Limits, colimits, sources, sinks, pullbacks, pushouts

Steven Klein

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Sources and Sinks

Definition

Let I be a class and **A** be a category. A *source* is a pair $(A, (f_i)_{i \in I})$ consisting of an **A**-object A and a family of **A**-morphisms $f_i : A \longrightarrow A_i$.

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We may regard an object A as an empty source (a source indexed by the empty set).

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Remark

We may regard an object A as an empty source (a source indexed by the empty set).

Moreover we may regard an Morphism $f : A \longrightarrow A_1$ as an source indexed by the set $\{1\}$.

Sources and Sinks

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Let I be a class and $\mathbf{A}$ be a category. A *sink* is a pair $(A, (f_i)_{i \in I})$ consisting of an $\mathbf{A}$-object A and a family of $\mathbf{A}$-morphisms $f_i : A_i \longrightarrow A$.

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The object A is called the *codomain of the sink*.
The family of objects $(A_i)_{i \in I}$ is called the *domain of the sink*.

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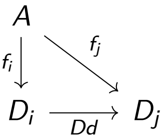
Example

A diagram in a category $\mathbf{A}$ with scheme $\bullet \rightrightarrows \bullet$ is essentially a pair of $\mathbf{A}$-Morphisms with common domain and common codomain.

Definition

Let $D : \mathbf{I} \longrightarrow \mathbf{A}$ be a diagram. An $\mathbf{A}$-source $(A, (f_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ with codomain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$ is called *natural for D* or *cone*, if for each $\mathbf{I}$-morphism $d : i \longrightarrow j$ the triangle



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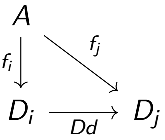

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Let $(L, (l_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ be a natural source for D with codomain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$. $(L, (l_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ is called a *limit*, if it fulfils the following universal property: For each natural source $(A, (f_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ with codomain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$ there exists a unique morphism $f : A \longrightarrow L$ with $f_i = l_i \circ f$ for each $i \in \mathrm{Ob}(\mathbf{I})$.

Example

We consider $\mathbf{I} = \mathbb{N}^{\mathrm{op}}$ the poset of non-negative integers with the opposite of the usual ordering as a category. A diagram $D : \mathbf{I} \longrightarrow \mathbf{A}$ is essentially a sequence

$$\ldots \xrightarrow{d_2} D_2 \xrightarrow{d_1} D_1 \xrightarrow{d_0} D_0$$

of $\mathbf{A}$-morphisms where

$$D(n+1 \longrightarrow n) = d_n, D(n+2 \longrightarrow n) = d_n \circ d_{n+1},$$

etc..

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A natural source for D is a source $(A, (f_n)_{n\in\mathbb{N}})$ with codomain $(D_n)_{n\in\mathbb{N}}$ with $f_n = d_n \circ f_{n+1}$ for each $n \in \mathbb{N}$.

Example

Now, let **A** be **Set**. A limit of the diagram D is a source $(L, (l_n)_{n\in\mathbb{N}})$ with codomain $(D_n)_{n\in\mathbb{N}}$, where L is the set of all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in D_n$ and $d_n(x_{n+1}) = x_n$ for each $n \in \mathbb{N}$ and where l_m is a restriction of the projection $\pi_m : \prod_{n\in\mathbb{N}} D_n \longrightarrow D_m$ for each m.

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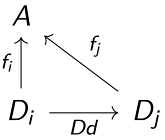
Such an limit is called a *projective limit* or an *inverse limit*.

Limits and colimits

Definition

Let $D : \mathbf{I} \longrightarrow \mathbf{A}$ be a diagram. An $\mathbf{A}$-sink $(A, (f_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ with domain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$ is called *natural for D* or *cocone*, if for each $\mathbf{I}$-morphism $d : i \longrightarrow j$ the triangle



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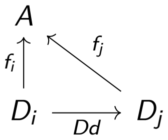

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Let $(L, (l_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ be a natural sink for D with domain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$. $(L, (l_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ is called a *colimit*, if it fulfils the following universal property: For each natural sink $(A, (f_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ with domain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$ there exists a unique morphism $f : L \longrightarrow A$ with $f_i = f \circ l_i$ for each $i \in \mathrm{Ob}(\mathbf{I})$.

Example

We consider $\mathbf{I} = \mathbb{N}$ the poset of non-negative integers as a category. A diagram $D : \mathbf{I} \longrightarrow \mathbf{A}$ is essentially a sequence

$$D_0 \xrightarrow{d_0} D_1 \xrightarrow{d_1} D_2 \xrightarrow{d_2} \dots$$

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etc..

A natural sink for D is a source $(A, (f_n)_{n \in \mathbb{N}})$ with domain $(D_n)_{n \in \mathbb{N}}$ with $f_n = f_{n+1} \circ d_n$ for each $n \in \mathbb{N}$.

Example

Now, let **A** be **Set**. We set $C := \bigcup_{i \in \mathbb{N}} (D_i \times \{i\})$ (the disjoint union of the D_i). We then define an equivalence relation $\sim$ on C by $(x, i) \sim (y, j)$ if and only if there exists a $k \geq i, j$ with $d_{ik}(x) = d_{jk}(y)$ where $d_{ij} := D(i \longrightarrow j)$.

Our limit is given by $(C/\sim, (p \circ \mu_i)_{i \in \mathbb{N}})$, where $p : C \longrightarrow C/\sim$ is the natural map from C onto the map of equivalence classes under $\sim$ and $\mu_i : D_i \longrightarrow C$ is the inclusion map.

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Such an colimit is called a *inductive limit* or a *direct limit*.

Theorem

Let $D : \mathbf{I} \longrightarrow \mathbf{A}$ be a diagram and $(L, (l_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ with codomain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$ be a limit of D. The following hold:
For each limit $(K, (k_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ with codomain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$ of D there exist an isomorphism $h : K \longrightarrow L$ such that for each $i \in \mathrm{Ob}(\mathbf{I})$

$$k_i = l_i \circ h$$

hold.

Proof.

First we show that we can cancel a limit $(L, (l_i)_{i \in \mathrm{Ob}(\mathbf{I})})$ from the left (i.e. for any pair $r, s : A \longrightarrow L$ of morphisms $l_i \circ r = l_i \circ s$ for each $i \in I$ implies $r = s$).

Since $(A, (l_i \circ r)_{i \in \mathrm{Ob}(\mathbf{I})})$ and $(A, (l_i \circ s)_{i \in \mathrm{Ob}(\mathbf{I})})$ are natural sources with the same codomain as $(L, (l_i)_{i \in \mathrm{Ob}(\mathbf{I})})$, our conjecture follows directly out of the uniqueness requirement in the definition of limit.

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Since $(L, (l_i)_{i\in\mathrm{Ob}(\mathbf{I})})$ and $(K, (k_i)_{i\in\mathrm{Ob}(\mathbf{I})})$ are limits with the same codomain, there exist unique morphisms h and k with $k_i = l_i \circ h$, $l_i = k_i \circ k$ for each $i \in \mathrm{Ob}(\mathbf{I})$.

Therefore we get $k_i \circ \mathrm{id}_K = k_i \circ k \circ h$ and $l_i \circ \mathrm{id}_L = l_i \circ h \circ k$. This yield $\mathrm{id}_K = k \circ h$ and $\mathrm{id}_L = h \circ k$. □

Products and coproducts

Definition

A source $(P, (p_i)_{i \in I})$ with codomain $(A_i)_{i \in I}$ that has the property that for each source $(A, (f_i)_{i \in I})$ with codomain $(A_i)_{i \in I}$ there exists a unique morphism $f : A \longrightarrow P$ such that $f_i = p_i \circ f$ for each $i \in I$ is called *product*.

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A product with codomain $(A_i)_{i \in I}$ is called a *product of the family* $(A_i)_{i \in I}$.

Example

In the category **Set** let $(A_i)_{i \in I}$ be a family of sets indexed by a set I and let $\prod_{i \in I} A_i$ be its cartesian product. Then $(\prod_{i \in I} A_i, (\pi_i)_{i \in I})$, where $\pi_j : \prod_{i \in I} A_i \longrightarrow A_j$ is the natural projection, is a product of $(A_i)_{i \in I}$.

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In the categories **Vec** and **Grp** direct products together with the natural projections are products.

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In the category **Top** topological products (i.e. the cartesian product of the sets equipped with the product topology) together with the natural projections are products.

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In a partially ordered class considered as a category a source $(P, (p_i)_{i \in I})$ is a product if and only if P is a meet.

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A sink $(P, (p_i)_{i \in I})$ with domain $(A_i)_{i \in I}$ that has the property that for each sink $(A, (f_i)_{i \in I})$ with domain $(A_i)_{i \in I}$ there exist a unique morphism $f : P \longrightarrow A$ such that $f_i = f \circ p_i$ for each $i \in I$ is called *coproduct*.

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A coproduct with domain $(A_i)_{i \in I}$ is called a *coproduct of the family* $(A_i)_{i \in I}$.

Products and coproducts

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In the category **Set** let $(A_i)_{i \in I}$ be a family of sets indexed by a set I. Then $(\bigcup_{i \in I}(A_i \times \{i\}), (\mu_i)_{i \in I})$, where $\mu_j : A_j \longrightarrow \bigcup_{i \in I}(A_i \times \{i\})$ is the natural embedding, is a product of $(A_i)_{i \in I}$.

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In the category **Vec** direct sums together with the injections $\mu_i : A_j \longrightarrow \bigoplus_{i\in I} A_i$ are coproducts.

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In the category **Grp** free products together with the injections $\mu_i : A_j \longrightarrow *_{i \in I} A_i$ are coproducts.

Products and coproducts

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In a poset considered as a category, coproducts are joints.

Products and coproducts

Remark

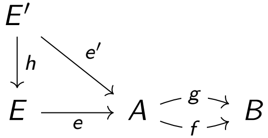
Let $D : \mathbf{I} \longrightarrow \mathbf{A}$ be a diagram with a discrete scheme. Every source with codomain $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$ is natural. A source is a limit of D if and only if it is a product of the family $(D_i)_{i \in \mathrm{Ob}(\mathbf{I})}$.

Equalizer and coequalizer

Definition

Let $f, g : A \longrightarrow B$ be two morphisms. A morphism $e : E \longrightarrow A$ with the properties

1. $f \circ e = g \circ e$ and
2. for any morphism $e' : E' \longrightarrow A$ with $f \circ e' = g \circ e'$ there exists a unique morphism h such that the triangle



commutes,

is called an *equalizer of f and g*.

Equalizer and coequalizer

Example

In the category **Set** (**Vec**, **Grp** or **Top**) let $f, g : A \longrightarrow B$ be morphisms. If $E = \{a \in A \mid f(a) = g(a)\}$ considered as a subset (linear subspace, subgroup, subspace) of A, then the inclusion from E to A is an equalizer of f and g.

Equalizer and coequalizer

Definition

Let $f, g : A \longrightarrow B$ be two morphisms. A morphism $c : B \longrightarrow C$ with the properties

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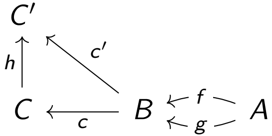

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is called an *coequalizer of f and g*.

Equalizer and coequalizer

Example

In the categoriey **Set** let $f, g : A \longrightarrow B$ be morphisms and let $\sim$ be the smallest equivalence relation on B such that $f(a) \sim g(a)$ for all $a \in A$. The natural map $p : B \longrightarrow B/\sim$ is a coequalizer of f and g.

Equalizer and coequalizer

Remark

Equalizer are limits of diagrams with the scheme $\bullet \rightrightarrows \bullet$.

Equalizer and coequalizer

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Equalizer are limits of diagrams with the scheme $\bullet \rightrightarrows \bullet$*.*
More precisely for a pair of morphisms $f, g : A \longrightarrow B$*, considered as a diagram* D *with scheme* $\bullet \rightrightarrows \bullet$ *a source* $(C, (e, h))$ *with codomain* (A, B) *is natural if* $g \circ e = h = f \circ e$*. Hence* e *is an equalizer of* f *and* g *if and only if the source* $(C, (e, f \circ e))$ *is a limit of* D*.*

Pullbacks and Pushouts

Definition

A commuting square

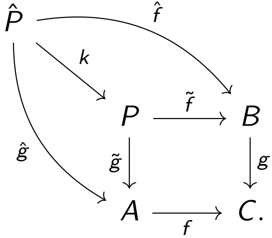
$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & B \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

is called a *pullback square* provided that for any commuting square

$$\begin{array}{ccc} \hat{P} & \xrightarrow{\hat{f}} & B \\ {\scriptstyle \hat{g}}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C, \end{array}$$

Pullbacks and Pushouts

There exists a unique morphism $k : \hat{P} \longrightarrow P$ for which the following diagram commutes



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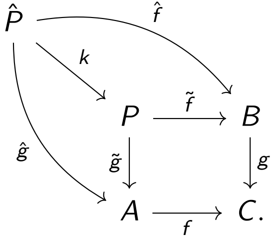

In this case the source $(P, (\tilde{g}, \tilde{f}))$ is called a *pullback* of the sink $(C, (f, g))$ and $\tilde{f}$ is called *pullback of f along g*.

Pullbacks and Pushouts

Remark

Pullbacks are limits of diagrams with scheme $\bullet \to \bullet \leftarrow \bullet$.

Pullbacks and Pushouts

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Pullbacks are limits of diagrams with scheme $\bullet \to \bullet \leftarrow \bullet$.
More precisely A square

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & B \\ \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C. \end{array}$$

is a pullback square if and only if the source $(P, (\tilde{g}, f \circ \tilde{g}, \tilde{f}))$ *is a limit of the sink* $(C, (f, g))$, *considered as a diagram in* $\mathbf{A}$ *with scheme* $\bullet \to \bullet \leftarrow \bullet$.

Pullbacks and Pushouts

Theorem

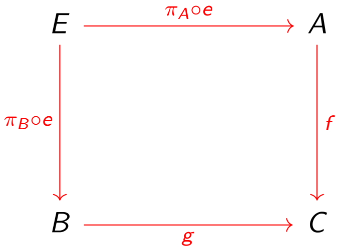
Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be morphisms.
If a source $(A \times B, (\pi_A, \pi_B))$ with codomain (A, B) is a product of A and B, and $e : E \longrightarrow A \times B$ is an equalizer of $f \circ \pi_A$ and $g \circ \pi_B$, then

$$\begin{array}{ccc} E & \xrightarrow{\pi_A \circ e} & A \\ {\scriptstyle \pi_B \circ e}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

is a pullback square.

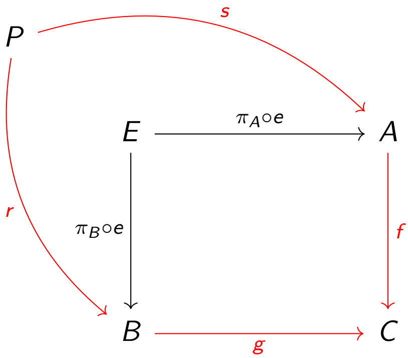
Pullbacks and Pushouts

The square commutes since $e : E \longrightarrow A \times B$ is an equalizer of $f \circ \pi_A$ and $g \circ \pi_B$.



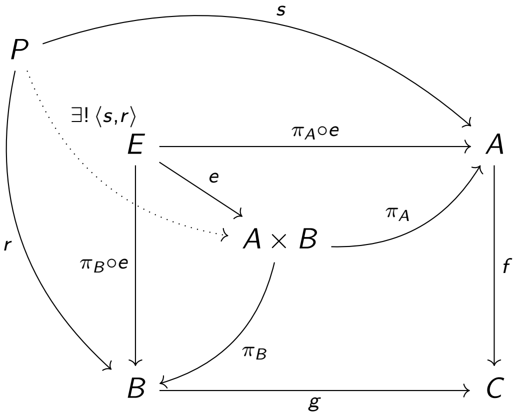
Pullbacks and Pushouts

Let P be an object and r, s be morphisms such that the following square commutes:



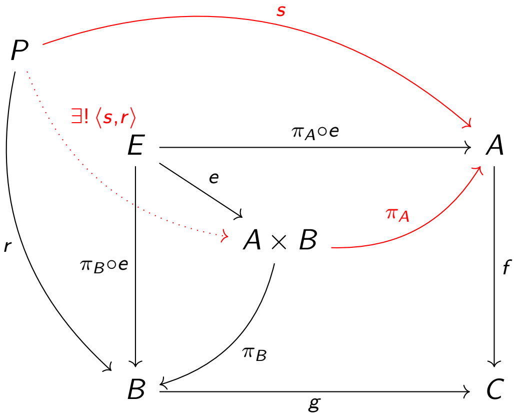
Pullbacks and Pushouts

Since $(A \times B, (\pi_A, \pi_B))$ is a product and $(P, (s, r))$ is a source with the same codomain as $(A \times B, (\pi_A, \pi_B))$, there exists a unique morphism $\langle s, r \rangle : P \longrightarrow A \times B$ with



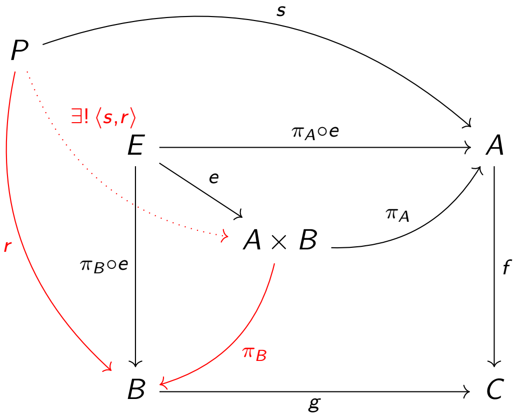
Pullbacks and Pushouts

$s = \pi_A \circ \langle s, r \rangle$ and



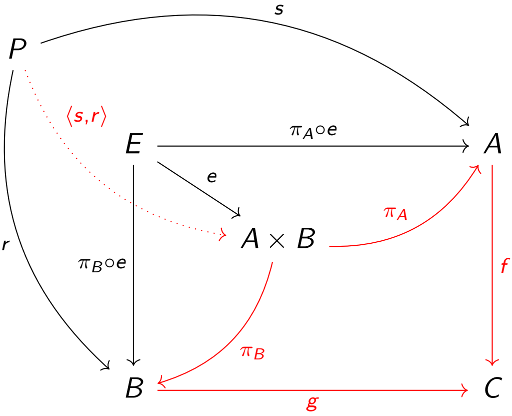
Pullbacks and Pushouts

$r = \pi_B \circ \langle s, r \rangle$.



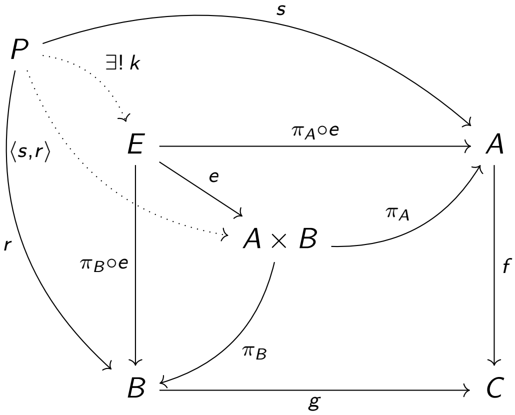
Pullbacks and Pushouts

We also have $f \circ s = g \circ r$ and consequently $f \circ \pi_A \circ \langle s, r\rangle = g \circ \pi_B \circ \langle s, r\rangle$.



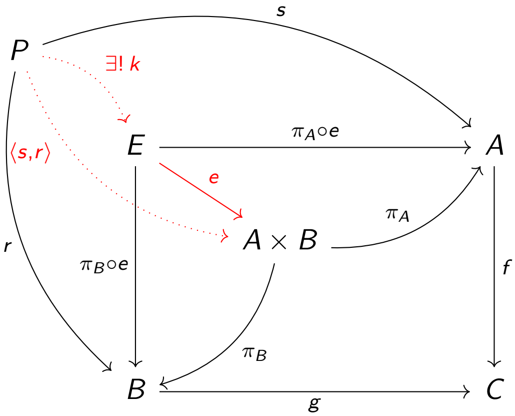
Pullbacks and Pushouts

Since e is an equalizer of $f \circ \pi_A$ and $g \circ \pi_B$, we infer the existence of a unique morphism $k : P \longrightarrow E$ with



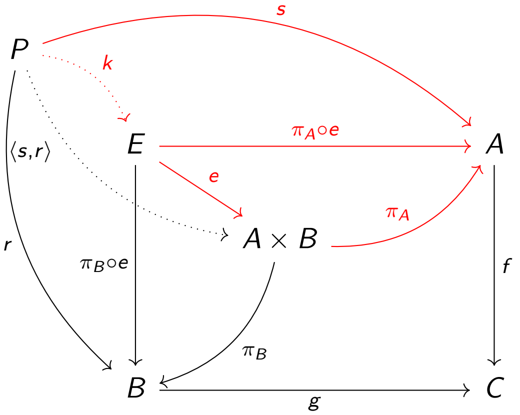
Pullbacks and Pushouts

$\langle s, r \rangle = e \circ k.$



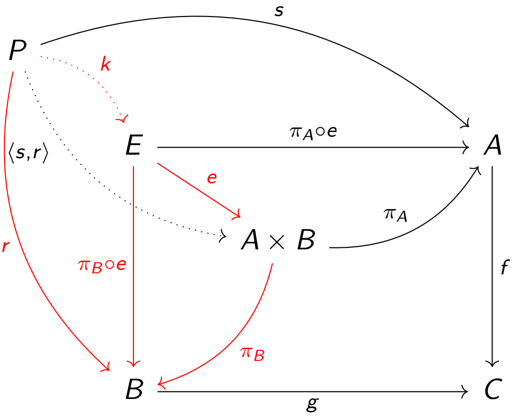
Pullbacks and Pushouts

Since we already know that $s = \pi_A \circ \langle s, r\rangle$ and $r = \pi_B \circ \langle s, r\rangle$, we we get $s = \pi_A \circ e \circ k$ and



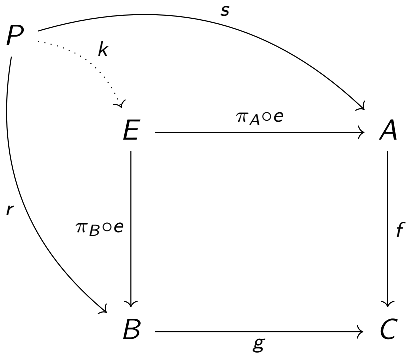
Pullbacks and Pushouts

$r = \pi_B \circ e \circ k.$



Pullbacks and Pushouts

We conclude that the following diagram commutes:



Pullbacks and Pushouts

We still have to show that k is unique with this property. Let t be another morphism with $s = \pi_A \circ e \circ t$ and $r = \pi_B \circ e \circ t$.

Then it holds $\pi_A \circ e \circ t = \pi_A \circ e \circ k$ and $\pi_B \circ e \circ t = \pi_B \circ e \circ k$. Since we can cancel products and equalizers from the left, we infer $k = h$.

Pullbacks and Pushouts

Example

Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be morphisms in **Set**, $P := \{(a, b) \in A \times B \mid f(a) = g(b)\}$ and let $\tilde{f} : P \longrightarrow A$ and $\tilde{g} : P \longrightarrow B$ be restrictions of the projections from $A \times B$. Then

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & A \\ {\scriptstyle \tilde{g}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

is a pullback square.

Pullbacks and Pushouts

Example

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is a pullback square.

We could construct similar examples in **Vec** or **Top**.

Pullbacks and Pushouts

Definition

A commuting square

$$\begin{array}{ccc} P & \xleftarrow{\tilde{f}} & B \\ {\scriptstyle \tilde{g}}\uparrow & & \uparrow{\scriptstyle g} \\ A & \xleftarrow[f]{} & C \end{array}$$

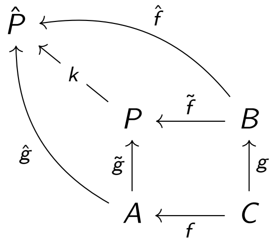
is called a *pushout square* provided that for any commuting square

$$\begin{array}{ccc} \hat{P} & \xleftarrow{\hat{f}} & B \\ {\scriptstyle \hat{g}}\uparrow & & \uparrow{\scriptstyle g} \\ A & \xleftarrow[f]{} & C \end{array}$$

Pullbacks and Pushouts

Definition

there exists a unique morphism $k : P \longrightarrow \hat{P}$ for which the following diagram commutes



Pullbacks and Pushouts

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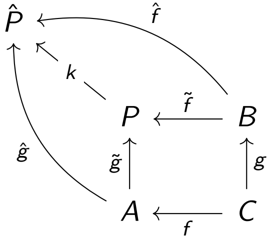

In this case the sink $(P, (\tilde{g}, \tilde{f}))$ is called a *pushout* of the source $(C, (f, g))$ and $\tilde{f}$ is called *pushout of f along g*.

Completeness

Definition

Let **A** be a category. If for each (finite) set-indexed family of objects in **A** there exists a product, then **A** is said to *have (finite) products*.

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If for each 2-sink in **A** there exists a pullback, then **A** is said to *have pullbacks*.

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Definition

Let **A** be a category. If for each finite diagram **A** there exists a limit, then **A** is said to be *finitely complete*.

Completeness

Definition

Let **A** be a category. If for each finite diagram **A** there exists a limit, then **A** is said to be *finitely complete*.

If for each small diagram **A** there exists a limit, then **A** is said to be *complete*.

Completeness

Example

The categories **Set**, **Top** and **Grp** are complete.

Completeness

Example

The categories **Set**, **Top** and **Grp** are complete.

The category of finite sets is finitely complete but not complete.

Completeness

Example

The categories **Set**, **Top** and **Grp** are complete.

The category of finite sets is finitely complete but not complete.

The category **Field** (objects are fields and morphisms are algebraic field extensions) is not finitely complete.

Completeness

Theorem

Let **A** *be a category. The following are equivalent*

1. **A** *is finitely complete;*
2. **A** *has finite products and equalizers;*
3. **A** *has pullbacks and a terminal object.*

Preservation of limits

Definition

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a functor. F is said to *preserve a limit* $\mathcal{L} = (L, (l_i))$ *of a diagram* $D : \mathbf{I} \longrightarrow \mathbf{A}$ provided that $(FL, (Fl_i))$ is a limit of the diagram $F \circ D : \mathbf{I} \longrightarrow \mathbf{B}$.

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If F preserves all limits of diagrams $D : \mathbf{I} \longrightarrow \mathbf{A}$, then F is said to *preserve all limits of a diagram* $D : \mathbf{I} \longrightarrow \mathbf{A}$ *with a scheme* $\mathbf{I}$.

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If F preserves all limits of diagrams $D : \mathbf{I} \longrightarrow \mathbf{A}$, then F is said to *preserve all limits of a diagram* $D : \mathbf{I} \longrightarrow \mathbf{A}$ *with a scheme* $\mathbf{I}$.

F is said to *preserve equalizers*, if F preserves all limits with the scheme $\bullet \rightrightarrows \bullet$. F is said to *preserve products*, if F preserves all limits over small discrete schemes. F is said to *preserve small limits*, if F preserves all limits over small schemes.

Preservation of limits

Example

For the category **Top**, the forgetful functor preserves limits.

Preservation of limits

Example

For the category **Top**, the forgetful functor preserves limits.

For the categories **Vec** and **Grp**, the forgetful functor preserves limits. It also preserves discrete colimits, but neither coproducts nor coequalizers.

Preservation of limits

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For the category **Top**, the forgetful functor preserves limits.

For the categories **Vec** and **Grp**, the forgetful functor preserves limits. It also preserves discrete colimits, but neither coproducts nor coequalizers.

The full embedding **Haus** $\longrightarrow$ **Top** preserves limits and coproducts, but not coequalizers.

Preservation of limits

Theorem

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a functor and $\mathbf{A}$ finitely complete. The following are equivalent:

1. *F preserves finite limits;*
2. *F preserves finite products and equalizers;*
3. *F preserves pullbacks and terminal objects.*