Category theory – basic reading

Why category theory?

Once upon a time, my opinion of category theory was the same as my opinion of Facebook: if I ignore it for long enough, hopefully it will go away. It is now my educated opinion that category theory will **not** go away, and in fact the language of category theory will continue to spread until it becomes the default foundation of mathematics. During this transition period there will be three kinds of mathematicians:

1. those young enough to be raised with the new language,
2. those willing to invest enough time and energy to learn the new language,
3. everyone else.

I see this transition as roughly analogous to the process that happened between 1830 and 1930, as Galois' ideas were slowly absorbed into the foundations of mathematics. These notes are written for people like myself who find category theory challenging, but who don't want to get left behind.

AHS:

1) universal language : similar constructions, transport problems / insights (products, "free", "closure"...)
2) clarifies notions : "natural transformation", "initial / universal objects"...
3) correct algebraic structure / tool (not only "meta")

important feature: duality ("co") (injective $\Leftrightarrow$ surjective, initial $\Leftrightarrow$ terminal ...)

Some aspects from the basic reading:

- Set theory: sets vs. classes, "small" $\leftrightarrow$ set
- category $\mathcal{A}$: Objects $A, A', B \ldots$

  morphisms $A \xrightarrow{f} B$, $\hom(A,B)$ (pairwise disjoint)

  $A \xrightarrow{id_A} A$, $id \circ f = f$, $f \circ id = f$

  composition $g \circ f : A \xrightarrow{f} B \xrightarrow{g} C$ (associative)

Ex.:
- **Set**: Obj: all sets, morphism: functions, id, comp. as usual
- **Vec**: (real) vector spaces, linear maps, id, comp ...
- **Grp**: groups, group homomorphisms, id, comp. ...
- **Top**: topol. spaces, continuous functions, ...
- **Monoid**: $(M, \bullet, e)$ monoid, $\mathcal{O} = \{M\}$, $\hom(M,M) = M$, $id_M = e$, $y \circ x := y \bullet x$
- **Posets**: $x \to y :\Leftrightarrow x \leq y$

- duality: $\mathcal{A}^{op}$ with objects as in $\mathcal{A}$, but $A \longleftarrow B$ as morphism, $f \circ^{op} g := g \circ f$

  Ex: • $(X, \leq)^{op} = (X, \geq)$

  • injective $\overset{op}{\longleftrightarrow}$ surjective [p. 26]

  duality principle: property P holds $\forall \mathcal{A} \iff P^{op}$ holds $\forall \mathcal{A}$

- functor $F: \mathcal{A} \to \mathcal{B}$: $\mathrm{Obj}_{\mathcal{A}} \ni A \mapsto F(A) \in \mathrm{Obj}_{\mathcal{B}}$, $(A \xrightarrow{f} A') \mapsto (F(A) \xrightarrow{F(f)} F(A'))$

  respecting id & comp. $\left(F(f \circ g) = F(f) \circ F(g),\ F(id_A) = id_{F(A)} \right)$

  Ex: $F: Vec^{op} \to Vec$, $V \mapsto \hat{V} = \hom(V, \mathbb{R})$, $(V \xleftarrow{f} W) \mapsto \left(\begin{array}{c} \hat{V} \to \hat{W} \\ g \mapsto g \circ f \end{array} \right)$

  covariant: $\forall A \in \mathcal{A}$ $\hom(A, \cdot): \mathcal{A} \to Set$,

  (contravariant: on $\cancel{\mathcal{A}}^{op}$)

  $\hom(A, B) \in Set$, $(B \xrightarrow{f} C) \mapsto \left[\begin{array}{c} \hom(A,B) \xrightarrow{\hom(A,f)} \hom(A,C) \\ g \longmapsto f \circ g \end{array} \right]$

  forgetful: $U: \mathcal{A} \to Set$ Ex: $U: Vec \to Set$

$F: \mathcal{A} \to \mathcal{B}$ Isomorphism: $\exists\, G: \mathcal{B} \to \mathcal{A}$: $F \circ G = id_{\mathcal{B}}$, $G \circ F = id_{\mathcal{A}}$

faithful: $F: \hom_{\mathcal{A}}(A, A') \to \hom_{\mathcal{B}}(F(A), F(A'))$ injective $\forall A, A'$

full: ... surjective

embedding: faithful & $F(A) = F(B) \Rightarrow A = B$ for objects

Ex: $U: \mathrm{Top} \to \mathrm{Set}$ faithful, ("$U(A) = A$"), not full, (morph. in Top continuous), no embedding ($(X, \tau_1) \neq (X, \tau_2)$)

$\mathrm{Set} \to \mathrm{Top}$ full embedding

$A \mapsto (A$, discrete topology$)$

- object-free def.: only specify morphisms, since objects arise from id_A
- subcategories: $Ob(\mathcal{A}) \subseteq Ob(\mathcal{B})$, $\hom_{\mathcal{A}}(A, A') \subseteq \hom_{\mathcal{B}}(A, A')$, id, comp. ...

"=": full subcateg.

Ex: Abelian groups $\subseteq$ Grp (full), Hausdorff spaces $\subseteq$ Top (full), ... "more conditions"

- skeleton: $\mathcal{A} \subseteq \mathcal{B}$ skeleton $:\Longleftrightarrow$ full, isomorphism-dense subcategory, no two distinct objects are isomorphic
  [Def. 4.9]

  4.14 Prop.: $\forall \mathcal{A}$ $\exists!$ skeleton + skeleton equiv. to $\mathcal{A}$
  ↖ up to isomorphism

  Cor: $\mathcal{A}, \mathcal{B}$ equivalent $\Longleftrightarrow$ skeletons isomorphic

- reflection: $\mathcal{A} \subseteq \mathcal{B}$ subcategory, $B \in \mathcal{B}$. $B \xrightarrow{r} A$ $\mathcal{A}$-reflection for B, if $\forall f\ \exists! g$

$$\begin{tikzcd} B \arrow[r, "r"] \arrow[dr, "f"'] & A \arrow[d, dashed, "g"] \\ & A' \in \mathcal{A} \end{tikzcd}$$

  Ex: • $\mathcal{A} = \text{Abel.grps} \subseteq \mathcal{B} = \text{Grp} \ni G \xrightarrow{r} G / \langle xy = yx,\ x,y \in G \rangle$ "quotient = refl."

  • $\mathcal{A}$ = complete metric spaces $\subseteq \mathcal{B}$ = Metric spaces $\ni (X,d) \xrightarrow{r} (\bar{X}, \bar{d})$ "completion"

- concrete category: category $\mathcal{A}$ & faithful ("forgetful") functor $U: \mathcal{A} \to \mathcal{X}$
  (+ concrete functors)
  "extra structure"
  $\mathcal{X} = \text{Set}$: "construct"

  Ex: TopVec with $U: \text{TopVec} \longrightarrow \text{Set}$
  (or TopGrp)
  or $U: \text{TopVec} \longrightarrow \text{Top}$
  or $U: \text{TopVec} \longrightarrow \text{Vec}$