Universal property

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- a morphism $(f, g) : (A, A') \to (B, B')$, where $f : B \to A$ and $g : A' \to B'$ are $\mathscr{C}$-morphisms, to the function

$$\mathrm{Hom}_{\mathscr{C}}(f, g) := h \mapsto g \circ h \circ f : \mathrm{Hom}_{\mathscr{C}}(A, A') \to \mathrm{Hom}_{\mathscr{C}}(B, B')$$

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- $\mathrm{Hom}_{\mathscr{C}}(A,-) : \mathscr{C} \to \mathsf{Set}$ which, for a $\mathscr{C}$-morphisms $f : B \to C$

  $$\mathrm{Hom}_{\mathscr{C}}(A,f) := \mathrm{Hom}_{\mathscr{C}}(\mathsf{id}_A,f) : \mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{C}}(A,C)$$

- Dually, $\mathrm{Hom}_{\mathscr{C}}(-,A) := \mathrm{Hom}_{\mathscr{C}^{op}}(A,-) : \mathscr{C}^{op} \to \mathsf{Set}$

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$$\begin{aligned}(F(g) \circ \alpha_B)(f) &= F(g)(F(f)(u)) = F(g \circ f)(u) \\ &= \alpha_C(g \circ f) = (\alpha_C \circ \mathrm{Hom}(A, g))(f)\end{aligned}$$

Yoneda lemma

Theorem (Yoneda)

The construction is a bijection

$$\mathrm{Hom}_{[\mathscr{C},\mathsf{Set}]}(\mathrm{Hom}_{\mathscr{C}}(A,-),F) \cong F(A)$$

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Now, let $\alpha : \mathrm{Hom}(A,-) \to F$ and let α' be the natural transformation associated to u_α

$$\alpha'_B(f) = F(f)(u_\alpha) = F(f)(\alpha_A(\mathsf{id}_A)) = \alpha_B(\mathrm{Hom}(A,f)(\mathsf{id}_A)) = \alpha_B(f)$$

so $\alpha' = \alpha$ $\square$

Yoneda embedding

Corollary

There is a bijection

$$\mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{[\mathscr{C}^{op},\mathsf{Set}]}(\mathrm{Hom}_{\mathscr{C}^{op}}(A,-),\mathrm{Hom}_{\mathscr{C}^{op}}(B,-))$$

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The associated functor $Y : \mathscr{C} \to [\mathscr{C}^{op}, \mathsf{Set}]$ is fully faithful.

- $Y(A) := \mathrm{Hom}_{\mathscr{C}}(-,A) : \mathscr{C}^{op} \to \mathsf{Set}$ for $A \in \mathscr{C}$
- $Y(f)_C := \mathrm{Hom}(\mathsf{id}_C, f) : \mathrm{Hom}(C,A) \to \mathrm{Hom}(C,B)$ for a $\mathscr{C}$-morphism $f : A \to B$

Definition (Representation)

A functor $F : \mathscr{C} \to \mathsf{Set}$ is called *representable* iff it is naturally isomorphic to $\mathrm{Hom}(A, -)$ for some $A \in \mathscr{C}$.

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An element $u \in F(A)$ has the universal property iff
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Proof.

Let (A, u) and (A', u') be universal elements; then there are unique $f : A \to A'$ and $f' : A' \to A$ such that $u' = F(f)(u)$ and $u = F(f')(u')$ so $u = F(f' \circ f)(u)$ and $u' = F(f \circ f')(u)$,

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now there must be an *unique* $h : A \to A$ such that $u = F(h)(u)$ i.e. id_A, so

$$f' \circ f = \mathsf{id}_A$$

Similarly for $f' \circ f = \mathsf{id}_{A'}$ □

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- Assuming $\alpha : \mathrm{Hom}(A, -) \to F$ is a natural isomorphism, then for $B \in \mathscr{C}$ and $v \in F(B)$ we have

$$F(\alpha_B^{-1}(v))(u) = \alpha_B(\alpha_B^{-1}(v)) = v$$

□

Example: subsets

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its universal element is $(\{0,1\},\{1\})$;
for any set X and $Y \in \mathcal{P}(X)$ (i.e. $Y \subseteq X$) there is a $f : X \to \{0,1\}$ such that $Y = \{x \mid f(x) \in \{1\}\}$

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Example: Free objects

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If $Free(X)$ exists for all X, then *Free* is an *adjoint* to U and $\eta : \mathsf{id}_{\mathsf{Set}} \to U \circ Free$ is called its unit.

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Definition (Bilinear mapping)

A mapping $\beta : U(A) \times U(B) \to U(C)$ is called bilinear when for all $a \in U(A)$ and $b \in U(B)$
both $\beta(a, -) : U(B) \to U(C)$ and $\beta(-, b) : U(A) \to U(B)$ are group homomorphisms.

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Its universal element is $(A \otimes B,\ (a, b) \mapsto a \otimes b)$;
for any abelian group G and bilinear $g : U(A) \times U(B) \to U(G)$ there is an $f : A \otimes B \to G$ such that $g = (a, b) \mapsto U(f)(a \otimes b)$

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Bonus: this *universal definition* works for R-Modules for a ring R, replacing bilinearity with R-bilinearity.

Generalized elements

Definition

We call a $\mathscr{C}$-morphism $a : T \to A$ a *generalized element* of A with form T.

Abusing notation we write $a \in A$ and, for $f : A \to B$, $f(a) := f \circ a \in B$

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- In **Set** generalized elements are T indexed families
- *points* are g.e. with form $\{*\}$
- *lines* are g.e. with form $(0, 1)$
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- if $1 \in \mathscr{C}$ is a final object, then the generalized elements with form 1 are called *global* elements

Generalized elements

Corollary

Let $f, g : A \to B$ be two $\mathscr{C}$-morphisms, if for all $a \in A$ we have $f(a) = g(a)$ then $f = g$

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Proof.

From representability of $\mathrm{Hom}(A, -)$, its universal element is (A, id_A), so for $\mathsf{id}_A \in A$ then

$$f = f(\mathsf{id}_A) = g(\mathsf{id}_A) = g$$

□