NONCOMMUTATIVE MAXIMAL ERGODIC INEQUALITIES ASSOCIATED WITH DOUBLING CONDITIONS

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ABSTRACT. This paper is devoted to the study of noncommutative maximal inequalities and ergodic theorems for group actions on von Neumann algebras. Consider a locally compact group G of polynomial growth with a symmetric compact subset V. Let α be a continuous action of G on a von Neumann algebra $\mathcal{M}$ by trace-preserving automorphisms. We then show that the operators defined by

$$A_n x = \frac{1}{m(V^n)} \int_{V^n} \alpha_g x \, dm(g), \quad x \in L_p(\mathcal{M}), n \in \mathbb{N}, 1 \le p \le \infty$$

is of weak type $(1,1)$ and of strong type (p,p) for $1 < p < \infty$. Consequently, the sequence $(A_n x)_{n \ge 1}$ converges almost uniformly for $x \in L_p(\mathcal{M})$ for $1 \le p < \infty$. Also we establish the noncommutative maximal and individual ergodic theorems associated with more general doubling conditions; and we prove the corresponding results for general actions on one fixed noncommutative L_p-space which are beyond the class of Dunford-Schwartz operators considered previously by Junge and Xu. As key ingredients, we also obtain the Hardy-Littlewood maximal inequality on metric spaces with doubling measures in the operator-valued setting. After the groundbreaking work of Junge and Xu on the noncommutative Dunford-Schwartz maximal ergodic inequalities, this is the first time that more general maximal inequalities are proved beyond Junge-Xu's setting. Our approach is based on the quantum probabilistic methods as well as the random walk theory.

1. INTRODUCTION

This paper studies maximal inequalities and ergodic theorems for group actions on noncommutative L_p-spaces. The connection between ergodic theory and von Neumann algebras goes back to the very beginning of the theory of operator algebras. However, the study of individual ergodic theorems in the noncommutative setting only took off with Lance's pioneering work [Lan76] in 1976. The topic was then extensively investigated in a series of works of Conze, Dang-Ngoc, Kümmerer, Yeadon and others (see [CDN78, Küm78, Yea77, Jaj85] and the references therein). Among them, Yeadon [Yea77] obtained a maximal ergodic theorem in the preduals of semifinite von Neumann algebras. But the corresponding maximal inequalities in L_p-spaces remained open until the celebrated work of Junge and Xu [JX07], which established the noncommutative analogue of the Dunford-Schwartz maximal ergodic theorem. This breakthrough motivates further research on noncommutative ergodic theorems, such as [AD06, Hu08, Bek08, Lit14, HS16]. Note that all these works essentially remain in the class of Dunford-Schwartz operators, that is, do not go beyond Junge-Xu's setting.

On the other hand, in the classical ergodic theory, a number of significant developments related to individual ergodic theorems for group actions have been established in the past years. In particular, Breuillard [Bre14] and Tessera [Tes07] studied the balls in groups of polynomial growth; they proved that for any invariant metric quasi-isometric to a word metric (such as invariant Riemannian metrics on connected nilpotent Lie groups), the balls are asymptotically invariant and satisfy the doubling condition, and hence satisfy the individual ergodic theorem. This settled a long-standing problem in ergodic theory since Calderón's classical paper [Cal53] in 1953. Also, Lindenstrauss [Lin01] proved the individual ergodic theorem for a tempered Følner sequences,

which resolves the problem of constructing pointwise ergodic sequences on an arbitrary amenable group. We refer to the survey paper [Nev06] for more details.

Thus it is natural to extend Junge-Xu's work to actions of more general amenable groups rather than the integer group. As in the classical case, the first natural step would be to establish the maximal ergodic theorems for doubling conditions. However, since we do not have an appropriate analogue of covering lemmas in the noncommutative setting, no significant progress has been made in this direction. In this paper we provide a new approach to this problem. This approach is based on both classical and quantum probabilistic methods, and allows us to go beyond the class of Dunford-Schwartz operators considered by Junge and Xu.

Our main results establish the noncommutative maximal and individual ergodic theorems for ball averages under the doubling condition. Let G be a locally compact group equipped with a right Haar measure m. Recall that for an invariant metric[1] d on G, we say that (G, d) satisfies the *doubling condition* if the balls $B_r := \{g \in G : d(g, e) \leq r\}$ satisfy

$$(1.1) \qquad m(B_{2r}) \leq Cm(B_r), \quad r > 0,$$

where C is a constant independent of r. We say that the balls are *asymptotically invariant* under right translation if for every $g \in G$,

$$(1.2) \qquad \lim_{r \to \infty} \frac{m((B_r g) \triangle B_r)}{m(B_r)} = 0,$$

where $\triangle$ denotes the usual symmetric difference of subsets. To state the noncommutative ergodic theorems, we consider a von Neumann algebra $\mathcal{M}$ equipped with a normal semifinite trace τ. We also consider an action α of G on the associated noncommutative L_p-spaces $L_p(\mathcal{M})$, under some mild assumptions clarified in later sections. In particular, if α is a continuous action of G on $\mathcal{M}$ by τ-preserving automorphisms of $\mathcal{M}$, then α extends to isometric actions on the spaces $L_p(\mathcal{M})$. The following is one of our main results:

Theorem 1.1. *Assume that (G, d) satisfies* (1.1) *and* (1.2). *Let α be a continuous action of G on $\mathcal{M}$ by τ-preserving automorphisms. Let A_r be the averaging operators*

$$A_r x = \frac{1}{m(B_r)} \int_{B_r} \alpha_g x\, dm(g), \quad x \in \mathcal{M}, r > 0.$$

Then $(A_r)_{r>0}$ is of weak type $(1, 1)$ and of strong type (p, p) for $1 < p < \infty$. Moreover for all $1 \leq p < \infty$, the sequence $(A_r x)_{r>0}$ converges almost uniformly for $x \in L_p(\mathcal{M})$.

Here we refer to Section 2.1 for the notion of weak and strong type (p, p) inequalities in the noncommutative setting. Also, the notion of almost uniform convergence is a noncommutative analogue of the notion of almost everywhere convergence. We refer to Definition 6.1 for the relevant definitions.

There exist a number of examples satisfying assumption (1.1) and (1.2) of the above theorem, for which we refer to [Bre14, Tes07, Nev06] as is quoted before. In particular, if we take G to be the integer group $\mathbb{Z}$ and d to be the usual word metric, then we recover the usual ergodic average $A_n = \frac{1}{2n+1} \sum_{k=-n}^{n} T^k$ for an invertible operator T, as is treated in [JX07]. More generally, we may consider groups of polynomial growth:

Theorem 1.2. *Assume that G is generated by a symmetric compact subset V and is of polynomial growth.*

(1) Fix $1 < p < \infty$. Let α be a strongly continuous and uniformly bounded action of G on $L_p(\mathcal{M})$ such that α_g is a positive map for each $g \in G$. Then the operators defined by

$$A_n x = \frac{1}{m(V^n)} \int_{V^n} \alpha_g x\, dm(g), \quad x \in L_p(\mathcal{M}), n \in \mathbb{N}$$

is of strong type (p, p). The sequence $(A_n x)_{n \geq 1}$ converges bilaterally almost uniformly for $x \in L_p(\mathcal{M})$.

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[1]In this paper we always assume that d is a measurable function on $G \times G$ and m is a Radon Borel measure with respect to (G, d).

(2) *Let α be a strongly continuous action of G on $\mathcal{M}$ by τ-preserving automorphisms. Then the operators defined by*

$$A_n x = \frac{1}{m(V^n)} \int_{V^n} \alpha_g x \, dm(g), \quad x \in \mathcal{M}, n \in \mathbb{N}$$

is of weak type $(1,1)$ and of strong type (p,p) for all $1 < p < \infty$. The sequence $(A_n x)_{n \geq 1}$ converges almost uniformly for $x \in L_p(\mathcal{M})$ for all $1 \leq p < \infty$.

The theorems rely on several key results obtained in this paper. The subjects that we address are as follows:

i) *Noncommutative transference principles.* Our first key ingredient is a noncommutative variant of Calderón's transference principle [Cal68, CW76, Fen98], given in Theorem 3.1 and Theorem 3.3. More precisely, we prove that for actions by an amenable group, in order to establish the noncommutative maximal ergodic inequalities, it suffices to show the inequalities for translation actions on operator-valued functions. We remark that the particular case of certain actions by $\mathbb{R}$ is also discussed in [Hon17] by the first author.

ii) *Noncommutative Hardy-Littlewood maximal inequalities on metric measure spaces.* As for the second key ingredient, we prove in Theorem 4.1 a noncommutative extension of Hardy-Littlewood maximal inequalities on metric measure spaces. For a doubling metric measure space (X, d, μ), denote by $B(x, r)$ the ball with center x and radius r with respect to the metric d. Our result asserts that the Hardy-Littlewood averaging operators on the $L_p(\mathcal{M})$-valued functions

$$A_r f(x) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu, \quad f \in L_p(X; L_p(\mathcal{M})), x \in X, r > 0$$

satisfy the weak type $(1,1)$ and strong type (p,p) inequalities. We remark that the classical argument via covering lemmas does not seem to fit into this operator-valued setting. Instead, our approach is based on the study of random dyadic systems by Naor, Tao [NT10] and Hytönen, Kaimera [HK12]. The key idea is to relate the desired inequality to noncommutative martingales, and to use the available results in quantum probability developed in [Cuc71, Jun02]. The approach is inspired by Mei's famous work [Mei03, Mei07] which asserts that the usual continuous BMO space is the intersection of several dyadic BMO spaces.

iii) *Domination by Markov operators.* In the study of ergodic theorems for actions by free groups or free abelian groups, it is a key fact that the associated ergodic averages can be dominated by the standard averaging operators of the form $\frac{1}{n} \sum_{k=1}^{n} T^k$ for some map T (see [Bru73, NS94]). Also, in [SS83], Stein and Strömberg apply the Markov semigroup with heat kernels to estimate the maximal inequalities on Euclidean spaces with large dimensions. In this paper, we build a similar result for groups of polynomial growth. Our approach is new and the construction follows easily from some typical Markov chains on these groups. More precisely, we show in Proposition 4.8 that for a group G of polynomial growth with a symmetric compact generating subset $V \subset G$, and for an action α of G, there exists a constant c such that

$$\frac{1}{m(V^n)} \int_{V^n} \alpha_g x \, dm(g) \leq \frac{c}{n^2} \sum_{k=1}^{2n^2} T^k x, \quad x \geq 0,$$

where $T = \frac{1}{m(V)} \int_V \alpha_g \, dm(g)$. The result will help us to improve the weak type inequalities in Theorem 1.2.

iv) *Individual ergodic theorems for L_p representations.* In the classical setting, the individual ergodic theorem holds for positive contractions on L_p-spaces with one fixed $p \in (1, \infty)$ ([IT64, Akc75]). The results can be also generalized for positive power-bounded operators and more general Lamperti operators (see for example [MRDlT88, Kan78, Tem15]). However in the noncommutative setting, the individual ergodic theorems on L_p-spaces were only known for operators which can be extended to $L_1 + L_\infty$. In this paper we will develop in Section 6 some new methods to prove the individual ergodic theorems for operators on one fixed L_p-space.

Apart from the above approach, we also provide in Section 5 an alternative proof of Theorem 1.1 for discrete groups of polynomial growth. Compared to the previous approach, this proof is much

more group-theoretical and has its own interests. It relies essentially on the concrete structure of groups of polynomial growth discovered by Bass, Gromov and Wolf.

We remark that although our results are stated in the setting of tracial L_p-spaces, a large number of the results can be extended to the general non-tracial case without difficulty. Since the standard methods for these generalizations are already well developed in [JX07, HJX10], we would like to leave the details to the reader, and restrict to the semifinite case for simplicity of exposition.

We end this introduction with a brief description of the organization of the paper. In the next section we recall some basics on noncommutative maximal operators as well as actions by amenable groups. Section 3 is devoted to the proof of the noncommutative Calderón transference principle. In Section 4 we prove the Hardy-Littlewood maximal inequalities mentioned above, and deduce the maximal inequalities in Theorem 1.1. We also use the similar ideas to establish the ergodic theorems for increasing sequences of compact subgroups (Theorem 4.7). In the last part of the section, we will provide an approach based on the random walk theory, which relates the ball averages to the classical ergodic averages of Markov operators. In Section 5 we provide an alternative group-theoretical approach to Theorem 1.2. In Section 6 we discuss the individual ergodic theorems, which proves the bilateral almost uniform convergences in Theorem 1.1. Also we give new results on almost uniform convergences associated with actions on one fixed L_p-space.

2. Preliminaries

2.1. **Noncommutative L_p-spaces and noncommutative maximal norms.** Throughout the paper, unless explicitly stated otherwise, $\mathcal{M}$ will always denote a semifinite von Neumann algebra equipped with a normal semifinite trace τ. Let $\mathcal{S}_+$ denote the set of all $x \in \mathcal{M}_+$ such that $\tau(\text{supp}\,x) < \infty$, where $\text{supp}\,x$ denotes the support of x. Let $\mathcal{S}$ be the linear span of $\mathcal{S}_+$. Given $1 \leq p < \infty$, we define
$$\|x\|_p = \left[\tau(|x|^p)\right]^{1/p}, \quad x \in \mathcal{S},$$
where $|x| = (x^*x)^{1/2}$ is the modulus of x. Then $(\mathcal{S}, \|\cdot\|_p)$ is a normed space, whose completion is the noncommutative L_p-space associated with $(\mathcal{M}, \tau)$, denoted by $L_p(\mathcal{M})$. As usual, we set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. Let $L_0(\mathcal{M})$ denote the space of all closed densely defined operators on H measurable with respect to $(\mathcal{M}, \tau)$ (H being the Hilbert space on which $\mathcal{M}$ acts). Then $L_p(\mathcal{M})$ can be viewed as closed densely defined operators on H. We denote by $L_0^+(\mathcal{M})$ the positive part of $L_0(\mathcal{M})$, and set $L_p^+(\mathcal{M}) = L_0^+(\mathcal{M}) \cap L_p(\mathcal{M})$. We refer to [PX03] for more information on noncommutative L_p-spaces.

For a σ-finite measure space (X, Σ, μ), we consider the von Neumann algebraic tensor product $L_\infty(X)\bar{\otimes}\mathcal{M}$ equipped with the trace $\int \otimes \tau$, where $\int$ denotes the integral against μ. For $1 \leq p < \infty$, the space $L_p(L_\infty(X)\bar{\otimes}\mathcal{M})$ isometrically coincides with $L_p(X; L_p(\mathcal{M}))$, the usual L_p-space of p-integrable functions from X to $L_p(\mathcal{M})$. In this paper we will not distinguish these two notions unless specified otherwise.

Maximal norms in the noncommutative setting require a specific definition. The subtlety is that $\sup_n |x_n|$ does not make any sense for a sequence $(x_n)_n$ of arbitrary operators. This difficulty is overcome by considering the spaces $L_p(\mathcal{M}; \ell_\infty)$, which are the noncommutative analogs of the usual Bochner spaces $L_p(X; \ell_\infty)$. These vector-valued L_p-spaces were first introduced by Pisier [Pis98] for injective von Neumann algebras and then extended to general von Neumann algebras by Junge [Jun02]. The descriptions and properties below can be found in [JX07, Section 2]. Given $1 \leq p \leq \infty$, $L_p(\mathcal{M}; \ell_\infty)$ is defined as the space of all sequences $x = (x_n)_{n \geq 0}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there are $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $y = (y_n) \subset L_\infty(\mathcal{M})$ such that
$$x_n = ay_nb, \quad \forall\, n \geq 0.$$
We then define
$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\left\{\|a\|_{2p} \sup_{n \geq 0} \|y_n\|_\infty \|b\|_{2p}\right\},$$
where the infimum runs over all factorizations as above. We will adopt the convention that the norm $\|x\|_{L_p(\mathcal{M};\ell_\infty)}$ is denoted by $\left\|\sup_n^+ x_n\right\|_p$. As an intuitive description, we remark that a

positive sequence $(x_n)_{n \geq 0}$ of $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M}; \ell_\infty)$ if and only if there exists a positive $a \in L_p(\mathcal{M})$ such that $x_n \leq a$ for any $n \geq 0$ and in this case,

$$\big\| \sup_n{}^+ x_n \big\|_p = \inf \{\|a\|_p : \ a \in L_p(\mathcal{M}), \ a \geq 0 \text{ and } x_n \leq a \text{ for any } n \geq 0\}.$$

Also, we denote by $L_p(\mathcal{M}; c_0)$ the closure of finite sequences in $L_p(\mathcal{M}; \ell_\infty)$ for $1 \leq p < \infty$. On the other hand, we may also define the space $L_p(\mathcal{M}; \ell_\infty^c)$, which is the space of all sequences $x = (x_n)_{n \geq 0}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there are $a \in L_p(\mathcal{M})$ and $y = (y_n) \subset L_\infty(\mathcal{M})$ such that

$$x_n = y_n a, \quad \forall \, n \geq 0.$$

And we define

$$\|x\|_{L_p(\mathcal{M}; \ell_\infty^c)} = \inf \big\{ \sup_{n \geq 0} \|y_n\|_\infty \|a\|_p \big\},$$

where the infimum runs over all factorizations as above. Similarly, we denote by $L_p(\mathcal{M}; c_0^c)$ be the closure of finite sequences in $L_p(\mathcal{M}; \ell_\infty^c)$. We refer to [Mus03, DJ04] for more information.

Indeed, for any index I, we can define the spaces $L_p(\mathcal{M}; \ell_\infty(I))$ of families $(x_i)_{i \in I}$ in $L_p(\mathcal{M})$ with similar factorizations as above. We omit the details and we will simply denote the spaces by the same notation $L_p(\mathcal{M}; \ell_\infty)$ and $L_p(\mathcal{M}; c_0)$ if non confusion can occur.

The following properties will be of use in this paper.

Proposition 2.1. (1) A family $(x_i)_{i \in I} \subset L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M}; \ell_\infty)$ if and only if

$$\sup_{J \text{ finite}} \Big\| \sup_{i \in J}{}^+ x_i \Big\|_p \ < \ \infty,$$

and in this case

$$\Big\| \sup_{i \in I}{}^+ x_i \Big\|_p = \sup_{J \text{ finite}} \Big\| \sup_{i \in J}{}^+ x_i \Big\|_p.$$

(2) Let $1 \leq p_0 < p_1 \leq \infty$ and $0 < \theta < 1$. Then we have isometrically

$$(2.1) \qquad\qquad L_p(\mathcal{M}; \ell_\infty) = (L_{p_0}(\mathcal{M}; \ell_\infty), L_{p_1}(\mathcal{M}; \ell_\infty))_\theta,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. If additionally $p_0 \geq 2$, then we have isometrically

$$(2.2) \qquad\qquad L_p(\mathcal{M}; \ell_\infty^c) = (L_{p_0}(\mathcal{M}; \ell_\infty^c), L_{p_1}(\mathcal{M}; \ell_\infty^c))_\theta,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Based on these notions we can discuss the noncommutative maximal inequalities.

Definition 2.2. Let $1 \leq p \leq \infty$ and let $S = (S_i)_{i \in I}$ be a family of maps from $L_p^+(\mathcal{M})$ to $L_0^+(\mathcal{M})$.

(1) For $p < \infty$, we say that S is of *weak type* (p, p) *with constant* C if there exists a constant $C > 0$ such that for all $x \in L_p^+(\mathcal{M})$ and $\lambda > 0$, there is a projection $e \in \mathcal{M}$ satisfying

$$\tau(1 - e) \leq \frac{C^p}{\lambda^p} \|x\|_p^p, \quad e S_i(x) e \leq \lambda e, i \in I.$$

(2) For $1 \leq p \leq \infty$, we say that S is of *strong type* (p, p) *with constant* C if there exists a constant $C > 0$ such that

$$\Big\| \sup_{i \in I}{}^+ S_i x \Big\|_p \leq C \|x\|_p, \quad x \in L_p(\mathcal{M}).$$

We will also need a reduction below for weak type inequalities.

Lemma 2.3 ([Hon17, Lemma 3.2]). *If for all finite subset $J \subset I$, $(S_i)_{i \in J}$ is of weak type (p, p) with constant C, then $(S_i)_{i \in I}$ is of weak type (p, p) with constant $4C$.*

The following noncommutative Doob inequalities will play a crucial role in our proof.

Lemma 2.4 ([Cuc71, Jun02]). *Let $(\mathcal{M}_n)_{n \in \mathbb{Z}}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\cup_{n \in \mathbb{Z}} \mathcal{M}_n$ is w*-dense in $\mathcal{M}$. Denote by $\mathbb{E}_n$ the τ-preserving conditional expectation from $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}_n)$. Then $(\mathbb{E}_n)_{n \in \mathbb{Z}}$ is of weak type $(1, 1)$ and strong type (p, p) for all $1 < p < \infty$.*

2.2. **Actions by amenable groups.** Unless explicitly stated otherwise, throughout G will denote a locally compact group with neutral element e, equipped with a fixed right invariant Haar measure m. For a Banach space E, we say that

$$\alpha : G \to B(E), \quad g \mapsto \alpha_g$$

is an action if $\alpha_g \circ \alpha_h = \alpha_{gh}$ for all $g, h \in G$. Let $(\mathcal{M}, \tau)$ be as before. For a fixed $1 \le p \le \infty$, we will be interested in actions $\alpha = (\alpha_g)_{g \in G}$ on $L_p(\mathcal{M})$ with the following conditions:

- ($\mathbf{A}_1^p$) Continuity: for all $x \in L_p(\mathcal{M})$, the map $g \mapsto \alpha_g x$ from G to $L_p(\mathcal{M})$ is continuous. Here we take the norm topology on $L_p(\mathcal{M})$ if $1 \le p < \infty$ and the w*-topology if $p = \infty$.
- ($\mathbf{A}_2^p$) Uniform boundedness: $\sup_{g \in G} \|\alpha_g : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| < \infty$.
- ($\mathbf{A}_3^p$) Positivity : for all $g \in G$, $\alpha_g x \ge 0$ if $x \ge 0$ in $L_p(\mathcal{M})$.

As a natural example, if α is an action on $\mathcal{M}$ satisfying the condition:

- ($\mathbf{A}'$) for all $x \in \mathcal{M}$, the map $g \mapsto \alpha_g x$ from G to $\mathcal{M}$ is continuous with respect to the w*-topology on $\mathcal{M}$; and for all $g \in G$, α_g is an automorphism of $\mathcal{M}$ (in the sense of $*$-algebraic structures) such that $\tau = \tau \circ \alpha_g$,

then α extends naturally to actions on $L_p(\mathcal{M})$ with conditions ($\mathbf{A}_1^p$)-($\mathbf{A}_3^p$) for all $1 \le p \le \infty$, still denoted by α (see e.g. [JX07, Lemma 1.1]). In this case for each $g \in G$, α_g is an isometry on $L_p(\mathcal{M})$. We refer to [Oli13, Oli12, Bek15] for other natural examples of group actions on noncommutative L_p-spaces.

Recall that G is said to be *amenable* if G admits a *Følner net*, i.e., a net $(F_i)_{i \in I}$ of measurable subsets of G with $m(F_i) < \infty$ such that for all $g \in G$,

$$\lim_i \frac{m((F_i g) \triangle F_i)}{m(F_i)} = 0.$$

Note that the above condition is a reformulation of the asymptotic invariance (1.2) for the general setting. It is known that $(F_i)_{i \in I}$ is a Følner net if for all compact measurable subsets $K \subset G$,

$$(2.3) \qquad \lim_i \frac{m(F_i K)}{m(F_i)} = 1.$$

Recall that G is a compactly generated group *of polynomial growth* if the compact generating subset $V \subset G$ satifies

$$m(V^n) \le kn^r, \quad n \ge 1,$$

where $k > 0$ and $r \in \mathbb{N}$ are constants independent of n. It is well-known that any group of polynomial growth is amenable and the sequence $(V^n)_{n \ge 1}$ satisfies the above Følner condition (see e.g. [Bre14, Tes07]). We refer to [Pat88] for more information on amenable groups.

Now let G be amenable and $(F_i)_{i \in I}$ be a Følner net in G. Let $1 < p < \infty$. Let $\alpha = (\alpha_g)_{g \in G}$ be an action of G on $L_p(\mathcal{M})$ satisfying ($\mathbf{A}_1^p$)-($\mathbf{A}_3^p$). Denote by A_i the corresponding averaging operators

$$A_i x = \frac{1}{m(F_i)} \int_{F_i} \alpha_g x \, dm(g), \quad x \in L_p(\mathcal{M}).$$

According to the mean ergodic theorem for amenable groups (see e.g. [ADAB$^+$10, Théorème 2.2.7]), we have a canonical splitting on $L_p(\mathcal{M})$:

$$L_p(\mathcal{M}) = \mathcal{F}_p \oplus \mathcal{F}_p^{\perp},$$

with

$$(2.4) \qquad \mathcal{F}_p = \{x \in L_p(\mathcal{M}) : \alpha_g x = x, g \in G\}, \quad \mathcal{F}_p^{\perp} = \overline{\mathrm{span}}\{x - \alpha_g x : g \in G, x \in L_p(\mathcal{M})\}.$$

Let P be the bounded positive projection from $L_p(\mathcal{M})$ onto $\mathcal{F}_p$. Then $(A_i x)_i$ converges to Px in $L_p(\mathcal{M})$ for all $x \in L_p(\mathcal{M})$.

Assume additionally that α extends to an action on $\cup_{1 \le p \le \infty} L_p(\mathcal{M})$ satisfying ($\mathbf{A}_1^p$)-($\mathbf{A}_3^p$) for *every* $1 \le p \le \infty$. Note that the convergence in $L_p(\mathcal{M})$ yields the convergence in measure in $L_0(\mathcal{M})$, and in particular for $x \in L_1^+(\mathcal{M}) \cap \mathcal{M}_+ \cap L_2(\mathcal{M})$ and for $p_0 = 1$ or $p_0 = \infty$,

$$\|Px\|_{p_0} \le \liminf_{n \to \infty} \|A_n x\|_{p_0} \le \sup_{g \in G} \|\alpha_g\|_{B(L_{p_0}(\mathcal{M}))} \|x\|_{p_0},$$

so by [JX07, Lemma 1.1] and [Yea77, Proposition 1], P admits a continuous extension on $L_1(\mathcal{M})$ and $\mathcal{M}$, still denoted by P. The splitting (2.4) is also true in this case. Note then however that $\mathcal{F}_\infty^\perp$ is the w*-closure of the space spanned by $\{x - \alpha_g x : g \in G, x \in \mathcal{M}\}$.

3. NONCOMMUTATIVE CALDERÓN'S TRANSFERENCE PRINCIPLE

In this section we discuss a noncommutative variant of Calderón's transference principle. Fix $1 \leq p < \infty$. Let G be a locally compact group and α be an action satisfying $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$ in the previous section. Let $(\mu_n)_{n \geq 1}$ be a sequence of Radon probability measures on G. We consider the following averages

$$(3.1) \qquad A_n x = \int_G \alpha_g x \, d\mu_n(g), \quad x \in L_p(\mathcal{M}), n \geq 1.$$

Also, let us consider the natural translation action of G on itself. We are interested in the following averages: for all $f \in L_p(G; L_p(\mathcal{M}))$,

$$(3.2) \qquad A_n' f(g) = \int_G f(gh) \, d\mu_n(h), \quad g \in G, n \geq 1,$$

where the integration denotes the usual integration of Banach space valued functions.

3.1. **Strong type inequalities.** We begin with the transference principle for strong type (p,p) inequalities.

Theorem 3.1. *Assume that G is amenable. Fix $1 \leq p < \infty$. If there exists a constant $C > 0$ such that*

$$\left\| \sup_n{}^+ A_n' f \right\|_p \leq C \|f\|_p, \quad f \in L_p(G; L_p(\mathcal{M})),$$

then there exists a constant $C' > 0$ depending on α such that

$$\left\| \sup_n{}^+ A_n x \right\|_p \leq C C' \|x\|_p, \quad x \in L_p(\mathcal{M}).$$

Proof. Note that we may take an increasing net of compact subsets $K_i \subset G$ such that $\lim_i \mu_n(K_i) = \mu_n(G)$. Then for

$$A_{n,i} x = \int_G \alpha_g x \chi_{K_i}(g) \, d\mu_n(g), \quad x \in L_p(\mathcal{M}), n \geq 1,$$

we have

$$A_{n,i} x \leq A_n x, \quad \lim_{i \to \infty} \|A_n x - A_{n,i} x\|_p = 0, \quad x \in L_p^+(\mathcal{M}).$$

So for $x, y \in L_p^+(\mathcal{M})$,

$$A_n x \leq y \quad \text{iff} \quad \forall i \,, \, A_{n,i} x \leq y.$$

Hence $\|\sup_{n,i}^+ A_{n,i} x_{n,i}\|_p = \|\sup_n^+ A_n x_n\|_p$. So without loss of generality we may assume that μ_n are of compact support.

We fix $x \in L_p(\mathcal{M})$ and $N \geq 1$. Choose a compact subset $K \subset G$ such that μ_n is supported in K for all $1 \leq n \leq N$. Since $\alpha_g : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ is positive for all $g \in G$, we see that $(\alpha_g \otimes \mathrm{Id})_{g \in G}$ extends to a uniformly bounded family of maps on $L_p(\mathcal{M}; \ell_\infty)$ (see e.g. [HJX10, Proposition 7.3]). So we may choose a constant $C' > 0$ such that

$$\left\| \sup_{1 \leq n \leq N}{}^+ A_n x \right\|_p = \left\| \sup_{1 \leq n \leq N}{}^+ \alpha_{g^{-1}} \alpha_g A_n x \right\|_p \leq C' \left\| \sup_{1 \leq n \leq N}{}^+ \alpha_g A_n x \right\|_p, \quad g \in G.$$

Let F be a compact subset. Then we have

$$(3.3) \qquad \left\| \sup_{1 \leq n \leq N}{}^+ A_n x \right\|_p^p \leq C'^p \frac{1}{m(F)} \int_F \left\| \sup_{1 \leq n \leq N}{}^+ \alpha_g A_n x \right\|_p^p dm(g).$$

We define a function $f \in L_p(G; L_p(\mathcal{M}))$ as

$$f(h) = \chi_{FK}(h) \alpha_h x, \quad h \in G.$$

Then for all $g \in F$,

$$(3.4) \qquad \alpha_g A_n x = \int_K \alpha_{gh} x d\mu_n(h) = \int_K f(gh) d\mu_n(h) = A_n' f(g).$$

We consider $(A_n' f)_{1 \leq n \leq N} \in L_p(L_\infty(G) \bar{\otimes} \mathcal{M}; \ell_\infty)$, and for any $\varepsilon > 0$ we take a factorization $A_n' f = a F_n b$ such that $a, b \in L_{2p}(L_\infty(G) \bar{\otimes} \mathcal{M})$, $F_n \in L_\infty(G) \bar{\otimes} \mathcal{M}$ and

$$\|a\|_{2p} \sup_{1 \leq n \leq N} \|F_n\|_\infty \|b\|_{2p} \leq \left\| (A_n' f)_{1 \leq n \leq N} \right\|_{L_p(L_\infty(G) \bar{\otimes} \mathcal{M}; \ell_\infty)} + \varepsilon.$$

Then we have

$$\int_G \left\| \sup_{1 \leq n \leq N}^+ A_n' f(g) \right\|_p^p dm(g) \leq \int_G \|a(g)\|_{2p}^p \sup_{1 \leq n \leq N} \|F_n(g)\|_\infty^p \|b(g)\|_{2p}^p dm(g)$$

$$\leq \|a\|_{2p}^p \sup_{1 \leq n \leq N} \|F_n\|_\infty^p \|b\|_{2p}^p \leq \left(\left\| (A_n' f)_{1 \leq n \leq N} \right\|_{L_p(L_\infty(G) \bar{\otimes} \mathcal{M}; \ell_\infty)} + \varepsilon \right)^p.$$

Since ε is arbitrarily chosen, we obtain

$$\int_G \left\| \sup_{1 \leq n \leq N}^+ A_n' f(g) \right\|_p^p dm(g) \leq \left\| \sup_{1 \leq n \leq N}^+ A_n' f \right\|_p^p.$$

Thus together with (3.3), (3.4) and the assumption we see that

$$\left\| \sup_{1 \leq n \leq N}^+ A_n x \right\|_p^p \leq \frac{C'^p}{m(F)} \int_F \left\| \sup_{1 \leq n \leq N}^+ A_n' f(g) \right\|_p^p dm(g) \leq \frac{C'^p}{m(F)} \left\| \sup_{1 \leq n \leq N}^+ A_n' f \right\|_p^p$$

$$\leq \frac{C^p C'^p}{m(F)} \|f\|_p^p = \frac{C^p C'^p}{m(F)} \int_{FK} \|\alpha_h x\|_p^p dm(h)$$

$$\leq \frac{C^p C'^p m(FK)}{m(F)} \|x\|_p^p.$$

Since G is amenable, for any $\varepsilon > 0$ we may choose the above subset F such that $m(FK)/m(F) \leq 1 + \varepsilon$. Therefore we get

$$\left\| \sup_{1 \leq n \leq N}^+ A_n x \right\|_p \leq CC'(1 + \varepsilon) \|x\|_p.$$

Note that N, ε, x are all arbitrarily chosen, so we establish the theorem. $\qquad \square$

Remark 3.2. Applying the same argument, we may obtain several variants of the above theorem.

(1) The sequence of measures $(\mu_n)_{n \geq 1}$ can be replaced by any family $(\mu_i)_{i \in I}$ of Radon probability measures for an arbitrary index set I.

(2) The positivity of the action α can be replaced by more general assumptions. It suffices to assume that

$$\sup_{g \in G} \|\alpha_g \otimes \mathrm{Id}\|_{B(L_p(\mathcal{M}; \ell_\infty))} < \infty.$$

If $\mathcal{M}$ is commutative, this is equivalent to say that the operators $(\alpha_g)_{g \in G}$ are regular with uniformly bounded regular norm ([MN91]). In the noncommutative setting, one may assume that $(\alpha_g)_{g \in G}$ are uniformly bounded decomposable maps, and we refer to [Pis95, JR04] for more details.

(3) One may also state similar properties for transference of linear operators; in this case the assumption on positivity of α can be ignored, and the semigroup actions can be included. We have the following noncommutative analogue of the transference result in [CW76, Theorem 2.4]. Assume that G and α satisfy one of the following conditions:

(a) G is an amenable locally compact group, and α satisfies $(\mathbf{A}_1^p)$ and $(\mathbf{A}_2^p)$;

(b) G is a discrete amenable semigroup or $G = \mathbb{R}_+$, α satisfies $(\mathbf{A}_1^p)$ and each α_g is an isometry on $L_p(\mathcal{M})$ (or more generally, there exist $K_1, K_2 > 0$ such that for all $g \in G$, we have $K_1 \|x\|_p \leq \|\alpha_g x\|_p \leq K_2 \|x\|_p$).

Let μ be a bounded Radon measure on G. Define

$$T_\mu(f) = \int_G f(gh) d\mu(h), \quad f \in L_p(G),$$

and

$$\widetilde{T}_\mu(x) = \int_G \alpha_g x d\mu(g), \quad x \in L_p(\mathcal{M}).$$

Then we have

$$\|\widetilde{T}_\mu\|_{B(L_p(\mathcal{M}))} \leq \sup_{g \in G} \|\alpha_g\|_{B(L_p(\mathcal{M}))} \|T_\mu \otimes \mathrm{Id}\|_{B(L_p(L_\infty(G)\bar{\otimes}\mathcal{M}))}.$$

3.2. **Weak type inequalities.** Now we discuss the transference principle for weak type (p, p) inequalities. In this case we will only consider the special case of group actions on von Neumann algebras. We assume that α is given by an action on $\mathcal{M}$ satisfying the condition $(\mathbf{A}')$ in Section 2.2.

Theorem 3.3. *Assume that G is amenable. Let $(A_n)_{n\geq 1}$ and $(A'_n)_{n\geq 1}$ be the associated sequences of maps given in* (3.1) *and* (3.2). *Fix $1 \leq p < \infty$. If the sequence $(A'_n)_{n\geq 1}$ is of weak type (p, p), then $(A_n)_{n\geq 1}$ is of weak type (p, p) too.*

Proof. As in the last subsection, we may assume without loss of generality that μ_n is of compact support. Assume that the sequence $(A'_n)_{n\geq 1}$ is of weak type (p, p) with constant C. By Lemma 2.3, it suffices to show that there exists a constant $C' > 0$ such that for all $\lambda > 0, x \in L_p^+(\mathcal{M}), N \geq 1$, there exists a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) \leq \frac{C'^p}{\lambda^p} \|x\|_p^p, \quad e(A_n x)e \leq \lambda, \quad 1 \leq n \leq N.$$

We fix $\lambda > 0, x \in L_p(\mathcal{M})$ and $N \geq 1$. Choose a compact subset $K \subset G$ such that μ_n is supported in K for all $1 \leq n \leq N$. Let F be a compact subset. We define a function $f \in L_p(G; L_p(\mathcal{M}))$ as

$$f(h) = \chi_{FK}(h)\alpha_h x, \quad h \in G.$$

Then for all $g \in F$,

$$(3.5) \qquad \alpha_g A_n x = \int_K \alpha_{gh} x d\mu_n(h) = \int_K f(gh)d\mu_n(h) = A'_n f(g).$$

Since the sequence $(A'_n)_{n\geq 1}$ is of weak type (p, p) with constant C, we may choose a projection $e \in L_\infty(G)\bar{\otimes}\mathcal{M}$ such that

$$\int_G \tau(e(g)^\perp)dg \leq \left(C\frac{\|f\|_{L_p(G;L_p(\mathcal{M}))}}{\lambda}\right)^p \quad \text{and} \quad e(A'_n f)e \leq \lambda, \quad 1 \leq n \leq N,$$

where we regard e as a measurable operator-valued function on G. Therefore by (3.5) for $g \in G$, we have

$$(3.6) \qquad (\alpha_{g^{-1}}e(g))(A_n x)(\alpha_{g^{-1}}e(g)) \leq \lambda, \quad n \geq 1.$$

Recall that each $\alpha_{g^{-1}}$ is a unital τ-preserving automorphism of $\mathcal{M}$. In particular, for an arbitrary $\varepsilon > 0$, we may choose $g_0 \in G$ and a projection $\tilde{e} := \alpha_{g_0^{-1}}e(g_0) \in \mathcal{M}$ such that

$$\tau(\tilde{e}^\perp) \leq \inf_{g \in G} \tau(e(g)^\perp) + \varepsilon \quad \text{and} \quad \tilde{e}(A_n x)\tilde{e} \leq \lambda, \quad n \geq 1.$$

Then we have

$$\begin{aligned}
\tau(\tilde{e}^\perp) &\leq \frac{1}{m(F)} \int_F \tau(e(g)^\perp)dg + \varepsilon \leq \frac{C^p}{\lambda^p m(F)} \|f\|_{L_p(G;L_p(\mathcal{M}))}^p + \varepsilon \\
&= \frac{C^p}{\lambda^p m(F)} \int_{FK} \|\alpha_h x\|_p^p dm(h) + \varepsilon \\
&\leq \frac{C^p m(FK)}{\lambda^p m(F)} \|x\|_p^p + \varepsilon.
\end{aligned}$$

Since G is amenable, for any $\varepsilon > 0$ we may choose the above subset F such that $m(FK)/m(F) \leq 1 + \varepsilon$. Therefore we get

$$\tau(\tilde{e}^\perp) \leq \frac{C^p(1 + \varepsilon)}{\lambda^p} \|x\|_p^p + \varepsilon, \quad \tilde{e}(A_n x)\tilde{e} \leq \lambda, \quad 1 \leq n \leq N.$$

Note that N, ε, x are all arbitrarily chosen, so we establish the theorem. $\qquad \square$

Remark 3.4. We remark that the above result also holds for other index sets. For example, $(\mu_n)_{n \geq 1}$ can be replaced by an one-parameter family $(\mu_r)_{r \in \mathbb{R}_+}$ such that $r \mapsto A_r x$ is continuous for all $x \in L_p(\mathcal{M})$. The only ingredient needed in the proof is that the condition (3.6) holds true almost everywhere on G.

4. Maximal inequalities: probabilistic approach

This section is devoted to the proof of the maximal inequalities in Theorem 1.1. To this end we will first establish a noncommutative Hardy-Littlewood maximal inequalities on doubling metric spaces.

4.1. **Hardy-Littlewood maximal inequalities on metric measure spaces.** Throughout a metric measure space (X, d, μ) refers to a metric space (X, d), equipped with a Radon measure μ. We denote $B(x, r) = \{y \in X : d(y, x) \leq r\}$, and we say that μ satisfies the *doubling condition* if there exists a constant $K > 0$ such that

(4.1) $\forall r > 0, x \in X, \quad \mu(B(x, 2r)) \leq K\mu(B(x, r)).$

In the sequel we always assume the non-degeneracy property $0 < \mu(B(x, r)) < \infty$ for all $r > 0$. The following theorem can be regarded as an operator-valued analogue of the Hardy-Littlewood maximal inequalities.

Theorem 4.1. *Let (X, d, μ) be a metric measure space. Suppose that μ satisfies the doubling condition. Let $1 \leq p < \infty$, and let A_r be the averaging operators*

$$A_r f(x) = \frac{1}{\mu(B(x, r))} \int_{B(x,r)} f d\mu, \quad f \in L_p(X; L_p(\mathcal{M})), x \in X, r > 0.$$

Then $(A_r)_{r \in \mathbb{R}_+}$ is of weak type $(1, 1)$ and of strong type (p, p) for $1 < p < \infty$.

The key ingredient of the proof is the following construction of random partitions of metric measure spaces, which is established in [HK12, Corollary 7.4].

Lemma 4.2. *Let (X, d) be a metric space and μ be a Radon measure on X satisfying the doubling condition. Then there exists a finite collection of families $\mathscr{P}^1, \mathscr{P}^2, \ldots, \mathscr{P}^N$, where each $\mathscr{P}^i := (\mathscr{P}_k^i)_{k \in \mathbb{Z}}$ is a sequence of partitions of X, such that the following conditions hold true:*

(1) *for each $1 \leq i \leq N$ and for each $k \in \mathbb{Z}$, the partition $\mathscr{P}_{k+1}^i$ is a refinement of the partition $\mathscr{P}_k^i$;*

(2) *there exists a constant $C > 0$ such that for all $x \in X$ and $r > 0$, there exist $1 \leq i \leq N, k \in \mathbb{Z}$ and an element $Q \in \mathscr{P}_k^i$ such that*

$$B(x, r) \subset Q, \quad \mu(Q) \leq C\mu(B(x, r)).$$

Remark 4.3. The lemma dates back to the construction of dyadic systems in the case of $X = \mathbb{R}^d$, which is due to Mei [Mei03, Mei07]. We remark that Mei's construction also works for the discrete space $\mathbb{Z}^d$ as follows. For $0 \leq i \leq d$ and $k \geq 0$, we set $\tilde{\mathscr{P}}_{-k}^i$ to be the following family of intervals in $\mathbb{Z}$,

$$\tilde{\mathscr{P}}_{-k}^i = \{[\alpha_k^{(i)} + m2^k, \alpha_k^{(i)} + (m+1)2^k) \cap \mathbb{Z} : m \geq 1\},$$

where $\alpha_k^{(i)} = \sum_{j=0}^{k-1} 2^j \xi_j^{(i)}$ modulo 2^k, with

$$\xi_{(d+1)n+l}^{(i)} = \delta_{i,l}, \quad n \geq 0, 0 \leq l \leq d.$$

And we set $\tilde{\mathscr{P}}_k^i = \tilde{\mathscr{P}}_0^i$ for all $k \geq 0$. Consider the usual word metric d and the counting measure μ on $X = \mathbb{Z}^d$. Then the the partitions

$$\mathscr{P}_k^i := \left(\tilde{\mathscr{P}}_k^i\right)^d, \quad k \in \mathbb{Z}, 0 \leq i \leq d$$

satisfy the conditions in Lemma 4.2, with constant $C \leq 2^{3d(d+2)}$.

Proof of Theorem 4.1. Let Σ be the σ-algebra of Borel sets on X. For $1 \leq i \leq N$ and $k \in \mathbb{Z}$, we define $\Sigma_k^i \subset \Sigma$ to be the σ-subalgebra generated by the elements of $\mathscr{P}_k^i$. Denote by $\mathbb{E}_k^i$ the conditional expectation from $L_\infty(X, \Sigma, \mu) \bar{\otimes} \mathcal{M}$ to $L_\infty(X, \Sigma_k^i, \mu|_{\Sigma_k^i}) \bar{\otimes} \mathcal{M}$. For each $x \in X$, let $\mathscr{P}_k^i(x)$ be the unique element of $\mathscr{P}_k^i$ which contains x. Then we have

$$\mathbb{E}_k^i g(x) = \frac{1}{\mu(\mathscr{P}_k^i(x))} \int_{\mathscr{P}_k^i(x)} g d\mu, \quad x \in X, g \in L_\infty(X, \Sigma, \mu) \bar{\otimes} \mathcal{M}.$$

By Lemma 2.4, there exists a constant $C > 0$ such that for $\lambda > 0$ and $f \in L_1^+(X; L_1(\mathcal{M}))$, there exists a projection $e_i \in L_\infty(X, \Sigma, \mu) \bar{\otimes} \mathcal{M}$ satisfying

$$\tau(e_i^\perp) \leq \frac{C}{\lambda} \|f\|_{L_1(X; L_1(\mathcal{M}))}, \quad e_i(\mathbb{E}_k^i f) e_i \leq \lambda, \quad k \in \mathbb{Z}.$$

Take $e = \wedge_i e_i$ to be the infimum of $(e_i)_{i=1}^N$, i.e., the projection onto $\cap_i e_i H$ (H being the Hilbert space on which $\mathcal{M}$ acts). Note that $(\wedge_i e_i)^\perp \leq \sum_i e_i^\perp$. Then we have

$$(4.2) \qquad \tau(e^\perp) \leq \frac{CN}{\lambda} \|f\|_{L_1(X; L_1(\mathcal{M}))}, \quad e(\mathbb{E}_k^i f) e \leq \lambda, \quad 1 \leq i \leq N, k \in \mathbb{Z}.$$

By Lemma 4.2, there exists a constant $C' > 0$ such that for each $x \in X$ and $r > 0$, there exist $1 \leq i \leq N$ and $k \in \mathbb{Z}$ such that $B(x, r) \subset \mathscr{P}_k^i(x), \mu(\mathscr{P}_k^i(x)) \leq C' \mu(B(x, r))$; in particular,

$$(4.3) \qquad A_r f(x) = \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f d\mu \leq C' \frac{1}{\mu(\mathscr{P}_k^i(x))} \int_{\mathscr{P}_k^i(x)} f d\mu = C' \mathbb{E}_k^i f(x).$$

Then together with (4.2) we see that

$$e(A_r f) e \leq C' \lambda, \quad r > 0.$$

Therefore $(A_r)_{r>0}$ is of weak type $(1, 1)$.

On the other hand, for $1 < p < \infty$, according to the proof of (4.3), we have for $f \in L_p^+(X; L_p(\mathcal{M}))$,

$$\|(A_r f)_{r>0}\|_{L_p(\mathcal{M} \bar{\otimes} L_\infty(X); \ell_\infty)} \leq C' \|(\sum_{i=1}^N \mathbb{E}_k^i f)_{k \in \mathbb{Z}}\|_{L_p(\mathcal{M} \bar{\otimes} L_\infty(X); \ell_\infty)}$$

$$\leq C' \sum_{i=1}^N \|(\mathbb{E}_k^i f)_{k \in \mathbb{Z}}\|_{L_p(\mathcal{M} \bar{\otimes} L_\infty(X); \ell_\infty)}.$$

Since each $(\mathbb{E}_k^i)_{k \in \mathbb{Z}}$ on the right hand side is of strong type (p, p) by Lemma 2.4, we see that $(A_r)_{r \in \mathbb{R}_+}$ is of strong type (p, p), as desired. $\square$

Remark 4.4. There is another approach of random partitions of metric measure spaces, which is proved by Naor and Tao [NT10, Lemma 3.1]. The construction is motivated by the study of Hardy-Littlewood maximal inequalities on large dimensional doubling spaces. Their result replaces the families $\mathscr{P}^1, \mathscr{P}^2, \ldots, \mathscr{P}^N$ in Lemma 4.2 by a infinite random collection $(\mathscr{P}^\omega)_{\omega \in \Omega}$ for a probability space (Ω, P) and assume a positive probability for the coverings of balls. In this case we may find a random family of martingales $\{(\mathbb{E}_k^\omega)_{k \in \mathbb{Z}}, \omega \in \Omega\}$ such that for some $k : \mathbb{R}_+ \to \mathbb{Z}$ and for some fixed constant c, we have

$$A_r f \leq c \int_\Omega \mathbb{E}_{k(r)}^\omega f(x) dP(\omega), \quad f \in L_p^+(X; L_p(\mathcal{M})).$$

This yields as well the strong type (p, p) inequalities of $(A_r)_{r \in \mathbb{R}_+}$ for $1 < p < \infty$. We omit the details.

4.2. **Maximal ergodic inequalities.** Based on the previous result we are now ready to deduce the following maximal ergodic theorems. We say that a metric d on G is *invariant* if $d(e, g) = d(h, hg)$ for all $g, h \in G$. We denote $B_r = B(e, r)$ for $r > 0$. As before we consider an action α on $L_p(\mathcal{M})$ satisfying the conditions $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$ for a fixed p in Section 2.2. The following establishes the maximal inequalities in Theorem 1.1.

Theorem 4.5. *Let G be an amenable locally compact group and d be an invariant metric on G. Assume that (G, d) satisfies the doubling condition* (1.1). *Fix $1 \leq p < \infty$. Let A_r be the averaging operators*

$$A_r x = \frac{1}{m(B_r)} \int_{B_r} \alpha_g x \, dm(g), \quad x \in L_p(\mathcal{M}), r > 0.$$

Then $(A_r)_{r>0}$ is of strong type (p, p) if $1 < p < \infty$. If moreover α satisfies the condition $(\mathbf{A}')$, then $(A_r)_{r>0}$ is of weak type $(1, 1)$.

Proof. By Theorem 3.1,Theorem 3.3 and the remarks following them, it suffices to prove the maximal inequalities for the averaging operators

$$A'_r f(g) = \frac{1}{m(B_r)} \int_{B_r} f(gh) \, dm(h), \quad f \in L_p(G; L_p(\mathcal{M})), g \in G, r > 0.$$

Since the condition (1.1) holds, G must be unimodular (see e.g. [Cal53]). In other words, the measure m is also invariant under left translation. Note that $gB_r = B(g, r)$ by the invariance of d. Thus $m(B_r) = m(gB_r) = m(B(g, r))$ and

$$A'_r f(g) = \frac{1}{m(B(g, r))} \int_{B(g, r)} f(h) \, dm(h).$$

And by Theorem 4.1, the right hand side is of weak type $(1, 1)$ and of strong type (p, p) for $1 < p < \infty$. Thus $(A'_r)_{r>0}$ is of weak type $(1, 1)$ and of strong type (p, p) for $1 < p < \infty$ as well. The theorem is proved. $\square$

Example 4.6. The theorem is noncommutative variants of classical results due to Wiener [Wie39], Calderón [Cal53] and Nevo [Nev06]. If G is a compactly generated group of polynomial growth, the theorem applies to a large class of invariant metrics on G, such as distance functions derived from an invariant Riemann metric or word metrics. We refer to [Nev06, Section 4 and 5] for more examples. Here we list several typical examples satifying the doubling condition:

(1) Let G be a compactly generated group of polynomial growth, and V be a symmetric compact generating subset. The the *word metric* defined by

$$\forall g, h \in G, \quad d(g, h) = \inf\{n \in \mathbb{N}, \, g^{-1}h \in V^n\}$$

satisfies (1.1) and (1.2). Note that the integer groups and finitely generated nilpotent groups are of polynomial growth.

i) The averaging operators

$$A_n x = \frac{1}{m(V^n)} \int_{V^n} \alpha_g x \, dm(g), \quad x \in L_p(\mathcal{M}), n \in \mathbb{N}$$

is of strong type (p, p) if $1 < p < \infty$. If moreover α satisfies the condition $(\mathbf{A}')$, then $(A_n)_{n\geq 1}$ is of weak type $(1, 1)$. This in particular establishes the maximal inequalities in Theorem 1.2.

ii) Let $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be a positive invertible operator with positive inverse such that $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$. Then

$$A_n = \frac{1}{2n+1} \sum_{k=-n}^{n} T^k, \quad n \in \mathbb{N}$$

is of strong type (p, p) if $1 < p < \infty$. If T is an automorphism of $\mathcal{M}$ which leaves τ invariant, then $(A_n)_{n\geq 1}$ is of weak type $(1, 1)$.

(2) Let G be a compactly generated group of polynomial growth and let d be a metric on G. If d is invariant under a co-compact subgroup of G and if d satisfies a weak kind of the existence of geodesics axiom (see [Bre14, Definition 4.1]), then (G, d) satisfies (1.1) and (1.2).

We remark that a natural generalization of the doubling condition (1.1) is given by Tempelman [Tem67] as follows. A sequence $(F_k)_{k \geq 1}$ of sets of finite measure in G satisfies *Tempelman's regular condition* if

$$m(F_k^{-1} F_k) < C m(F_k)$$

for some $C > 0$ independent of k. We refer to [Tem92, Chapter 5] for more details. It is unclear for us how to establish the noncommutative maximal inequalities in this setting. In the following we provide a typical example for which the inequalities hold true.

Theorem 4.7. *Let G be an increasing union of compact subgroups $(G_n)_{n \geq 1}$. Fix $1 \leq p < \infty$. Let $(A_n)_{n \geq 1}$ be the averaging operators*

$$A_n x = \frac{1}{m(G_n)} \int_{G_n} \alpha_g x \, dm(g), \quad x \in L_p(\mathcal{M}).$$

Then $(A_n)_{n \geq 1}$ is of strong type (p, p) if $1 < p < \infty$. If moreover α satisfies the condition $(\mathbf{A}')$, then $(A_n)_{n \geq 1}$ is of weak type $(1, 1)$.

Proof. By Theorem 3.1 and Theorem 3.3, it suffices to prove the maximal inequalities for the averaging operators

$$A_n' f(g) = \frac{1}{m(G_n)} \int_{G_n} f(gh) \, dm(h), \quad f \in L_p(G; L_p(\mathcal{M})), g \in G, n \geq 1.$$

Set Σ to be the σ-algebra of Borel sets on G. For each $n \geq 1$, we define $\Sigma_n \subset \Sigma$ to be the σ-subalgebra generated by the cosets of G_n

$$\{g G_n : g \in G\}.$$

We see that $\Sigma_{n+1} \subset \Sigma_n$ for all $n \geq 1$. Let $\mathbb{E}_n$ be the conditional expectation from $L_\infty(G, \Sigma, m) \bar{\otimes} \mathcal{M}$ to $L_\infty(G, \Sigma_n, m|_{\Sigma_n}) \bar{\otimes} \mathcal{M}$. Then it is easy to see

$$\mathbb{E}_n = A_n', \quad n \geq 1.$$

According to Lemma 2.4, we see that $(\mathbb{E}_n)_{n \geq 1}$ is of weak type $(1, 1)$ and of strong type (p, p) for $1 < p < \infty$. This yields the desired inequalities. $\qquad \square$

4.3. **A random walk approach.** In this subsection we provide an alternative approach to maximal inequalities for groups of polynomial growth. This approach is based on a Gaussian lower bound of random walks on groups ([HSC93]). Independent of the previous approaches, in this method we do not need the results on dyadic decompositions of the group, nor do we use transference principle. The key observation is that we may relate the ball averages on groups with the ergodic averages of a Markov operator.

Proposition 4.8. *Let G be a locally compact group of polynomial growth and let V be a compact generating set. Let α be a strongly continuous action of G on an ordered Banach space E such that $\alpha_g x \geq 0$ for all $g \in G$ and $x \in E_+$. Define an operator T on E by*

$$T x = \frac{1}{m(V)} \int_V \alpha_g x \, dm(g), \quad x \in E.$$

Then there exists a constant c only depending on G such that

$$\frac{1}{m(V^n)} \int_{V^n} \alpha_g x \, dm(g) \leq \frac{c}{n^2} \sum_{k=1}^{2n^2} T^k x, \quad x \in E_+.$$

We remark that for actions by abelian semigroups, it is known by Brunel [Bru73] that the ergodic averages of multi-operators can be related to averages of some Markov operators; Nevo and Stein [NS94] showed that similar observations hold for spherical averages of free group actions. These results play an essential role in the proof of ergodic theorems therein. In the case of groups of polynomial growth, our construction of Markov operators is different from theirs, which is inspired by [SS83]. The argument is relatively easy, and is based on the Markov chains on groups of polynomial growth.

To prove the proposition, we consider a locally compact group G and a measure μ on G. For an integer k, we denote by $\mu^{\star k}$ the k-th convolution of μ, that is, the unique measure ν on G satisfying

$$\int_G f d\nu = \int_{\prod_{i=1}^k G} f(g_1 \cdots g_k) d\mu(g_1) \cdots d\mu(g_k), \quad f \in C_0(G).$$

If f is the density function of μ, we still denote by $f^{\star k}$ the density function of $\mu^{\star k}$.

In the following d will denote the word metric with respect to V introduced in Example 4.6(1), and $B_r = \{x \in G : d(e, x) \le r\}$ for $r > 0$.

Lemma 4.9 ([HSC93]). *Let G be a locally compact group of polynomial growth and let V be a compact generating set. Let f be the density function of a symmetric continuous probability measure on G such that $\mathrm{supp}(f)$ is bounded and $V \subset \mathrm{supp}(f)$. Then there exists a constant $c > 0$ such that for any integer k,*

$$f^{\star k}(g) \ge \frac{c e^{-d(e,g)^2/k}}{m(B_{\sqrt{k}})}, \quad g \in B_k.$$

Lemma 4.10. *Let G, V and f be as in the previous lemma. Then there exists a constant $c > 0$ such that for any integer n,*

$$\frac{\chi_{B_n}}{m(B_n)} \le \frac{c}{2n^2} \sum_{k=1}^{2n^2} f^{\star k}.$$

Proof. It suffices to prove the inequality in the lemma for sufficiently large n. By the previous lemma, there exists $c > 0$ such that

$$f^{\star k}(g) \ge \frac{c}{m(B_{\sqrt{k}})}, \quad g \in B_{\sqrt{k}}(e).$$

Therefore, for $g \in B_n(e)$,

$$\frac{1}{2n^2} \sum_{k=1}^{2n^2} f^{\star k}(g) \ge \frac{1}{2n^2} \sum_{k=(n+1)^2}^{2n^2} f^{\star k}(g) \ge \frac{1}{2n^2} \sum_{k=(n+1)^2}^{2n^2} \frac{c}{m(B_{\sqrt{k}})}$$

$$\ge \frac{c(n^2 - 2n - 1)}{2n^2 m(B_{\sqrt{2}n})} \ge \frac{c'}{m(B_n)},$$

where $c' > 0$ is a constant only depending on the doubling condition of G. $\qquad\square$

Proof of Proposition 4.8. We apply Lemma 4.10 with $f = \chi_V/m(V)$. Then we obtain for $x \in E_+$,

$$\frac{1}{m(V^n)} \int_{V^n} \alpha_g x \, dm(g) \le \frac{c}{2n^2} \sum_{k=1}^{2n^2} \int_G \alpha_g x f^{\star k}(g) dm(g).$$

By the definition of $f^{\star k}$, we have

$$\int_G \alpha_g x f^{\star k}(g) dm(g) = \int_{\prod_{i=1}^k G} \alpha_{g_1 \cdots g_k} x f(g_1) \cdots f(g_k) dm(g_1) \cdots dm(g_k).$$

Recall that α is a group action and $f = \chi_V/m(V)$, so we obtain

$$\int_G \alpha_g x f^{\star k}(g) dm(g) = \frac{1}{m(V)^k} \int_{\prod_{i=1}^k V} \alpha_{g_1} \cdots \alpha_{g_k} x \, dm(g_1) \cdots dm(g_k) = T^k x.$$

Therefore we establish the desired inequality. $\qquad\square$

Proposition 4.8 allows us to deduce maximal ergodic theorems for group actions from well-known ergodic theorems for Markov operators. As a corollary, we may obtain the maximal inequalities in Example 4.6 (1). Moreover, for maximal ergodic inequalities on L_1, it is not necessary to assume that α is an action by automorphisms as in $(\mathbf{A}')$:

Corollary 4.11. *Let G and V be as above. Let α be a continuous τ-preserving action of G on $\mathcal{M}$ such that α_g is a positive isometry on $\mathcal{M}$ for each $g \in G$. Then α extends to an action on $L_1(\mathcal{M})$. The operators defined by*

$$A_n x = \frac{1}{m(V^n)} \int_{V^n} \alpha_g x \, dm(g), \quad x \in L_1(\mathcal{M}), n \in \mathbb{N}$$

is of weak type $(1,1)$.

Proof. Note that the operator

$$Tx = \frac{1}{m(V)} \int_V \alpha_g x \, dm(g), \quad x \in \mathcal{M}$$

is a positive contraction on $\mathcal{M}$, which preserves τ. Then it is well-known that the averages $\frac{1}{n} \sum_{k=1}^n T^k$ is of weak type $(1,1)$ (see [Yea77]). Thus by Proposition 4.8, $(A_n)_{n \geq 0}$ is of weak type $(1,1)$ as well. $\qquad\square$

5. Maximal inequalities: group-theoretic approach

In this section we provide an alternative approach to Theorem 4.5 in the case where G is a finitely generated discrete group of polynomial growth, d is the word metric, and $p \neq 1$. The argument follows from a structural study of nilpotent groups.

We first recall some well-known facts on the structure of nilpotent groups. Let G be a discrete finitely generated nilpotent group with lower central series

$$G = G_1 \supset G_2 \supset \cdots \supset G_K \supset G_{K+1} = \{e\}.$$

Each quotient group G_i/G_{i+1} is an abelian group of rank r_i, that is, there is a group isomorphism

$$\pi_i : G_i/G_{i+1} \to F_i \times \mathbb{Z}^{r_i}$$

with a finite abelian group F_i. It was shown in [Bas72] that G is of polynomial growth. We summarize below some facts in the argument of [Bas72]. We may choose a finite generating set T of G such that

$$[T, T] = \{s^{-1}t^{-1}st : s, t \in T\} \subset T$$

and take

$$T_i = G_i \cap T, \quad 1 \leq i \leq K + 1.$$

Then

$$G_i = \langle T_i \rangle, \quad 1 \leq i \leq K + 1.$$

For each $1 \leq j \leq K$, we order the elements in $T_j \setminus T_{j+1}$ as

$$(5.1) \qquad T_j \setminus T_{j+1} = \{t_1^{(j)}, t_2^{(j)}, \ldots, t_{r_j}^{(j)}, t_{r_j+1}^{(j)} \ldots, t_{l_j}^{(j)}\}$$

so that $\pi_j([t_1^{(j)}]), \ldots, \pi_j([t_{r_j}^{(j)}])$ are the generators of $\mathbb{Z}^{r_j}$. Let N_j be the index of the subgroup $\langle t_1^{(j)}, t_2^{(j)}, \ldots, t_{r_j}^{(j)} \rangle G_{j+1}$ in G_j.

By a *word* in a subset $T' \subset T$ we mean a sequence of elements $w = (s_1, s_2, \ldots, s_n)$ with $s_1, s_2, \ldots, s_n \in T'$, and we denote by $|w| = s_1 s_2 \cdots s_n \in G$ the resulting group element in G. If $w = (s_1, s_2, \ldots, s_n)$ is a word in T' and if $T'' \subset T'$, we let $\deg_{T''}(w)$ be the cardinality of $\{k : 1 \leq k \leq n, s_k \in T''\}$. We say that

$$\deg^{(j)}(w) \leq (d_j, d_{j+1}, \ldots, d_K) \coloneqq d$$

if we have $\deg_{T_j \setminus T_{i+1}}(w) \leq d_i$ for all $j \leq i \leq K$. Denote by $G_j(d)$ the set of all words w in T_j such that $\deg^{(j)}(w) \leq d$, and by $G_j'(d)$ the subset of the words $w \in G_j(d)$ of the form

$$w = (t_1^{(j)}, \ldots, t_1^{(j)}, \ldots, t_{l_j}^{(j)}, \ldots, t_{l_j}^{(j)}, v)$$

where v is a word in T_{j+1} and the element $t_k^{(j)}$ does not appear more than N_j times in w for $r_j < k \leq l_j$.

The key observation in [Bas72] for proving the polynomial growth of G is as follows (see the assertions (6) and (7) in [Bas72, p.613]).

Lemma 5.1. *Let $c > 0$ be a constant and $m \geq 1$. For each $1 \leq j \leq K$, we have*

$$\{g \in G : g = |w|, w \in G_j(cm^j, \ldots, cm^K)\} \subset \{g \in G : g = |w|, w \in G_j'(c'm^j, \ldots, c'm^K)\}$$

where $c' > 0$ is a constant only depending on c and G_j.

In particular, we take $g \in T^n$. Hence g corresponds to a word w in $T = T_1$ such that

$$g = |w|, \ w \in G_1(m_1, \ldots, m_K), \ m := \max\{m_1, \ldots, m_K\} \leq n.$$

Using the lemma inductively, we may find another word $w' = (w_1, \ldots, w_K)$ in T (where each w_j in the bracket stands for a subword in $T_j \setminus T_{j+1}$) and a constant $c > 0$ such that

$$g = |w'|, \ (w_j, \ldots, w_K) \in G_j'(cn, \ldots, cn^K), \quad 1 \leq j \leq K.$$

In other words, we have the following observation.

Lemma 5.2. *Let $n \in \mathbb{N}$. Each element $g \in T^n$ can be written in the following form*

$$g = (t_1^{(1)})^{n_{11}} \cdots (t_{l_1}^{(1)})^{n_{1l_1}} \cdots (t_1^{(K)})^{n_{K1}} \cdots (t_{l_K}^{(K)})^{n_{Kl_K}},$$

where there exists a constant $c > 0$ such that for $1 \leq j \leq K$,

$$n_{jk} \leq cn^j, \ \text{if } 1 \leq k \leq r_j,$$

and

$$n_{jk} \leq N_j, \ \text{if } r_j < k \leq l_j.$$

In [Bas72] and [Wol68] it is proved that G satisfies the following strict polynomial growth condition.

Lemma 5.3. *We have two constants $c_1, c_2 > 0$ such that*

$$c_1 n^{d(G)} \leq |T^n| \leq c_2 n^{d(G)}$$

where $|\ |$ denotes the cardinality of a subset and

$$d(G) = \sum_{j=1}^{K} jr_j.$$

Note that the upper bound in the above lemma follows directly from Lemma 5.2.

Now we will prove the following maximal inequalities, which are particular cases studied in Theorem 4.5 and Example 4.6 (1).

Proposition 5.4. *Let G be a finitely generated discrete group of polynomial growth and let $S \subset G$ be a finite generating set. Fix $1 < p < \infty$ and let α be an action $\alpha = (\alpha_g)_{g \in G}$ of G on $L_p(\mathcal{M})$ which satisfies $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$. We consider the following averaging operators*

$$A_n = \frac{1}{|S^n|} \sum_{g \in S^n} \alpha_g, \quad n \geq 1,$$

where $|\ |$ denotes the cardinality of a subset. Then $(A_n)_{n \geq 1}$ is of strong type (p, p).

The proposition relies on the following characterization of groups of polynomial growth by Gromov [Gro81].

Lemma 5.5. *Any finitely generated discrete group of polynomial growth contains a finitely generated nilpotent subgroup of finite index.*

We also need the following fact.

Lemma 5.6. *Let G be a finitely generated group of polynomial growth. Let H be a normal subgroup of G of finite index. Then H is finitely generated. Let $U \subset G$ be a finite system of representatives of the cosets G/H with $e \in U$. Let $T \subset H$ be a finite generating set of H. Write $V = U \cup T$. Then there exists an integer N such that*

$$\forall m \in \mathbb{N}, \quad V^m \subset UT^{(3N+1)m}.$$

Proof. This is given in the proof of [Wol68, Theorem 3.11]. Let N be an integer large enough such that for all $u_1^\epsilon, u_2^\eta \in U$ with $\epsilon, \eta \in \{\pm 1\}$, there exist $u \in U$ and $t \in T^N$ satisfying $u_1^\epsilon u_2^\eta = ut$. Then N satisfies the desired condition. $\qquad\square$

Now we deduce the desired result.

Proof of Proposition 5.4. By Theorem 3.1, it suffices to consider the case where α is an action on $L_p(G; L_p(\mathcal{M}))$ by translation. By Lemma 5.5, we may find a nilpotent subgroup $H \subset G$ of finite index. As is explained in [Wol68, 3.11], H can be taken normal by replacing H with $\cap_{g \in G} gHg^{-1}$. Now let $T = \{t_k^{(j)} : 1 \le k \le l_j, 1 \le j \le K\}$ be a finite generating set of the nilpotent group H, where T and the indices k, j are chosen in the same manner as in (5.1). Also, let U and V be given as in the previous lemma. Consider $x \in L_p^+(G; L_p(\mathcal{M}))$ and write

$$\tilde{A}_n x = \frac{1}{|V^n|} \sum_{g \in V^n} \alpha_g x, \quad n \in \mathbb{N}.$$

Since the operators α_g extends to positive operators on $L_p(G; L_p(\mathcal{M}))$, by Lemma 5.2 and Lemma 5.6, there exists a constant $c > 0$ for all $n \in \mathbb{N}$,

$$\sum_{g \in V^n} \alpha_g x \le \sum_{h \in U} \alpha_h \sum_{t \in T^{(3N+1)n}} \alpha_t x$$

$$\le \sum_{h \in U} \alpha_h \sum_{1 \le j \le K} \sum_{\substack{1 \le n_{jk} \le cn^j \\ 1 \le k \le r_j}} \sum_{\substack{1 \le n_{jk} \le N_j \\ r_j < k \le l_j}} \alpha_{t_1^{(1)}}^{n_{11}} \cdots \alpha_{t_{l_1}^{(1)}}^{n_{1l_1}} \cdots \alpha_{t_1^{(K)}}^{n_{K1}} \cdots \alpha_{t_{l_K}^{(K)}}^{n_{Kl_K}} x.$$

Recall that by Lemma 5.3 we may find a constant $c' > 0$ such that

$$|V^n| \ge c' n^{\sum_{j=1}^K j r_j}.$$

So we may find a constant $c'' > 0$ satisfying

$$\tilde{A}_n x \le c'' \sum_{h \in U} \alpha_h \frac{1}{n^{\sum_{j=1}^K j r_j}} \sum_{1 \le j \le K} \sum_{\substack{1 \le n_{jk} \le cn^j \\ 1 \le k \le r_j}} \sum_{\substack{1 \le n_{jk} \le N_j \\ r_j < k \le l_j}} \alpha_{t_1^{(1)}}^{n_{11}} \cdots \alpha_{t_{l_1}^{(1)}}^{n_{1l_1}} \cdots \alpha_{t_1^{(K)}}^{n_{K1}} \cdots \alpha_{t_{l_K}^{(K)}}^{n_{Kl_K}} x.$$

Note that by [JX07, Theorem 4.1], for each $1 \le j \le K$ and $1 \le k \le r_j$ there exists a constant C_p only depending on p such that

$$\left\| \sup_n^+ \frac{1}{cn^j} \sum_{l=1}^{cn^j} \alpha_{t_k^{(j)}}^l x \right\|_p \le C_p \|x\|_p, \quad x \in L_p(G; L_p(\mathcal{M})).$$

Applying the inequality iteratively, we obtain a constant $C_p' > 0$ such that

$$\left\| \sup_n^+ \tilde{A}_n x \right\|_p \le C_p' \|x\|_p, \quad x \in L_p(G; L_p(\mathcal{M})).$$

Since S and V are both finite, we may find two integers k and k' with

$$S \subset V^k, \quad V \subset S^{k'}.$$

So the strong type (p, p) inequality for A_n follows as well. $\qquad\square$

6. Individual ergodic theorems

In this section we apply the maximal inequalities to study the pointwise ergodic convergence in Theorem 1.1 and Theorem 1.2.

We will use the following analogue for the noncommutative setting of the usual almost everywhere convergence. The definition is introduced by Lance [Lan76] (see also [Jaj85]).

Definition 6.1. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $x_n, x \in L_0(\mathcal{M})$. $(x_n)_{n \geq 1}$ is said to converge *bilaterally almost uniformly* (*b.a.u.* in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) < \varepsilon \quad \text{and} \quad \lim_{n \to \infty} \|e(x_n - x)e\|_\infty = 0,$$

and it is said to converge *almost uniformly* (*a.u.* in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) < \varepsilon \quad \text{and} \quad \lim_{n \to \infty} \|(x_n - x)e\|_\infty = 0.$$

In the case of classical probability spaces, the definition above is equivalent to the usual almost everywhere convergence in terms of Egorov's theorem.

Now let G be an amenable locally compact group and $(F_n)_{n \geq 1}$ be a Følner sequence in G. Let $1 \leq p \leq \infty$. Assume that $\alpha = (\alpha_g)_{g \in G}$ is an action on $L_p(\mathcal{M})$ which satisfies $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$. Denote by A_n the corresponding averaging operators

$$A_n x = \frac{1}{m(F_n)} \int_{F_n} \alpha_g x \, dm(g), \quad x \in L_p(\mathcal{M}).$$

We keep the notation $\mathcal{F}_p \subset L_p(\mathcal{M})$ and P introduced in Section 2.2.

We will be first interested in the case where α extends to an action on $L_1(\mathcal{M}) + \mathcal{M}$. In this case the argument for b.a.u. convergences is standard, which is adapted from [JX07, Yea77, Hon17]. The following lemma from [DJ04] will be useful.

Lemma 6.2. *Let $1 \leq p < \infty$. If $(x_n) \in L_p(\mathcal{M}; c_0)$, then x_n converges b.a.u. to 0. If $(x_n) \in L_p(\mathcal{M}; c_0^c)$ with $2 \leq p < \infty$, then x_n converges a.u. to 0.*

We will also use the following noncommutative analogue of the Banach principle given by [Lit17] and [CL16, Theorem 3.1].

Lemma 6.3. *Let $1 \leq p < \infty$ and let $S = (S_n)_{n \geq 1}$ be a sequence of additive maps from $L_p^+(\mathcal{M})$ to $L_0^+(\mathcal{M})$. Assume that S is of weak type (p, p). Then the set*

$$\mathcal{C} = \{x \in L_p(\mathcal{M}) : (S_n x)_{n \geq 1} \text{ converges a.u.}\}$$

is closed in $L_p(\mathcal{M})$.

Proposition 6.4. *Assume that $\alpha = (\alpha_g)_{g \in G}$ is an action well-defined on $\cup_{1 \leq p \leq \infty} L_p(\mathcal{M})$ and satisfies $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$ for every $1 \leq p \leq \infty$. Let $(A_n)_{n \geq 1}$ be as above and let $1 \leq p_0 < p_1 \leq \infty$. Assume that $(A_n)_{n \geq 1}$ is of strong type (p, p) for all $p_0 < p < p_1$.*

(1) *For all $x \in L_p(\mathcal{M})$ with $p_0 < p \leq 2p_0$, $(A_n x - Px)_{n \geq 1} \in L_p(\mathcal{M}; c_0)$, and hence $(A_n x)_{n \geq 1}$ converges b.a.u. to Px;*

(2) *for all $x \in L_p(\mathcal{M})$ with $2p_0 < p < p_1$, $(A_n x - Px)_{n \geq 1} \in L_p(\mathcal{M}; c_0^c)$, and hence $(A_n x)_{n \geq 1}$ converges a.u. to Px.*

Proof. According to the splitting (2.4) and the discussion after it, we know that

$$S = \text{span}\{x - \alpha_g x : g \in G, x \in L_1(\mathcal{M}) \cap \mathcal{M}\}$$

is dense in $\overline{(\text{Id} - P)(L_p(\mathcal{M}))}$ for all $1 \leq p < \infty$. Also, observe that for all $x \in S$,

$$(6.1) \qquad \lim_{n \to \infty} A_n x = 0 \text{ a.u.}, \quad x \in S.$$

To see this, take an arbitrary $x \in S$ of the form $x = y - \alpha_{g_0} y$ for some $g_0 \in G$ and $y \in L_1(\mathcal{M}) \cap \mathcal{M}$. Then

$$A_n x = \frac{1}{m(F_n)} \int_{F_n} (\alpha_g y - \alpha_{g g_0} y) \, dm(g)$$

$$= \frac{1}{m(F_n)} \int_{F_n \setminus (F_n g_0 \cap F_n)} \alpha_g y \, dm(g) - \frac{1}{m(F_n)} \int_{F_n g_0 \setminus (F_n g_0 \cap F_n)} \alpha_g y \, dm(g).$$

Therefore according to $(\mathbf{A}_2^\infty)$,

$$\|A_n x\|_\infty \leq \frac{m(F_n \bigtriangleup F_n g_0)}{m(F_n)} \|y\|_\infty \sup_{g \in G} \|\alpha_g\|_{B(\mathcal{M})}, \tag{6.2}$$

which converges to 0 as $n \to \infty$ according to the Følner condition. This therefore yields the a.u. convergence of $(A_n x)_n$ in (6.1), as desired.

Now we prove the assertion (1). Take $x \in L_p(\mathcal{M})$. Since S is dense in $\overline{(\mathrm{Id} - P)(L_p(\mathcal{M}))}$, there are $x_k \in S$ such that

$$\lim_{k \to \infty} \|x - Px - x_k\|_p = 0.$$

Since $(A_n)_{n \geq 1}$ is of strong type (p, p), there exists a constant $C > 0$ independent of x such that

$$\left\|\left(A_n x - Px - A_n x_k\right)_n\right\|_{L_p(\mathcal{M};\ell_\infty)} \leq C\|x - Px - x_k\|_p.$$

Thus

$$\lim_{k \to \infty} \left(A_n x_k\right)_n = \left(A_n x - Px\right)_n \quad \text{in} \quad L_p(\mathcal{M}; \ell_\infty).$$

Since $L_p(\mathcal{M}; c_0)$ is closed in $L_p(\mathcal{M}; \ell_\infty)$, it suffices to show $\left(A_n x_k\right)_n \in L_p(\mathcal{M}; c_0)$ for all k. To this end we take an arbitrary $z \in S$ of the form $z = y - \alpha_{g_0} y$ for some $g_0 \in G$ and $y \in L_1(\mathcal{M}) \cap \mathcal{M}$. Take some $p_0 < q < p$. Note that $z \in L_q(\mathcal{M})$ and that $(A_n)_n$ is of strong type (q, q) by assumption, so $\left(A_n(z)\right)_n$ belongs to $L_q(\mathcal{M}; \ell_\infty)$. Then by (2.1) and (6.2), for any $m < n$,

$$\left\| \sup_{m \leq j \leq n}^+ A_j z \right\|_p \leq \sup_{m \leq j \leq n} \left\|A_j z\right\|_\infty^{1 - \frac{q}{p}} \left\| \sup_{m \leq j \leq n}^+ A_j z \right\|_q^{\frac{q}{p}}$$

$$\leq \sup_{m \leq j \leq n} \left(\frac{m(F_j \bigtriangleup F_j g_0)}{m(F_j)} \|y\|_\infty\right)^{1 - \frac{q}{p}} \left\| \sup_{m \leq j \leq n}^+ A_j z \right\|_q^{\frac{q}{p}}.$$

Thus $\left\| \sup^+_{j \geq m} A_j z \right\|_p$ tends to 0 as $m \to \infty$. Therefore the finite sequence $(A_1 z, ..., A_m z, 0, ...)$ converges to $(A_n z)_n$ in $L_p(\mathcal{M}; \ell_\infty)$ as $m \to \infty$. As a result $\left(A_n(z)\right)_n \in L_p(\mathcal{M}; c_0)$, as desired.

The assertion (2) is similar. It suffices to note that by the classical Kadison inequality [Kad52],

$$(A_n x)^2 \leq A_n(x^2) \sup_{g \in G} \|\alpha_g\|_{B(\mathcal{M})}, \quad x \in L_p(\mathcal{M}) \cap \mathcal{M}, x \geq 0,$$

and hence by the strong type (p, p) inequality and the definition of $L_p(\mathcal{M}; \ell_\infty^c)$, there exists a constant C such that

$$\|(A_n x)_{n \geq 1}\|_{L_p(\mathcal{M};\ell_\infty^c)} \leq \|((A_n x)^2)_{n \geq 1}\|_{L_{p/2}(\mathcal{M};\ell_\infty)}^{1/2} \leq C\|x\|_p, \quad x \in L_p^+(\mathcal{M}).$$

Then a similar argument yields that $(A_n x - Px)_{n \geq 1} \in L_p(\mathcal{M}; c_0^c)$. $\qquad\square$

Remark 6.5. The above argument certainly works as well for a Følner sequence $(F_r)_{r>0}$ indexed by $r \in \mathbb{R}_+$ provided that $r \mapsto A_r x$ is continuous for $x \in L_p(\mathcal{M})$.

As a corollary we obtain the individual ergodic theorems for actions on $L_1(\mathcal{M}) + \mathcal{M}$. We complete the proof of Theorem 1.1.

Theorem 6.6. *Let d be an invariant metric on G. Assume that (G, d) satisfies* (1.1) *and* (1.2). *Let α be an action of G well-defined on $\cup_{1 \leq p \leq \infty} L_p(\mathcal{M})$ and satisfies $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$ for every $1 \leq p \leq \infty$. Denote*

$$A_r x = \frac{1}{m(B_r)} \int_{B_r} \alpha_g x \, dm(g), \quad x \in L_p(\mathcal{M}), r > 0.$$

Then $(A_r x)_{r>0}$ converges a.u. to Px as $r \to \infty$ for all $1 < p < \infty$.

Moreover, if α is a continuous action of G on $\mathcal{M}$ by τ-preserving automorphisms (i.e. α satisfies $(\mathbf{A}')$), then $(A_r x)_{r>0}$ converges a.u. to Px as $r \to \infty$ for all $x \in L_1(\mathcal{M})$.

Proof. Note that for $1 \leq p \leq 2$ and $2 < p' < \infty$, $L_p(\mathcal{M}) \cap L_{p'}(\mathcal{M})$ is dense in $L_p(\mathcal{M})$, and $(A_r x)_{r>0}$ converges a.u. to Px for all $x \in L_{p'}(\mathcal{M})$ according to Proposition 6.4. Then the theorem is an immediate consequence of Lemma 6.3 and Theorem 4.5. $\qquad\square$

For word metrics on groups of polynomial growth, it is well-known that the associated balls satisfy the Følner condition (see [Bre14, Tes07]). Together with Corollary 4.11 we obtain the following result. This also proves the a.u. convergence on $L_1(\mathcal{M})$ stated in Theorem 1.2.

Theorem 6.7. *Assume that G is of polynomial growth, and is generated by a symmetric compact subset V. Let α be an action of G well-defined on $\cup_{1 \leq p \leq \infty} L_p(\mathcal{M})$ and satisfies $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$ for every $1 \leq p \leq \infty$. Denote*

$$A_n x = \frac{1}{m(V^n)} \int_{V^n} \alpha_g x \, dm(g), \quad x \in L_1(\mathcal{M}), n \in \mathbb{N}$$

Then $(A_n x)$ converges a.u. to Px for all $1 < p < \infty$.

Moreover, if α is a continuous τ-preserving action of G on $\mathcal{M}$ such that α_g is a positive isometry on $\mathcal{M}$ for each $g \in G$, then $(A_n x)$ converges a.u. to Px for all $x \in L_1(\mathcal{M})$.

Also it is obvious that an increasing sequence of compact subgroups always satisfies the Følner condition. Together with Theorem 4.7 we obtain:

Theorem 6.8. *Let G be an increasing union of compact subgroups $(G_n)_{n \geq 1}$. Let α be an action of G well-defined on $\cup_{1 \leq p \leq \infty} L_p(\mathcal{M})$ and satisfies $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$ for every $1 \leq p \leq \infty$. Denote*

$$A_n x = \frac{1}{m(G_n)} \int_{G_n} \alpha_g x \, dm(g), \quad x \in L_p(\mathcal{M}), n \in \mathbb{N}.$$

Then $(A_n x)$ converges a.u. to Px for all $1 < p < \infty$.

Moreover, if α is a continuous action of G on $\mathcal{M}$ by τ-preserving automorphisms (i.e. α satisfies $(\mathbf{A}')$), then $(A_n x)$ converges a.u. to Px for all $x \in L_1(\mathcal{M})$.

Note that all the above arguments relies on the assumption that the action α extends to an uniformly bounded action on $L_\infty(\mathcal{M})$ with condition $(\mathbf{A}_1^\infty)$-$(\mathbf{A}_3^\infty)$, though our strong type (p, p) inequalities in previous sections do not require this assumption. Also, in general this assumption does not hold for bounded representations on one fixed L_p-space. In the following Theorem 6.11 we will give a stronger result for Følner sequences associated with doubling conditions. This also completes the proof of Theorem 1.2.

Lemma 6.9. *Let (X, d, μ) be a metric measure space satisfying the doubling condition (4.1). Take $i \in \mathbb{N}$ and $k \leq 2^i$. Then there exists $2^i \leq r_i < 2^{i+1}$ such that*

$$\mu(B(x, r_i + k) \setminus B(x, r_i)) \leq C k \mu(B(x, r_i))/r_i,$$

where C only depends on the doubling constant.

Proof. The result and the argument are adapted from [Tes07, Proposition 17]. For each $r \in \mathbb{N}$, we denote

$$S(x, r) = B(x, r + k) \setminus B(x, r).$$

Then

$$\cup_{n=0}^{\left[\frac{2^i}{k}\right]} S(x, 2^i + nk) \subset B(x, 2^{i+1}).$$

Therefore

$$\frac{2^i}{k} \inf_{0 \leq n \leq \left[\frac{2^i}{k}\right]} \mu(S(x, 2^i + nk)) \leq \mu(B(x, 2^{i+1})).$$

Thus the lemma follows thanks to the doubling condition (4.1). $\square$

Lemma 6.10 ([Bre14, Tes07]). *Let G be a locally compact group of polynomial growth, generated by a symmetric compact subset V. Then*

$$\lim_{n \to \infty} \frac{m(V^n)}{n^{d(G)}} = c$$

where $d(G)$ is the rank of G and c is a constant only depending on the metric d. And there exist $\delta > 0$ and a constant C such that

$$\frac{m(V^{n+1} \setminus V^n)}{m(V^n)} \leq C n^{-\delta}, \quad n \geq 1.$$

Theorem 6.11. *Fix $1 < p < \infty$. Let $\alpha = (\alpha_g)_{g \in G}$ be an action on $L_p(\mathcal{M})$ which satisfies $(\mathbf{A}_1^p)$-$(\mathbf{A}_3^p)$.*

(1) Assume that there exists an invariant metric d on G and that (G, d) satisfies (1.1) and (1.2). Denote

$$A_r x = \frac{1}{m(B_r)} \int_{B_r} \alpha_g x \, dm(g), \quad x \in L_p(\mathcal{M}), r > 0.$$

Then there exists a lacunary sequence $(r_k)_{k \geq 1}$ with $2^k \leq r_k < 2^{k+1}$ such that $(A_{r_k} x)_{k \geq 1}$ converges b.a.u. to Px for all $x \in L_p(\mathcal{M})$. If additionally $p \geq 2$, $(A_{r_k} x)_{k \geq 1}$ converges a.u. to Px for all $x \in L_p(\mathcal{M})$.

(2) Assume that G is a locally compact group of polynomial growth, generated by a symmetric compact subset V. Then the sequence

$$A_n x = \frac{1}{m(V^n)} \int_{V^n} \alpha_g x \, dm(g), \quad n \in \mathbb{N},$$

converges b.a.u. to Px for all $x \in L_p(\mathcal{M})$.

(3) Assume that G is an increasing union of compact subgroups $(G_n)_{n \geq 1}$. Then the sequence

$$A_n x = \frac{1}{m(G_n)} \int_{G_n} \alpha_g x \, dm(g), \quad n \in \mathbb{N}.$$

converges a.u. to Px for all $x \in L_p(\mathcal{M})$.

Proof. (1) By Lemma 6.9, there exists $(r_i)_{i \geq 1}$ such that $2^i \leq r_i \leq 2^{i+1}$ and such that

$$m(B_{r_i} \setminus B_{r_i - (3/2)^i}) \leq C(3/2)^i m(B_{r_i})/r_i.$$

That is to say,

$$(6.3) \qquad \frac{m(B_{r_i} \setminus B_{r_i - (3/2)^i})}{m(B_{r_i})} \leq C(3/4)^i$$

We show that $(A_{r_i} x)_{i \geq 1}$ converges b.a.u. to Px. By (2.4) and Lemma 6.2, it suffices to show that $(A_{r_i} x)_{i \geq 1} \in L_p(\mathcal{M}; c_0)$ for $x \in \mathcal{F}_p^\perp$. By Theorem 4.5, it is enough to consider the case where $x = y - \alpha_{g_0} y$ with $y \in L_p^+(\mathcal{M})$ and $g_0 \in G$. Denote $|g_0| = d(e, g_0)$. Note that

$$A_r x = A_r^1 y - A_r^2 y$$

where

$$A_r^1 y = \frac{1}{m(B_r)} \int_{B_r \setminus (B_r \cap B_r g_0)} \alpha_g y \, dm(g), \quad A_r^2 y = \frac{1}{m(B_r)} \int_{(B_r g_0) \setminus (B_r \cap B_r g_0)} \alpha_g y \, dm(g).$$

By (6.3) we have for i so that $(3/2)^i \geq |g_0|$,

$$\|A_{r_i}^1 y\|_p \leq \frac{m(B_{r_i} \setminus B_{r_i - |g_0|})}{m(B_{r_i})} \|y\|_p \leq C(3/4)^i \|y\|_p.$$

On the other hand, for any $m \leq j \leq n$,

$$A_{r_j}^1 y = [(A_{r_j}^1 y)^p]^{1/p} \leq \Big[\sum_{m \leq i \leq n} (A_{r_i}^1 y)^p \Big]^{1/p},$$

and by the previous argument

$$\Big\| \Big[\sum_{m \leq i \leq n} A_{r_i}^1 y^p \Big]^{1/p} \Big\|_p \leq \Big[\sum_{m \leq i \leq n} \|A_{r_i}^1 y\|_p^p \Big]^{1/p} \leq C(3/4)^m \|y\|_p.$$

Hence $\|(A_{r_j}^1 y)_{m \leq j \leq n}\|_{L_p(\mathcal{M}; c_0)}$ tends to 0 as $m, n \to \infty$. Similarly, $(A_{r_i}^2 y)_{i \geq 1}$ converges in the same manner. Therefore $(A_{r_i} x)_{i \geq 1} \in L_p(\mathcal{M}; c_0)$, as desired.

Moreover if $p \geq 2$,

$$(A_{r_j}^1 y)^2 = [(A_{r_j}^1 y)^p]^{2/p} \leq \Big[\sum_{m \leq i \leq n} (A_{r_i}^1 y)^p \Big]^{2/p},$$

and hence we can find contractions $u_j \in L_\infty(\mathcal{M})$ such that for m large enough,

$$A_{r_j}^1 y = u_j \Big[\sum_{m \leq i \leq n} (A_{r_i}^1 y)^p \Big]^{1/p} \quad \text{with} \quad \left\| \Big[\sum_{m \leq i \leq n} (A_{r_i}^1 y)^p \Big]^{1/p} \right\|_p \leq C(3/4)^m \|y\|_p.$$

Therefore $\|(A_{r_j}^1 y)_{m \leq j \leq n}\|_{L_p(\mathcal{M}; c_0^c)}$ tends to 0 as $m, n \to \infty$, and $(A_{r_i}^1 y)_{i \geq 1}$ converges a.u. to 0 according to Lemma 6.2. Similarly, $(A_{r_i}^2 y)_{i \geq 1}$ convergences in the same manner. Thus we obtain that $(A_{r_i} x)_{i \geq 1}$ converges a.u. to 0. Then by (2.4) and Lemma 6.3, $(A_{r_i} x)_{i \geq 1}$ converges a.u. for all $x \in L_p(\mathcal{M})$.

(2) We keep the notation x, y, g_0 in (1), and denote as before

$$A_k^1 y = \frac{1}{m(V^k)} \int_{(V^k) \backslash (V^k \cap V^k g_0)} \alpha_g y \, dm(g), \quad y \in L_p(\mathcal{M}), k \geq 1.$$

Write $\delta' = [\delta^{-1}] + 1$. Note that by Lemma 6.10, there exists a constant $C > 0$ such that for $y \in L_p(\mathcal{M})$ and $k \geq 1$,

$$(6.4) \qquad \left\| A_{k^{\delta'}}^1 y \right\|_p \leq \frac{m(V^{k^{\delta'}} \backslash V^{k^{\delta'} - |g_0|})}{m(V^{k^{\delta'}})} \|y\|_p \leq \frac{C |g_0| \|y\|_p}{k},$$

where $|g_0| = d(e, g_0)$ and d refers to the word metric defined in Example 4.6(1). Hence

$$\left\| \Big[\sum_{m \leq k \leq n} A_{k^{\delta'}}^1 y^p \Big]^{1/p} \right\|_p \leq \Big[\sum_{m \leq k \leq n} \|A_{k^{\delta'}}^1 y\|_p^p \Big]^{1/p} \leq C |g_0| \Big(\sum_{m \leq k \leq n} \frac{1}{k^p} \Big)^{1/p} \|y\|_p.$$

Then by the similar argument as in (1) we see that $(A_{n^{\delta'}} x)_{n \geq 1}$ converges b.a.u. to Px for all $x \in L_p(\mathcal{M})$, and if $p \geq 2$, $(A_{n^{\delta'}} x)_{n \geq 1}$ converges a.u. to Px.

For the general case, we consider $x \in L_p^+(\mathcal{M})$. For each k, let $n(k)$ be the number such that $n(k)^{\delta'} \leq k < (n(k) + 1)^{\delta'}$. Then

$$\frac{m(V^{n(k)^{\delta'}})}{m(V^k)} A_{n(k)^{\delta'}} x \leq A_k x \leq \frac{m(V^{(n(k)+1)^{\delta'}})}{m(V^k)} A_{(n(k)+1)^{\delta'}} x.$$

Also note that according to Lemma 6.10, $m(V^{n(k)^{\delta'}})/m(V^k)$ tends to 1. Therefore it is easy to see from the definition of b.a.u. convergence that $A_k x$ converges b.a.u. to Px.

(3) Note that for $x = y - \alpha_{g_0} y$ with $y \in L_p(\mathcal{M})$ and $g_0 \in G$, we have for n large enough so that $G_n \ni g_0$,

$$A_n x = \frac{1}{m(G_n)} \int_{G_n} \alpha_g y \, dm(g) - \int_{G_n} \alpha_{g g_0} y \, dm(g) = 0.$$

That is to say, $A_n x$ converges a.u. to 0 as $n \to \infty$. Then by (2.4) and Lemma 6.3, we see that $A_n x$ converges a.u. for all $x \in L_p(\mathcal{M})$. $\qquad \square$

In particular, the above arguments give the individual ergodic theorem for positive invertible operators on L_p-spaces.

Corollary 6.12. *Let $1 < p < \infty$. Let $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be a positive invertible operator with positive inverse such that $\sup_{n \in \mathbb{Z}} \|T^n\| < \infty$. Denote*

$$A_n = \frac{1}{2n + 1} \sum_{k=-n}^{n} T^k, \quad n \in \mathbb{N}.$$

Then $(A_n x)_{n \geq 1}$ converges b.a.u to Px for all $x \in L_p(\mathcal{M})$. If additionally $p \geq 2$, $(A_n x)_{n \geq 1}$ converges a.u to Px.

Proof. The assertion follows from the proof of Theorem 6.11(2) for the case where G equals the integer group $\mathbb{Z}$. It suffices to notice that in this case we may choose $\delta = 1$ in Lemma 6.10 and take $\delta' = 1$ in (6.4). $\qquad \square$

Remark 6.13. Note that the above result is not true for $p = 1$, even for positive invertible isometries on classical L_1-spaces (see for example [IT64]). So it is natural to assume $p \neq 1$ in the above discussions.

The following conjecture for mean bounded maps is still open. The result for classical L_p-spaces is given by [MRDlT88].

Conjecture 6.14. *Let* $1 < p < \infty$. *Let* $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ *be a positive invertible operator with positive inverse such that* $\sup_{n \in \mathbb{Z}} \| \frac{1}{2n+1} \sum_{k=-n}^{n} T^k \| < \infty$. *Denote*

$$A_n = \frac{1}{2n+1} \sum_{k=-n}^{n} T^k, \quad n \in \mathbb{N}.$$

Then $(A_n x)_{n \geq 1}$ *converges b.a.u to* Px *for all* $x \in L_p(\mathcal{M})$. *If additionally* $p \geq 2$, $(A_n x)_{n \geq 1}$ *converges a.u to* Px.

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