}{2i} > 0\}$$

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Subordination Formulation

Theorem (Belinschi, Mai, Speicher 2013)

Let x and y be selfadjoint operator-valued random variables free over $\mathcal{B}$. Then there exists a Fréchet analytic map $\omega\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ so that

$$G_{x+y}(b) = G_x(\omega(b)) \text{ for all } b \in \mathbb{H}^+(\mathcal{B}).$$

Moreover, if $b \in \mathbb{H}^+(\mathcal{B})$, then $\omega(b)$ is the unique fixed point of the map

$$f_b\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B}), \quad f_b(w) = h_y(h_x(w) + b) + b,$$

and

$$\omega(b) = \lim_{n\to\infty} f_b^{\circ n}(w) \qquad \text{for any } w \in \mathbb{H}^+(\mathcal{B}).$$

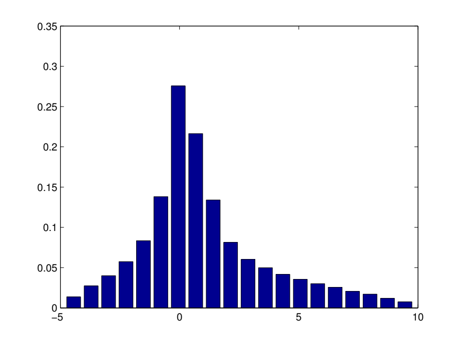
where

$$\mathbb{H}^+(\mathcal{B}) := \{b \in \mathcal{B} \mid \frac{b - b^*}{2i} > 0\}, \qquad h(b) := \frac{1}{G(b)} - b$$

Example

$P(X,Y) = XY + YX + X^2$

for independent X, Y; X is Gaussian and Y is Wishart



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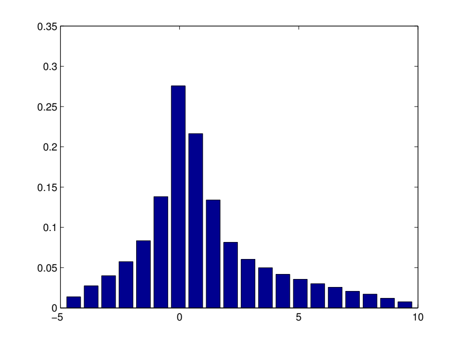

$p(x,y) = xy + yx + x^2$

for free x, y; x is semicircular and y is Marchenko-Pastur

Example

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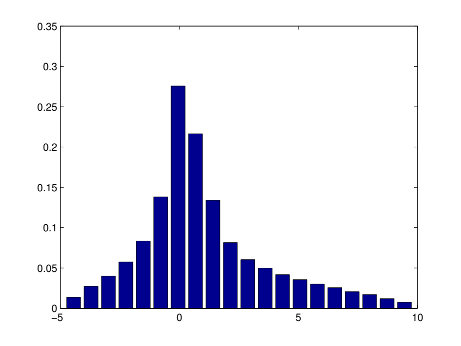
$$\hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$



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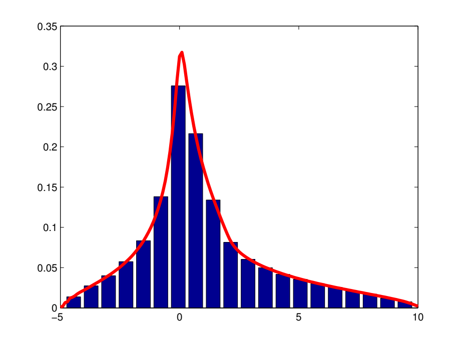

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Can We Calculate Free Deterministic Equivalents?

Problem

Usually, our free deterministic equivalents are polynomials in free variables. Can we calculate their distribution out of the knowledge of the distribution of each variable?

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Yes, we can!

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This yields:

- some fixed point equation for the wanted Stieltjes transform
- the uniqueness of the solution of this fixed point equations, within the class of Stieltjes transforms, is given by our analytic theory of operator-valued free convolution

Section 6

Conclusion

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Literature

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