Absence of Algebraic Relations and of Zero Divisors Under the Assumption of Finite Non-Microstates Free Fisher Information

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(joint work with Tobias Mai and Moritz Weber)

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Section 1

Motivation

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- free entropy $\chi(x_1, \dots, x_n)$, $\chi^*(x_1, \dots, x_n)$
- free Fisher information $\Phi^*(x_1, \dots, x_n)$
- more refined free dimension versions $\delta(x_1, \dots, x_n)$, $\delta^*(x_1, \dots, x_n)$

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We expect that

- finiteness of χ or Φ^*, or
- $\delta^{(*)}(x_1, \dots, x_n) = n$

correspond to "smoothness properties" of $x_1, \dots, x_n$ – roughly, they should behave like generators of the free group

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So for such $x_1, \ldots, x_n$ we might think of

- $\mathrm{vN}(x_1, \ldots, x_n)$ as a free group factor $L(\mathbb{F}_n)$
- $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ as the group algebra $\mathbb{C}\mathbb{F}_n$

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and, in particular, we expect

- there are no non-trivial polynomial relations between the $x_1, \ldots, x_n$, i.e., for any $P \in \mathbb{C}\langle X_1, \ldots, X_n \rangle$:

$$P(x_1, \ldots, x_n) = 0 \qquad \Longrightarrow \qquad P = 0$$

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- there are no non-trivial zero-divisors, i.e., for any $p = P(x_1, \dots, x_n) \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ and any $w \in \mathrm{vN}(x_1, \dots, x_n)$ we have:

$$pw = 0 \qquad \Longrightarrow \qquad p = 0 \quad \text{or} \quad w = 0$$

A random matrix perspective on zero-divisors

$P(x_1, \dots, x_n)$ arises quite canonically, in distribution, as the limit of polynomials in n random matrices; the absence of zero-divisors means in particular, that this asymptotic eigenvalue distribution has no atoms

Problem

When do we expect no atoms for the limit eigenvalue distribution of polynomials in matrices?

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- A canonical conjecture was that this happens, when the matrices are asymptotically free and each variable is "smooth" (e.g., semicircular). This was proved recently by Shlyakhtenko and Skoufranis.

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Remark

- A canonical conjecture was that this happens, when the matrices are asymptotically free and each variable is "smooth" (e.g., semicircular). This was proved recently by Shlyakhtenko and Skoufranis.
- We will, however, show that the relevant feature is not freeness, but finiteness of free Fisher information.

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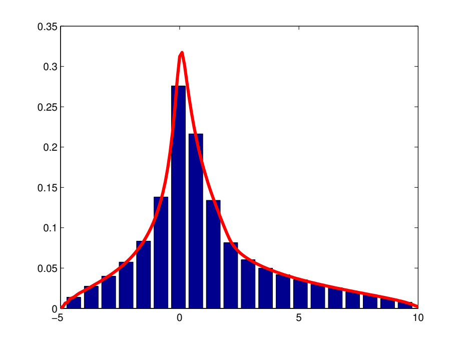
Example

$P(X,Y) = XY + YX + X^2$

for independent X, Y; X is Gaussian and Y is Wishart



$p(x,y) = xy + yx + x^2$

for free x, y; x is semicircular and y is Marchenko-Pastur

Section 2

Absence of algebraic relations

Non-commutative derivatives

Let $\mathbb{C}\langle X_1, \dots, X_n \rangle$ be the $*$-algebra of non-commutative polynomials in n self-adjoint (formal) variables $X_1, \dots, X_n$.

For $j = 1, \dots, n$, the non-commutative derivative with respect to X_j

$$\partial_j : \mathbb{C}\langle X_1, \dots, X_n \rangle \to \mathbb{C}\langle X_1, \dots, X_n \rangle \otimes \mathbb{C}\langle X_1, \dots, X_n \rangle$$

is a derivation in the sense that

$$\partial_j(P_1 P_2) = (\partial_j P_1)(1 \otimes P_2) + (P_1 \otimes 1)(\partial_j P_2),$$

which is uniquely determined by $\partial_j X_i = \delta_{i,j} 1 \otimes 1$ for $i = 1, \dots, n$.

More explicitly, ∂_j acts on monomials P as

$$\partial_j P = \sum_{P = P_1 X_j P_2} P_1 \otimes P_2.$$

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Conjugate variables

Let (M,τ) be a W^*-probability space with a faithful normal tracial state τ.

Definition

Let $x_1,\dots,x_n \in M$ be self-adjoint and $\xi_1,\dots,\xi_n \in L^2(M,\tau)$.

- If, for $j=1,\dots,n$ and for all $P \in \mathbb{C}\langle X_1,\dots,X_n\rangle$,

$$(\tau\otimes\tau)((\partial_j P)(x_1,\dots,x_n)) = \tau(\xi_j P(x_1,\dots,x_n))$$

  holds, we say that

  $(\xi_1,\dots,\xi_n)$ *satisfies the conjugate relations for* $(X_1,\dots,X_n)$.
- If, in addition, $\xi_1,\dots,\xi_n \in L^2(X_1,\dots,X_n,\tau)$, we say that

  $(\xi_1,\dots,\xi_n)$ is the conjugate system for $(X_1,\dots,X_n)$.

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Free Fisher information

Definition (Voiculescu, 1998)

Let (M, τ) be a tracial W^*-probability space and let self-adjoint elements $x_1, \dots, x_n \in M$ be given.
We define their (non-microstates) free Fisher information $\Phi^*(x_1, \dots, x_n)$

- by
$$\Phi^*(x_1, \dots, x_n) := \sum_{j=1}^{n} \|\xi_j\|_2^2$$
if a conjugate system $(\xi_1, \dots, \xi_n)$ for $(x_1, \dots, x_n)$ exists.
- by
$$\Phi^*(x_1, \dots, x_n) := \infty,$$
otherwise.

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The power of having no algebraic relations

Note: Usually it is assumed from the very beginning that $x_1, \dots, x_n$ do not satisfy any algebraic relation.

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- $\partial_j : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathbb{C}\langle x_1, \dots, x_n \rangle \otimes \mathbb{C}\langle x_1, \dots, x_n \rangle$ can also be defined as an (unbounded) operator on $L^2(x_1, \dots, x_n, \tau)$

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- the condition for the conjugate system $(\xi_1, \dots, \xi_n)$ can then be stated as

$$\xi_i = \partial_i^* 1 \otimes 1$$

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- the fact that $1 \otimes 1 \in \operatorname{dom} \partial_i^*$ implies that

$$\mathbb{C}\langle x_1, \ldots, x_n\rangle \otimes \mathbb{C}\langle x_1, \ldots, x_n\rangle \in \operatorname{dom} \partial_i^*,$$

and thus that ∂_i^* is densely defined, i.e., ∂_i is closable

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More elaborated investigations usually rely on this nice analytic properties, i.e., the absence of algebraic relations between $x_1, \dots, x_n$ is crucial

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... initiated by writing a book with Jamie Mingo and trying to get the basics on conjugate variables and free Fisher information right ....

Theorem (Mai, Speicher, Weber, 2014)

If there are elements $\xi_1, \dots, \xi_n \in L^2(M, \tau)$, such that $(\xi_1, \dots, \xi_n)$ satisfies the conjugate relations for $(x_1, \dots, x_n)$, i.e.

$$(\tau \otimes \tau)((\partial_j P)(x_1, \dots, x_n)) = \tau(\xi_j P(x_1, \dots, x_n)), \qquad j = 1, \dots, n$$

for any $P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, then the following holds true:

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for any $P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, then the following holds true:

(a) For $j = 1, \dots, n$, there is a unique derivation

$$\hat{\partial}_j : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathbb{C}\langle x_1, \dots, x_n \rangle \otimes \mathbb{C}\langle x_1, \dots, x_n \rangle$$

which satisfies $\hat{\partial}_j(x_i) = \delta_{j,i} 1 \otimes 1$ for $i = 1, \dots, n$.

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which satisfies $\hat{\partial}_j(x_i) = \delta_{j,i} 1 \otimes 1$ for $i = 1, \dots, n$.

(b) $x_1, \dots, x_n$ do not satisfy any non-trivial algebraic relation, i.e. there exists no non-zero $P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$ such that $P(x_1, \dots, x_n) = 0$.

Proof.

(a) to show: $P(x_1,\dots,x_n)=0 \quad \Longrightarrow \quad (\partial_j P)(x_1,\dots,x_n)=0.$

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Consider P with $P(x)=0$. For all $P_1,P_2\in\mathbb{C}\langle X_1,\dots,X_n\rangle$, we have then

$$0=\tau(\xi_j\cdot(P_1PP_2)(x))=(\tau\otimes\tau)([\partial_j(P_1PP_2)](x))$$

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Now

$$\begin{aligned}[\partial_j (P_1 P P_2)](x) &= (\partial_j P_1)(x) \cdot 1 \otimes (PP_2)(x) \\ &\quad + P_1(x) \otimes 1 \cdot (\partial_j P)(x) \cdot 1 \otimes P_2(x) + (P_1 P)(x) \otimes 1 \cdot (\partial_j P_2)(x) \\ &= P_1(x) \otimes 1 \cdot (\partial_j P)(x) \cdot 1 \otimes P_2(x).\end{aligned}$$

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$$\begin{aligned}[\partial_j(P_1PP_2)](x)&=(\partial_j P_1)(x)\cdot 1\otimes(PP_2)(x)\\ &\quad+P_1(x)\otimes 1\cdot(\partial_j P)(x)\cdot 1\otimes P_2(x)+(P_1P)(x)\otimes 1\cdot(\partial_j P_2)(x)\\ &=P_1(x)\otimes 1\cdot(\partial_j P)(x)\cdot 1\otimes P_2(x).\end{aligned}$$

Thus

$$(\tau\otimes\tau)[\sum_k P_1^{(k)}(x)\otimes 1\cdot(\partial_j P)(x)\cdot 1\otimes P_2^{(k)}(x)]=0$$

for all $\sum_k P_1^{(k)}\otimes P_2^{(k)}\in\mathbb{C}\langle X_1,\dots,X_n\rangle\otimes\mathbb{C}\langle X_1,\dots,X_n\rangle$. □

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$$\Delta_j := (\tau \otimes \mathrm{id}) \circ \hat{\partial}_j : \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}\langle x_1, \ldots, x_n \rangle.$$

$$P(x_1, \ldots, x_n) = 0 \quad \Longrightarrow \quad \forall j = 1, \ldots, n : \; (\Delta_j P)(x_1, \ldots, x_n) = 0$$

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$$P(x_1, \dots, x_n) = 0 \quad \Longrightarrow \quad \forall j = 1, \dots, n : (\Delta_j P)(x_1, \dots, x_n) = 0$$

Take $0 \neq P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$ such that $P(x_1, \dots, x_n) = 0$ holds. Write

$$P(X_1, \dots, X_n) = a_0 + \sum_{k=1}^{d} \sum_{i_1, \dots, i_k = 1}^{n} a_{i_1, \dots, i_k} X_{i_1} \dots X_{i_k},$$

where $d \geq 1$ denotes the total degree of P. Then

$$a_{i_1, \dots, i_d} = (\Delta_{i_d} \dots \Delta_{i_1} P)(x_1, \dots, x_n) = 0.$$

Remark

- So instead of assuming algebraic freeness and then working with free Fisher information, we define the latter in general and can use it to decide on algebraic freeness.

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- So instead of assuming algebraic freeness and then working with free Fisher information, we define the latter in general and can use it to decide on algebraic freeness.
- According to an earlier result of Dabrowski (proven under the additional assumption of algebraic freeness), we conclude that $\Phi^*(x_1, \ldots, x_n) < \infty$ also excludes *analytic relations* in the sense that there is no non-commutative power series $P \neq 0$ on a polydisk

$$D_R := \{(y_1, \ldots, y_n) \in M^n | \, \forall j = 1, \ldots, n : \; \|y_j\| < R\}, \quad R > 0,$$

such that $(x_1, \ldots, x_n) \in D_R$ and $P(x_1, \ldots, x_n) = 0$.

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Section 3

Absence of zero divisors

Let (M, τ) be a tracial W^*-probability space. Consider self-adjoint $x_1, \dots, x_n \in M$ and assume that

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Let $P \neq 0$ be a non-commutative polynomial.

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Let $P \neq 0$ be a non-commutative polynomial.

Theorem (Mai, Speicher, Weber, 2014)

There exists no $w = w^* \in \text{vN}(x_1, \dots, x_n)$, $w \neq 0$, such that

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Corollary (Mai, Speicher, Weber, 2014)

The distribution $\mu_{P(x_1,\dots,x_n)}$ of $P(x_1, \dots, x_n)$, for P not a constant, does not have atoms.

Recall that the *distribution* μ_y of $y = y^* \in M$ is determined by

$$\tau(y^k) = \int_{\mathbb{R}} t^k \, d\mu_y(t) \qquad \text{for } k = 0, 1, 2, \dots.$$

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The key observation

Proposition

Let $p = P(x_1, \dots, x_n) \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ be given. For any

$$u = u^*, v = v^* \in \mathrm{vN}(x_1, \dots, x_n),$$

the following implication holds true:

$$pu = 0 \quad \text{and} \quad vp = 0 \implies \forall i = 1, \dots, n: \; v \otimes 1 \cdot \partial_i p \cdot 1 \otimes u = 0.$$

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This follows formally by differentiating $0 = vpu$:

$$0 = \partial_i(vpu) = \partial_i v \cdot 1 \otimes pu + v \otimes 1 \cdot \partial_i p \cdot 1 \otimes u + vp \otimes 1 \cdot \partial_i u$$

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However, v and p are in general not in the domain of ∂_i, so this has to be justified by approximating u and v by polynomials and controling the norms of the derivatives of the approximations.

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This relies on

- Voiculescu's formulas for the action of ∂_j^*: for $Q \in \mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$ we have

$$\partial_j^*(Q) = m_{\xi_j}(Q) - m_1(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\partial_j \otimes \mathrm{id} + \mathrm{id} \otimes \partial_j)(Q),$$

where, for $\eta \in L^2(M,\tau)$, we define $m_\eta : M \otimes M \to L^2(M,\tau)$ by

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- norm estimates of Dabrowski:

$$\|(\mathrm{id} \otimes \tau)(\partial_j P)\|_2 \leq 2\|\xi_j\|_2\|P\|,$$
$$\|(\tau \otimes \mathrm{id})(\partial_j P)\|_2 \leq 2\|\xi_j\|_2\|P\|.$$

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Theorem (Mai, Speicher, Weber, 2014)

Consider polynomial $P \neq 0$. There exists no $w = w^* \in \mathrm{vN}(x_1, \dots, x_n)$, $w \neq 0$, such that

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But: if $qw = 0$ there is also a non-trivial v with $vq = 0$. Then we have $vqw = 0$ and we can continue our reduction. □

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Thank you!

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